GEOMETRY OF DUAL SPACES OF REDUCTIVE GROUPS
(non-archimedean case)

By

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Introduction

J. M. G. Fell introduced in [10] the notion of the non-unitary dual space of a locally compact group. By his definition, it is a topological space. The main part of that paper deals with the basic properties of the topology of non-unitary dual space. This paper can be considered mainly as a continuation of such investigations, in the case of a p-adic reductive group G. We write down proofs of the basic properties of this topological space for such G. Since in the last twenty years after the appearance of [10] the topology of non-unitary dual has not attracted much attention, we begin with a longer introduction explaining motivations and reasons for writing this paper.

Let G be a connected reductive group over a non-archimedean local field F. The set of all equivalence classes of topologically irreducible unitary representations of G is denoted by $\hat{G}$. The set $\hat{G}$ is in a natural way a topological space. The topology may be described in terms of approximation of matrix coefficients. In the study of some topological space a first reduction may be to decompose it into disjoint union of open and closed subsets. In [26] such a decomposition of $\hat{G}$ was obtained. One could see easily that the decomposition in [26] was not the finest possible of such type. This author realized that the right setting for the decomposition in [26] of $\hat{G}$ is not $\hat{G}$ but another topological space. Note that the problem of understanding decompositions of $\hat{G}$ into open and closed subsets is closely related to the problem of finding connected components of $\hat{G}$. On the other hand, the problem of picking up all connected components of $\hat{G}$ is closely related to the unitarizability problem for G. Progress in the unitarizability problem today is limited by the lack of more detailed information about the non-unitary dual. Therefore it is not very expectable to understand at this stage of the development of the theory, what connected components of $\hat{G}$ are. However, we shall see that in the case of the non-unitary dual we can describe the connected components.

Let $\tilde{G}$ be the set of all equivalence classes of irreducible smooth representations of G. Then $\hat{G}$ is in a natural way in one-to-one correspondence with the subset of

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all unitarizable classes in $\tilde{G}$ which possess a G-invariant inner product. We identify $\hat{G}$ with this subset of $\tilde{G}$. One defines a topology on $\tilde{G}$ again by approximation of matrix coefficients. The induced topology on $\hat{G} \subseteq \tilde{G}$ is the old one. Now connected components of $\tilde{G}$ are open (and closed). The subsets of the decomposition of $\hat{G}$ in [26] are actually intersections of connected components of $\tilde{G}$ with $\hat{G}$. Thus the decomposition of $\hat{G}$ in [26] is the best possible in the sense of the non-unitary dual $\tilde{G}$. At this point let us say that $\tilde{G}$ with the above topology is in a natural way homeomorphic to the non-unitary dual of G in the sense of [10] (although this is not quite evident). The result of [26] about convergence of sequences in $\hat{G}$ also holds in $\tilde{G}$. Therefore, a natural background of most of the results of [26] is the non-unitary dual as a topological space.

The Bernstein center is a reason to consider $\tilde{G}$ as a topological space. In the description of the Bernstein center certain subsets of $\tilde{G}$ appear which are called connected components (see 2.1 of [4]). This paper verifies that the connected components of [4] are really connected components of the topological space $\tilde{G}$ with the topology of [10].

We have mentioned before that understanding of some important questions about the topology of $\hat{G}$ does not seem to be very near while in the case of $\tilde{G}$, as we will see, we have explicit answers: description of connected components, description of isolated representations modulo center, Hausdorffization of $\tilde{G}$, etc. The understanding of these questions for $\tilde{G}$ may also help to get an idea about analogous questions for $\hat{G}$. At this point let us recall an analogous situation in the case of algebraic varieties. There are lots of questions about algebraic varieties which are much more difficult for varieties of rational points. By the way, there are questions about the topological space $\tilde{G}$ which are more difficult than such questions for $\hat{G}$, as we will indicate later. For example, in [31] $\mathrm{GL}(n,F)^\wedge$ is described as an abstract topological space. On the other hand, we will see why is it not likely that there exists such an explicit description of the topological space $\mathrm{GL}(n,F)^\sim$.

Another motivation for a study of this topology is the unitarizability problem. In fact, very often one works essentially with the topological spaces $\tilde{G}$ in completeness arguments in the case of low semi-simple split ranks. In such considerations, it is a useful fact that $\hat{G}$ is a closed subset of $\tilde{G}$.

The topological space $\tilde{G}$ is a geometric object which naturally appears in considerations of some problems of the representation theory of G. The structure of a topological space is not a very rich one, so one may ask whether this rather simple structure contains interesting facts of the representation theory of G. Let us look at one example. Let $P = MN$ be a parabolic subgroup of G and let σ be an irreducible smooth representation of M. We denote by $\mathrm{Ind}_P^G(\sigma)$ a smoothly induced representation of G from P by σ (the induction that we consider is normalized). An important problem of the theory of non-unitary representations is to have some information about reducibility of $\mathrm{Ind}_P^G(\sigma)$ and if reducibility appears, to have some information about what the irreducible subquotients are. The importance of this problem comes from among other things, the unitarizability problem (let us recall that originally the theory of non-unitary representations was introduced because of unitary ones). More precisely, if $\mathrm{Ind}_P^G(\sigma)$ is an end of some complementary series, then all irreducible subquotients lie in $\hat{G}$ ([18]). So if one wants to have all of $\hat{G}$, one needs to pick up all irreducible subquotients of $\mathrm{Ind}_P^G(\sigma)$ when these are ends of complementary series. Suppose now that we have some $\pi \in \hat{G}$. Suppose that we know that $\sigma \in \tilde{M}$ is Hermitian and that π is a subquotient of $\mathrm{Ind}_P^G(\sigma)$. Then the irreducibility of $\mathrm{Ind}_P^G(\sigma)$ implies that σ is unitarizable (for archimedean fields see Proposition 3.4 of [21] or Lemma 3.11 of [15]). Therefore, one needs also to pick up all representations of G obtainable in this manner. Let us indicate how the topology of $\tilde{G}$ contains desired information in the case of $\mathrm{GL}(n, F)$. Let $P = MN$ be a parabolic subgroup in $\mathrm{GL}(n, F)$ and σ a smooth irreducible representation of M. Then there exists a sequence (ψ_n) of non-trivial unramified characters of M which converges to the trivial character such that all $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\psi_n \sigma)$ are irreducible (it is easy to find such a sequence (ψ_n)). Set $\pi_n = \mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\psi_n \sigma)$. Now the following facts hold:

(1) The representation $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\sigma)$ is irreducible if and only if the sequence (π_n) in $\mathrm{GL}(n, F)^{\sim}$ has exactly one limit point.

(2) The set of all composition factors of $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\sigma)$ equals the set of all limits of the sequence (π_n) in $\mathrm{GL}(n, F)^{\sim}$.

At this point it is natural to recall the Kazhdan–Lusztig type multiplicity formulas (which are not yet formulated for general p-adic reductive G, as far as this author knows). They give in principle an algorithm which gives answers to such questions (for fixed σ). But a problem is that those σ which are interesting in the unitarizability problem are pretty far from the standard representations considered in Kazhdan–Lusztig type formulas, so an algorithm in general involves many representations which do not appear in the final answer. In the unitarizability problem one needs usually to treat families of representations. The facts that we wrote about $\mathrm{Ind}_P^G(\sigma)$'s indicate that the topology of $\tilde{G}$ contains very interesting and highly non-trivial information. That also suggests that we will not have such an explicit description of the topology of $\mathrm{GL}(n, F)^{\sim}$ as we have of $\mathrm{GL}(n, F)^{\wedge}$ in [30].

The following reason for paying attention to the topology of $\tilde{G}$ is a connection with cohomological properties of representations. We shall explain it in more detail. It is known that (for general locally compact group) if $\pi \in \hat{G}$ has non-trivial first cohomology group then for each neighborhood U of π in $\hat{G}$ and for each neighborhood V of the trivial representation 1_G

$$(U \cap V) \setminus \{1_G\} \neq 0$$

(see [32]). One may hope that there exists a connection of such type also for higher cohomology groups (see 9.2 in Chapter III of [12]). But here the situation is different as we can see from the following simple example. Let G be a simple group over F of split rank k greater than 1. Then 1_G is isolated ([16]) and the k-th cohomology group of $\mathrm{St}(G)$ is non-trivial, where $\mathrm{St}(G)$ denotes the Steinberg representation of G ([6], [8]). So we have a quite different situation than in the case of the first cohomology groups. But 1_G and $\mathrm{St}(G)$ are inseparable in $\tilde{G}$: each neighborhood U of 1_G in $\tilde{G}$ intersects non-trivially each neighborhood V of $\mathrm{St}(G)$ in $\tilde{G}$ (moreover $(U \cap V)\setminus\{1_G, St(G)\} \neq \emptyset$). Therefore, there are reasons to consider the topology of $\tilde{G}$ even if one is interested only in cohomology groups of elements of $\hat{G}$. Also, one may be interested in cohomology groups of elements of $\tilde{G}$, and then there is no possiblility to remain in $\hat{G}$. Proceeding further we shall not restrict ourselves to cohomology groups. We shall consider the groups $\mathrm{Ext}^n_G(\pi_1, \pi_2)$ of all classes of n-extensions of π_1 and π_2 by smooth representations. Restriction to cohomology groups would confine us to a very small domain in $\tilde{G}$ close to 1_G, and it would sharply restrict the connection with the unitarizability problem. We will see now that there exists a relation between cohomological properties of representations and the topology of $\tilde{G}$ of the type suggested by A. Guichardet in II, 9.2. of [12] (taking $\tilde{G}$ instead of $\hat{G}$). Namely, $\pi_1, \pi_2 \in \tilde{G}$ and $\mathrm{Ext}^n(\pi_1, \pi_2) \neq 0$ for some $n \geqq 0$ implies that π_1 and π_2 are inseparable. It would be interesting to relate n to the topology. Motivated by the situation with the cohomology groups, one could expect that $\pi_1, \pi_2 \in \hat{G}$ and $\mathrm{Ext}^1(\pi_1, \pi_2) \neq 0$ implies that π_1 and π_2 are inseparable in $\hat{G}$, i.e., that the non-triviality of the first extension groups is detectable already in $\hat{G}$.

We shall explain now how interesting only a part of that topology can be, namely the part on $\hat{G}$. The fact that $\hat{G}$ is closed in $\tilde{G}$ is very useful here. Let E be any locally compact non-discrete field. Crucial unitary representations in the description of unitary duals of general linear groups over E are representations $u(\delta, n)$ indexed by an irreducible square integrable representation δ of some $\mathrm{GL}(m, E)$ and a positive integer n (see the introduction of [28]). For a unitary character δ of $E^\times$ we have $u(\delta, n) = \delta \circ \det_n$. For an archimedean field E the only representations $u(\delta, n)$ which are not of this type are $u(\delta, n)$ for square-integrable $\delta \in \mathrm{GL}(2, \mathbf{R})^\wedge$. In order to solve the problem of classification of all irreducible unitary representations of $\mathrm{GL}(m, \mathbf{R})$ with non-trivial $(\mathfrak{g}, K)$-cohomology, B. Speh introduced in [22] representations $I(k)$. Her representations $I(k)$ are closely related to the representations $u(\delta, n)$. She proved by global (adelic) methods that the $I(k)$ are unitary. All irreducible unitary representations of $\mathrm{GL}(n, \mathbf{R})$ with non-trivial cohomology groups are "built" from these representations (for a precise statement see [22]). Thus the study of cohomologically non-trivial irreducible unitary representations of $\mathrm{GL}(n, \mathbf{R})$ directs attention to the representations $u(\delta, n)$. Cohomologically non-trivial irreducible unitary representations for real reductive groups are now classified (see [33], [34], and also [35]). Cohomological non-triviality for irreducible unitary representations is much more infrequent in the non-archimedean case. The only such representations for GL(n, F) can be St(GL(n, F)) and $1_{GL(n,F)}$ (this is also the case for any simple G, by [6] or [8]). Therefore the study of irreducible unitary cohomologically non-trivial representations of GL(n, F) does not direct our attention to any new irreducible unitary representation (the Steinberg representation being square-integrable may be assumed as a part of the theory of non-unitary dual, having in mind the Langlands classification). But consideration of inseparable points in GL$(n, F)^\wedge$ directs attention to the representations $u(\delta, n)$ and this is actually how we came to consider them. Since consideration of inseparable points for GL$(n, \mathbf{R})$-groups focuses attention on the representations $u(\delta, n)$ as well, we may say that topology, unlike cohomology, is a unifying factor between unitarizability problems for GL(n) over non-archimedean and archimedean fields. By the way, let us note that in the non-archimedean case consideration of the problem of classifying of pairs $\pi_1, \pi_2 \in$ GL$(n, F)^\wedge$ such that Ext$^1(\pi_1, \pi_2) \neq 0$ would also lead to the representations $u(\delta, n)$ (see Proposition 6.5 of [30]). But such a problem is certainly more difficult than the problem of classifying all elements of $\hat{G}$ with non-trivial cohomology groups.

The topological space $\hat{G}$ is a geometrical object and develops its own intuition. What is in some sense perhaps more important than the construction of the representations $u(\delta, n)$ is that this intuition led to the opinion that the $u(\delta, n)$ are sufficient for the classification of the unitary dual of GL(n, F). (Recall that for GL$(n, \mathbf{C})$ all building blocks of the unitary dual had been known for several years, but even a conjecture that they were enough for the whole of GL$(n, \mathbf{C})^\wedge$ was missing.)

We hope that this paper is a step toward an understanding of the natural structure of $\tilde{G}$.

Now we shall describe the contents of this paper according to sections. The first section fixes notation and recalls the notion of the Bernstein center and its basic properties. We introduce the Bernstein center in the following way. Let $\mathscr{Z}(G)$ be the center of the category Alg(G) (i.e. $\mathscr{Z}(G)$ is the ring of all natural endomorphisms of the identity functor). For each $(\pi, V) \in \tilde{G}$ and $T \in \mathscr{Z}(G)$, T acts as multiplying by a scalar which we will denote by $\chi_\pi(T)$. We shall call χ_π an infinitesimal character of π. Let $\Omega(G)$ be the set of all infinitesimal characters of elements in $\tilde{G}$. There is a natural mapping

$$\nu : \tilde{G} \to \Omega(G).$$

One supplies $\Omega(G)$ with the topology of pointwise convergence. This $\Omega(G)$ may be described by unramified characters of Levi subgroups in the following way. Let M be a Levi subgroup of a parabolic subgroup P of G. Let Unr(M) be the group of all unramified characters of M. This is a commutative algebraic group (isomorphic to some $(\mathbf{C}^\times)^k$). Fix an irreducible cuspidal representation ρ of M. Let $\psi \in \mathrm{Unr}(M)$. For an irreducible subquotient π of $\mathrm{Ind}_P^G(\psi\rho)$, χ_π does not depend on π so we are able to define a mapping

$$\nu_\rho : \mathrm{Unr}(M) \to \Omega(G).$$

Now $\nu_\rho(\mathrm{Unr}(M))$ is a connected component of $\Omega(G)$ and it is an open and closed subset of $\Omega(G)$.

In the second section we recall the definition of the topology of $\hat{G}$. Then we give simple and natural proofs of the results of [26] using the Bernstein center (these proofs were announced in the second section of [28]). The first result is that $\nu \mid \hat{G}$ is continuous. This means that convergence in $\hat{G}$ implies convergence of infinitesimal characters. This is proved for real reductive groups by P. Bernet and J. Dixmier in [1]. Let $\Omega \subseteq \Omega(G)$ be a connected component. Set

$$\tilde{G}_\Omega = \nu^{-1}(\Omega) \quad \text{and} \quad \hat{G}_\Omega = \tilde{G}_\Omega \cap \hat{G}.$$

A direct consequence of the continuity of ν is that the $\hat{G}_\Omega$ are open and closed subsets (which define a partition of $\hat{G}$). Let $P = MN$ be a parabolic subgroup of G and τ a smooth representation of M of finite length. Using a fundamental property of the topology of $\hat{G}$ concerning locally quasi-compactness, one obtains that the set of all $\psi \in \mathrm{Unr}(M)$ such that $\mathrm{Ind}_P^G(\psi\tau)$ has an irreducible unitarizable subquotient is a compact subset of $\mathrm{Unr}(M)$. We provide a proof of the last result which does not use general results on the topology of $\hat{G}$. It is in the third section.

We fix a topology on $\tilde{G}$ in the fourth section: a representation $\pi \in \tilde{G}$ belongs to the closure of $X \subseteq \tilde{G}$ if for each matrix coefficient c of π, each $\alpha > 0$, and each compact subset K of G there exists a representation $\sigma \in X$ with a matrix coefficient c_1 such that $|c(g) - c_1(g)| < \alpha$ for any $g \in K$. This topology on $\tilde{G}$ induces the standard topology on $\hat{G}$. It is proved here that $\nu : \tilde{G} \to \Omega(G)$ is continuous, which implies that $\tilde{G}_\Omega$ is an open and closed subset for $\Omega \subseteq \Omega(G)$ a connected component (Theorem 4.2). Moreover, we prove that the $\tilde{G}_\Omega$ are connected components of the topological space $\tilde{G}$ (Theorem 4.5). In the rest of the fourth section we deal with the connection of the topology of $\tilde{G}$ and decompositions of the category $\mathrm{Alg}(G)$ into direct sums of subcategories. Each direct sum decomposition $\mathrm{Alg}(G) = \oplus C_i$ induces in a natural way a partition of $\tilde{G} = \cup X_i$. For different direct sum decompositions one will obtain different partitions. A partition comes from the direct sum decomposition if and only if it is a partition into open and closed subsets. In this way the direct sum decomposition obtained in [3] corresponds to the partition of $\tilde{G}$ into connected components and therefore it is the finest possible such decomposition.

Description of the topology of $\hat{G}$ in terms of characters is very useful for getting finer information about the topology. The complete description for $\hat{G}$ is obtained in [18] by D. Miličić. We show in the fifth section that an analogous description holds for $\tilde{G}$. Since with non-unitary representations positive definite functors are not associated and C^*-algebra representations do not appear, we use methods of J. M. G. Fell from [11] instead of the methods of [18]. Let $H(G)$ be the Hecke algebra of G, let $H(G)'$ be the space of all linear forms on $H(G)$ supplied with the topology of pointwise convergence. Let $H(G)'_+$ be the set of all finite sums of characters of irreducible smooth representations. For $\varphi_1, \varphi_2 \in H(G)'$ set $\varphi_1 \leqq \varphi_2$ if and only if $\varphi_2 - \varphi_1 \in H(G)'_+$. For $X \subseteq \tilde{G}$ let Θ_X be the set of all characters of elements of X. Then the topology of $\tilde{G}$ is described by characters in the following way: $\pi \in \mathrm{Cl}\, X$ if and only if there exists $\varphi \in \mathrm{Cl}(\Theta_X)$ such that $\Theta_\pi \leqq \varphi$ (Theorem 5.4). We use ideas of Fell from [11], in the proof of this criterion.

Let us note that D. Miličić suggested in Remark 3.7 of his Ph.D. thesis (University of Zagreb, 1973) that it would be interesting to prove such a description of the topology of the non-unitary dual for real semi-simple groups. As far as we know this has not yet been done. In this paper we prove it for p-adic reductive groups (originally, we planned to deal in this paper with both cases, non-archimedean and archimedean, but that would have made this paper considerably longer).

Theorem 5.5 gives a few other equivalent descriptions of the topology of $\tilde{G}$. This theorem will be used later to show that the topology of $\tilde{G}$ is the same as the topology introduced in [10]. After showing that $\nu : \tilde{G} \to \Omega(G)$ is a closed mapping, we show that $(\tilde{G}, \Omega(G), \nu)$ is a Hausdorffization of (in general non-Hausdorff) topological space $\tilde{G}$ (for a precise statement see Theorem 5.7). Theorem 5.7 can be expressed in the following way: the set of all infinitesimal characters of irreducible smooth representations with pointwise convergence forms a Hausdorffization of $\tilde{G}$. Proposition 5.10 describes pairs of representations in $\tilde{G}$ which are inseparable: π_1 and π_2 in $\tilde{G}$ are inseparable if and only if $\nu(\pi_1) = \nu(\pi_1)$. Recall that $\nu(\pi_1) = \nu(\pi_2)$ if and only if $\chi_{\pi_1} = \chi_{\pi_2}$. Now $\pi_1, \pi_2 \in \tilde{G}$ and $\mathrm{Ext}^*(\pi_1, \pi_2) \neq 0$ implies that π_1 and π_2 are not separable. At the end of the fifth section we obtain that $\hat{G}$ is a closed subset of $\tilde{G}$.

The sixth section deals with isolated representations modulo center. For $\pi \in \tilde{G}$ let ω_π be the central character of π and let $\tilde{G}_{\omega_\pi}$ be the set of all $\sigma \in G$ such that $\omega_\pi = \omega_\sigma$. We say that $\pi \in \tilde{G}$ (resp. $\pi \in \hat{G}$) is isolated modulo center in $\tilde{G}$ (resp. in $\hat{G}$) if $\{\pi\}$ is an open subset of $\tilde{G}_{\omega_\pi}$ (resp. $\tilde{G}_{\omega_\pi} \cap \hat{G}$). Now Proposition 6.3 characterizes isolated representations modulo center in $\tilde{G}$: $\pi \in \tilde{G}$ is isolated modulo center in $\tilde{G}$ if and only if it is cuspidal. So irreducible cuspidal representations of G are characterized as isolated representations modulo center in $\tilde{G}$. We also obtain a topological characterization of projective and injective reprsentations in $\tilde{G}$: $\pi \in \tilde{G}$ is an injective and projective object in the category $\mathrm{Alg}(G)$ if and only if $\{\pi\}$ is an open subset of $\tilde{G}$.

Having in mind the Jacquet subrepresentation theorem, we have an understanding of $\tilde{G}$ in terms of isolated representations modulo center in non-unitary duals of Levi subgroups. It would be interesting to obtain such an understanding for $\hat{G}$ and isolated representations modulo center in unitary duals of Levi subgroups. Up to now this is available only for $\mathrm{GL}(n, F)$. Recall that we also have such an understanding of tempered representations in $\hat{G}$. Here one considers "being isolated" with respect to the Plancherel measure in the reduced dual space (see [20]).

Let $P = MN$ be a parabolic subgroup of G and τ a smooth representation of M of finite length. In the seventh section we prove that the set of all $\psi \in \mathrm{Unr}(M)$, such that $\mathrm{Ind}_P^G(\psi\tau)$ has an irreducible subquotient with bounded matrix coefficient, is relatively compact (Theorem 7.1).

In the eighth section we prove that $\tilde{G}$ is naturally equivalent as a topological space to the non-unitary dual space as it was introduced by J. M. G. Fell in [10].

One of the main tools that we use in the proofs of this paper is the Bernstein center and the facts about it. Some proofs can be carried out without use of the Bernstein center (for example, the results of the second section were proved in [26] without the Bernstein center).

I wish to thank F. Rodier who brought my attention to the Bernstein center and its connection with [25] and [26]. Conversations with various people helped me to refine some of the ideas presented in this paper. Among them let me mention H. Kraljević, D. Miličić and P. Sally. The interest of P. Gerardin was an important motivating factor for writing this paper.

In this paper we make no distinction between a representation and its class when this does not cause confusion. The characteristic function of a subset X of a set Y will be denoted by ch_X.

1. Notation and Bernstein center

We shall recall some facts about the Bernstein center following mainly [3] and [4].[†]

Let F be a non-archimedean local field and G the group of rational points of a connected reductive group defined over F. The modulus of F will be denoted by $|\ \ |_F$.

By $\tilde{G}$ we shall denote the set of all equivalence classes of irreducible smooth representations of G, and by $\hat{G}$ the set of all unitarizable classes in $\tilde{G}$. The group of all unramified characters of G will be denoted by $\mathrm{Unr}(G)$. The subgroup of all

[†] Concerning the Bernstein center we often follow the notation of the first version of [4].

unitary characters will be denoted by $\mathrm{Unr}^u(G)$. $\mathrm{Unr}(G)$ is in the natural way an abelian complex algebraic group, isomorphic to $(\mathbf{C}^\times)^k$, where k is the dimension of the maximal split torus in the center $Z(G)$ of G. The topology obtained in this way on $\mathrm{Unr}(G)$ coincides with the topology of pointwise convergence and with the topology of uniform convergence over compacts, of characters in $\mathrm{Unr}(G)$. A minimal parabolic subgroup P_0 of G will be fixed and by a standard parabolic subgroup we shall mean a standard parabolic subgroup with respect to P_0. For each parabolic subgroup P a Levi subgroup M will be fixed. We have the decomposition $P = MN$ where N is the unipotent radical of P.

A cuspidal pair (M, ρ) of G is a pair consisting of the Levi subgroup M of G and of an irreducible cuspidal representation ρ of M (more precisely, a class). Conjugation of cuspidal pairs by elements of G is defined in the natural way. The set of all cuspidal pairs modulo conjugation is denoted by $\Omega(G)$. Let $(M, \rho) \in \Omega(G)$. The image of the map

$$\nu_\rho \cdot \mathrm{Unr}(M) \to \Omega(G), \qquad \psi \to (M, \psi\rho),$$

is called a connected component of $\Omega(G)$. Since a connected component is a quotient of $\mathrm{Unr}(M)$, it has in the natural way the structure of a complex affine algebraic variety. The set $\Omega(G)$ is a disjoint union of connected components and this decomposition will be denoted by $\Omega(G) = \bigcup \Omega$. Now $\Omega(G)$ may be considered as a complex algebraic variety, being the union of infinitely many connected components, and it has the natural topology. In that topology connected components are open, closed, and connected sets. Thus these connected components are just connected components of $\Omega(G)$ as a topological space. Note that the points of $\Omega(G)$ have countable bases of neighborhoods (moreover, $\Omega(G)$ has a countable basis of open sets).

Now we are going to describe $\Omega(G)$ more explicitly.

Let A_0 be the maximal split torus in G contained in P_0. Let $W = W(A_0)$ be the Weyl group of A_0, i.e. the quotient of the normalizer of A_0 by the centralizer of A_0.

By $\mathscr{P}$ we shall denote some set of representatives for the relation "being associated" among standard parabolic subgroups. We shall fix one such $\mathscr{P}$. Let $\mathscr{M} = \{M; P = MN \in \mathscr{P}\}$. In the sequel, for $(M, \rho) \in \Omega(G)$ we shall always assume that $M \in \mathscr{M}$. Note that now if $(M_1, \rho_1) = (M_2, \rho_2)$ in $\Omega(G)$ with $M_i \in \mathscr{M}$, $i = 1, 2$, then $M_1 = M_2$. Thus the projection on the first coordinate

$$\Omega(G) \to \mathscr{M}$$

is well defined.

Let $(M, \rho) \in \Omega(G)$, $M \in \mathscr{M}$ and let Ω be the connected component in which (M, ρ) is contained. Set $D = \mathrm{Unr}(M)\rho \subseteq \tilde{M}$. Put

$$\mathrm{Unr}(M)^\rho = \{\psi \in \mathrm{Unr}(M); \psi\rho \cong \rho\}.$$

Note that $\mathrm{Unr}(M)^\rho$ is a subgroup of $\mathrm{Unr}(M)$ and this subgroup is finite. The set D can be considered as a connected component of $\Omega(M)$ and it is isomorphic to $\mathrm{Unr}(M)/\mathrm{Unr}(M)^\rho$. It is easy to see that $\mathrm{Unr}(M)^\rho$ does not depend on $\rho \in D$, so we shall write also $\mathrm{Unr}(M)^D$ for $\mathrm{Unr}(M)^\rho$ when $\rho \in D$.

The normalizer of M in G acts on $\tilde{M}$. We define

$$W^D = \{w \in W;\ wMw^{-1} = M \text{ and } wD = D\}.$$

Now W^D acts analytically on D and on $\mathrm{Unr}(M)$. The action on $\mathrm{Unr}(M)$ is by algebraic automorphisms and preserves $\mathrm{Unr}(M)^\rho$.

Finally, the connected component Ω of $\Omega(G)$ which contains (ρ, M) equals

$$W^D \backslash D \cong W^D \backslash (\mathrm{Unr}(M)/\mathrm{Unr}(M)^\rho).$$

Note that $\mathrm{Unr}(M)/\mathrm{Unr}(M)^\rho$ is again isomorphic to some $(\mathbf{C}^\times)^k$. Also the regular functions on $W^D \backslash D$ are just regular functions on D invariant for W^D.

The continuous mappings

$$\mu_1 : \mathrm{Unr}(M) \to D, \qquad \mu_2 : D \to \Omega$$

are open and so the composition

$$\nu_\rho : \mathrm{Unr}(M) \to \Omega \subseteq \Omega(G)$$

is open. Therefore, simple topological considerations imply that if (x_n) converges to x where $x_n, x \in \Omega$, then there exists a convergent sequence (ψ_n) in $\mathrm{Unr}(M)$ such that $\nu_\rho(\psi_n) = x_n$ for all n and $\nu_\rho(\lim_n \psi_n) = x$.

Suppose that X is a compact subset of Ω. Let (y_n) be a sequence in $\mu_2^{-1}(X)$. Then $(\mu_2(y_n))$ has convergent subsequence, say $(\mu_2(y_{n(k)}))_k$. By the above remark there exists a sequence $w_k \in W^D$ such that $(w_k y_{n(k)})_k$ converges. Since W^D is finite, there exists a convergent subsequence of $(y_{n(k)})_k$. Thus $\mu_2^{-1}(X)$ is compact. Analogously $\mu_1^{-1}(\mu_2^{-1}(X)) = \nu_\rho^{-1}(X)$ is compact. Thus $\nu_\rho^{-1}(X)$ is compact whenever $X \subseteq \Omega$ is compact.

For $M \in \mathcal{M}$, $P(M)$ will denote the unique $P \in \mathcal{P}$ such that $P = MN$. If τ is a smooth representation of M then $\mathrm{Ind}_{P(M)}^G(\tau)$ will denote the (smoothly) induced representation of G from $P(M)$ by τ. The induction that we consider is normalized.

We define

$$\nu : \tilde{G} \to \Omega(G)$$

in the following way. Let $\pi \in \tilde{G}$. Then there exists a cuspidal pair (M, ρ) (unique up to conjugation) such that π is a subquotient of $\mathrm{Ind}_{P(M)}^G(\rho)$. Set $\nu(\pi) = (M, \rho) \in \Omega(G)$. Clearly, for $x \in \Omega(G)$, $\nu^{-1}(x)$ is finite. Also ν is a surjective mapping. If Ω is a connected component in $\Omega(G)$, then $\nu^{-1}(\Omega)$ will be called a connected component of $\tilde{G}$ and denoted by $\tilde{G}_\Omega$.

An invariant measure dg on G will be fixed.

We shall denote by $H(G)$ the Hecke algebra of G, i.e. the convolution algebra of all complex-valued locally constant compactly supported functions on G. Equivalently, $H(G)$ is the convolution algebra of all locally constant compactly supported measures on G. For an open compact subgroup K of G, $H(G, K)$ will denote the subalgebra of $H(G)$ of all K-biinvariant functions.

Let $\mathfrak{Z}(G)$ be the space of all invariant distributions T on G such that $T * f$ is compactly supported for each $f \in H(G)$. Clearly, $\mathfrak{Z}(G) * H(G) \subseteq H(G)$. We shall call $\mathfrak{Z}(G)$ the Bernstein center of G.

Let (π, V) be a smooth representation of G. The action of $\mathfrak{Z}(G)$ on V is defined in the following way. Let $T \in \mathfrak{Z}(G)$ and $v \in V$. Choose $f \in H(G)$ so that $\pi(f)v = v$; here $\pi(f)$ denotes the operator $\int_G f(g)\pi(g)dg$. Define

$$\pi(T)v = \pi(T * f)v.$$

Now $\pi(T * f) = \pi(T)\pi(f)$ for $T \in \mathfrak{Z}(G)$ and $f \in H(G)$. Suppose that (π, V) is irreducible. Then each $T \in \mathfrak{Z}(G)$ acts as the multiplication by some constant. This constant will be denoted by $\chi_\pi(T)$. We shall call χ_π an infinitesimal character of π. In this way we obtain a function

$$\hat{T} : \pi \mapsto \chi_\pi(T), \qquad \hat{T} : \tilde{G} \to \mathbf{C}.$$

We shall denote the algebra of all regular functions on $\Omega(G)$ by $\mathbf{C}[\Omega(G)]$. For $x \in \Omega(G)$ let $e_x : \mathbf{C}[\Omega(G)] \to \mathbf{C}$ be the character $e_x(\varphi) = \varphi(x)$. Let $(\pi_i, V_i) \in \tilde{G}$ for $i = 1, 2$. If $\nu(\pi_1) = \nu(\pi_2)$, then $\chi_{\pi_1} = \chi_{\pi_2}$. Thus, we can define for $T \in \mathfrak{Z}(G)$

$$\Omega(G) \to \mathbf{C}, \qquad x \mapsto \chi_\pi(T)$$

if $\nu(\pi) = x$. This mapping, which will be denoted by $\hat{T}$, is a regular function on $\Omega(G)$. By construction we have

$$\chi_\pi(T) = e_{\nu(\pi)}(\hat{T}).$$

The basic fact about the Bernstein center is that the mapping

$$T \to \hat{T}, \qquad \mathfrak{Z}(G) \to \mathbf{C}[\Omega(G)]$$

is an isomorphism of vector spaces. In fact, it is an algebra isomorphism with suitably defined multiplication in $\mathfrak{Z}(G)$ (see [3]).

At the end of this paragraph we shall give a few simple observations about the topology of $\Omega(G)$.

1.1. Remarks. (i) Let (x_n) be a sequence in $\Omega(G)$. Note that (x_n) is a convergent sequence if and only if $(e_{x_n}(\varphi)) = (\varphi(x_n))$ is a convergent sequence for each $\varphi \in \mathbf{C}[\Omega(G)]$. Also (x_n) converges to $x \in \Omega(G)$ if and only if $(e_{x_n}(\varphi))$ converges to $e_x(\varphi)$ for each $\varphi \in \mathbf{C}[\Omega(G)]$.

(ii) A set $X \subseteq \Omega(G)$ is relatively compact if and only if the set

$$\{e_x(\varphi) : x \in X\} \subseteq \mathbf{C}$$

is bounded, for any $\varphi \in \mathbf{C}[\Omega(G)]$.

(iii) For $(\rho, M) \in \Omega(G)$ the mapping

$$\nu_\rho : \mathrm{Unr}(M) \to \Omega(G), \qquad \psi \mapsto (M, \psi\rho)$$

is proper, i.e. $\nu_g^{-1}(X)$ is compact if $X \subseteq \Omega(G)$ is compact.

We shall say that the sequence (χ_{π_n}) of infinitesimal characters converges pointwise to χ_π, π_n, $\pi \in \tilde{G}$, if $(\chi_{\pi_n}(T))$ converges to $\chi_\pi(T)$ for each $T \in \mathscr{Z}(G)$. By previous remarks (χ_{π_n}) converges pointwise to χ_π if and only if $(\nu(\pi_n))$ converges to $\nu(\pi)$ (see also the end of the proof of Lemma 2.1).

2. Application of the Bernstein center to the topology of unitary dual

We shall denote the space of all continuous complex-valued functions on G by $C(G)$. The subspace of all locally constant functions in $C(G)$ is denoted by $C^\infty(G)$. The space $C(G)$ (and its subspaces) are supplied with the open-compact topology. The closure operator will be denoted by Cl. This space has a countable basis of open sets, so closure may be described by convergent sequences. Let (f_n) be a sequence in $C(G)$ and $f \in C(G)$. Then (f_n) converges to f if and only if for any compact subset X of G, $(f_n \mid X)$ converges uniformly to $f \mid X$.

For a smooth representation (π, V) of G, $(\tilde{\pi}, \tilde{V})$ will denote the (smooth) contragradient representation of (π, V). The canonical pairing on $V \times \tilde{V}$ will be denoted by $\langle\ ,\ \rangle$. Let $v \in V$, $\tilde{v} \in \tilde{V}$. The function

$$c_{v,\tilde{v}} : g \to \tilde{v}(\pi(g)v) = \langle \pi(g)v, \tilde{v} \rangle$$

is called a matrix coefficient of (π, V). If (π, V) is preunitary, i.e. if V is supplied with an inner product $(\ ,\ )$ which is invariant for the action of G, then we set $c_{v,w}(g) = (\pi(g)v, w)$, $v, w \in V$. This is a matrix coefficient of (π, V) and all matrix coefficients may be obtained in this way if (π, V) is admissible.

For a set X of smooth representations of G set

$$\mathscr{F}(X) = \{c_{v,\tilde{v}} : (\pi, V) \in X, v \in V, \tilde{v} \in \tilde{V}\}.$$

If X is a set of preunitary smooth representations of G then we define

$$\mathscr{F}^+(X) = \{c_{v,v}; (\pi, V) \in X, v \in V\}$$

where $c_{v,v}(g) = (\pi(g)v, v)$ as above. If $X = \{(\pi, V)\}$ then we shall write simple $\mathscr{F}((\pi, V))$ or $\mathscr{F}(\pi)$ instead of $\mathscr{F}(\{(\pi, V)\})$, and analogously for $\mathscr{F}^+$.

Let (π, V) be an admissible smooth representation of G and let (Π, H) be a continuous representation of G on a Hilbert space H such that the subrepresentation of G on smooth vectors in H is isomorphic to (π, V). We define $c_{v,w}(g) =$

$(\Pi(g)v, w)$ for $v, w \in H$ (here we do not suppose that $(\ ,\ )$ is G-invariant). Using Proposition 4.2.2.1 of [36], one has

$$\{c_{v,w};\, v,\, w \in H\} \subseteq \mathrm{Cl}\, \mathscr{F}(\pi).$$

Suppose that (Π, H) is unitary. Then

$$\{c_{v,v};\, v \in H\} \subseteq \mathrm{Cl}\, \mathscr{F}^+(\pi).$$

Recall that $\hat{G}$ is in the natural bijection with the set of all equivalence classes of irreducible unitary representations and that the latter set is supplied with the topology defined by uniform convergence of positive definite functions associated with unitary representations.

From the observations that we have made above, this topology can be described in the following way dealing only with smooth representations. The closure $\mathrm{Cl}\, X$ in $\hat{G}$ of $X \subseteq \hat{G}$ satisfies: if $\pi \in \hat{G}$, then

$$\pi \in \mathrm{Cl}\, X \Leftrightarrow \mathscr{F}^+(\pi) \subseteq \mathrm{Cl}(\mathscr{F}^+(X))$$

(Proposition 1.8.1.5 of [9]). This is one of the standard definitions of the topology of $\hat{G}$.

We start to consider the connection of the topology of $\hat{G}$ and the Bernstein center.

2.1. Lemma. *Let (π_n, V_n) be a sequence in $\tilde{G}$ and $(\pi, V) \in \tilde{G}$. Let $v_n \in V_n$, $\tilde{v}_n \in \tilde{V}_n$, $v \in V$ and $\tilde{v} \in \tilde{V}$. Suppose that $v \neq 0$, $\tilde{v} \neq 0$ and that the sequence $(c_{v_n,\tilde{v}_n})$ converges uniformly on compacts to $c_{v,\tilde{v}}$. Then (χ_{π_n}) converges pointwise to χ_π and $(\nu(\pi_n))$ converges to $\nu(\pi)$ in $\Omega(G)$.*

Proof. Choose $f \in H(G)$ so that $\langle \pi(f)v, \tilde{v}\rangle \neq 0$. Since $c_{v_n,\tilde{v}_n}$ converges uniformly on compacts to $c_{v,\tilde{v}}$, then

$$\int_G f(g)\langle \pi_n(g)v_n, \tilde{v}_n\rangle dg = \langle \pi_n(f)v_n, \tilde{v}_n\rangle$$

converges to $\langle \pi(f)v, \tilde{v}\rangle$, i.e.

$$\lim_n \langle \pi_n(f)v_n, \tilde{v}_n\rangle = \langle \pi(f)v, \tilde{v}\rangle \neq 0.$$

Take $T \in \mathscr{Z}(G)$ arbitrarily. Since $T * f \in H(G)$, in the same way we have

$$\lim_n \langle \pi_n(T * f)v_n, \tilde{v}_n\rangle = \langle \pi(T * f)v, \tilde{v}\rangle$$

and further

$$\lim_n [\chi_{\pi_n}(T)\langle \pi_n(f)v_n, \tilde{v}_n\rangle] = \chi_\pi(T)\langle \pi(f)v, \tilde{v}\rangle.$$

Now we have directly $\lim_n \chi_{\pi_n}(T) = \chi_\pi(T)$. This proves the first statement of the lemma.

Recall that $\chi_{\pi_n}(T) = e_{\nu(\pi_n)}(\hat{T})$ and $\chi_\pi(T) = e_{\nu(\pi)}(\hat{T})$. Thus $\lim_n e_{\nu(\pi_n)}(\hat{T}) = e_{\nu(\pi)}(\hat{T})$ for any $T \in \mathscr{Z}(G)$. Now Remarks 1.1(i) and the fact that $\mathbf{C}[\Omega(G)] = \{\hat{T}; T \in \mathscr{Z}(G)\}$ imply that $(\nu(\pi_n))$ converges to $\nu(\pi)$.

We can prove now some of results of [26] in simple and natural way. An analogy of a part of the following theorem is proved in the real case in [1].

2.2. Theorem. (i) *The mapping*

$$\nu \mid \hat{G} : \hat{G} \to \Omega(G)$$

is continuous. We shall denote this map by ν^u.

(ii) *Let* $\Omega \subseteq \Omega(G)$ *be a connected component of* $\Omega(G)$. *Set*

$$\hat{G}_\Omega = \hat{G} \cap \tilde{G}_\Omega.$$

Then $\hat{G}_\Omega$ *is open and closed subset of* $\hat{G}$. *In other words, the intersection of a connected component in* $\tilde{G}$ *with* $\hat{G}$ *is an open and closed subset of* $\hat{G}$.

(iii) *Let* (π_n) *be a sequence in* $\hat{G}$ *which converges to* $\pi \in \hat{G}$. *Then* (χ_{π_n}) *converges pointwise to* χ_π *and* $(\nu(\pi_n))$ *converges to* $\nu(\pi)$. *Also there exist a cuspidal pair* (M, ρ) *of* G *and a convergent sequence of unramified characters* (ψ_n) *in* $\mathrm{Unr}(M)$ *such that* π_n *is a subquotient of* $\mathrm{Ind}_{P(M)}^G(\psi_n\rho)$ *for all except finitely many* n, *and that* π *is a subquotient of* $\mathrm{Ind}_{P(M)}^G(\psi\rho)$, *where* $\psi = \lim_n \psi_n$.

Proof. (i) Let $X \subseteq \hat{G}$ and $\pi \in \mathrm{Cl}\, X$. It is enough to prove that $\nu(\pi) \in \mathrm{Cl}(\nu(X))$. Take a non-zero matrix coefficient $c_{v,\tilde{v}}$ of π. By the definition of the topology there exist a sequence (π_n) in X and a sequence of matrix coefficients $c_{v_n,\tilde{v}_n}$ of π_n converging uniformly on compacts to $c_{v,\tilde{v}}$. Now Lemma 2.1 implies that (χ_{π_n}) converges pointwise to χ_π. Thus $(\nu(\pi_n))$ converges to $\nu(\pi)$ and so $\nu(\pi) \in \mathrm{Cl}(\nu(X))$.

(ii) Let $\Omega \subseteq \Omega(G)$ be a connected component of $\Omega(G)$. Then $(\nu^u)^{-1}(\Omega(G))$ is open and closed in $\hat{G}$ by (i). This proves (ii) since $\hat{G}_\Omega$ equals $(\nu^u)^{-1}(\Omega)$ where Ω is a connected component of $\Omega(G)$.

(iii) The first part is again a direct consequence of Lemma 2.1. The second part is a direct consequence of the first part and the fact that

$$\nu_\rho : \mathrm{Unr}(M) \to \Omega(G)$$

is an open mapping.

2.3. Remarks. (i) We do not need to consider hypersequences in (iii) of the last theorem since for the closure operator in $\hat{G}$ it is enough to consider sequences (see 3.3.4 of [9]). A proof of this fact will also be contained in the second part of this paper which deals with the topology of non-unitary dual.

(ii) Let $\Omega \subseteq \Omega(G)$ be a connected component. We can choose a cuspidal pair $x = (M, \rho)$ belonging to Ω such that ρ is unitarizable. For a "majority" of Ω in $\Omega(G)$, $\hat{G}_\Omega$ will be just $\{\mathrm{Ind}_{P(M)}^G(\psi\rho); \chi \in \mathrm{Unr}^u(G)\}$. Therefore, in this case $\hat{G}_\Omega$ will be just the connected component of the topological space $\hat{G}$ and this component is open and closed. But in the most interesting cases $\hat{G}_\Omega$ will have more connected components and understanding of these components of $\hat{G}_\Omega$ is one of the crucial points of the unitarizability problem for G.

The proof of the following simple fact can be extracted from the proof of Theorem 3.1 of [26]. We include the proof here for completeness.

2.4. Lemma. *Let $\tilde{G}_\Omega$ be a connected component in $\tilde{G}$. Then there exists an open compact subgroup K of G such that for any $(\pi, V) \in \tilde{G}_\Omega$, the space V^K of K-invariant vectors in V is different from zero.*

Proof. Let (M, ρ) be a cuspidal pair belonging to Ω. Choose a congruence subgroup K of G with decomposition

$$K = K_- K_0 K_+$$

with respect to $P(M)$ such that all representations

$$w\rho, \qquad w \in W(M) = \{w \in W; wMw^{-1} = M\}$$

have non-zero vectors invariant for K_0.

Let $(\pi, V) \in G_\Omega$. Then there exists $\psi \in \mathrm{Unr}(M)$ such that (π, V) is a subquotient of $\mathrm{Ind}_{P(M)}^G(\psi\rho)$. By Corollary 7.2.2 of [7], there exists a $w_0 \in W(M)$ such that (π, V) is isomorphic to a subrepresentation of $\mathrm{Ind}_{P(M)}^G(w_0(\psi\rho))$. Thus Frobenius reciprocity implies for the intertwining space of the Jacquet modulus (coinvariants for N where $P(M) = MN$) that

$$\mathrm{Hom}_M(V_N, \delta_{P(M)}^{1/2}(w_0(\psi\rho))) \neq 0.$$

Thus $V_N \neq 0$. The exactness of the Jacquet functor implies that we have an injection

$$0 \to V_N \to \mathrm{Ind}_{P(M)}^G(w_0(\psi\rho))_N.$$

Since $\{w(w_0(\psi\rho)); w \in W(M)\}$ is the Jordan–Hölder sequence of $\mathrm{Ind}_{P(M)}^G w_0(\psi\rho))_N$, there exists $w_1 \in W(M)$ so that we have a surjection

$$V_N \to \psi_1(w_1\rho) \to 0$$

where $\psi_1 \in \mathrm{Unr}\, M$. By the choice of K the space of K_0-fixed vectors of $\psi_1(w_1\rho)$ is non-zero. Now the exactness of the functor of K_0-invariants implies

$$(V_N)^{K_0} \neq 0.$$

Since the canonical mapping

$$V^K \to (V_N)^{K_0}$$

is surjective by Theorem 3.3.3 of [7] we have that $V^K \neq 0$.

We can prove now in a simple manner the following slight generalization of Theorem 3.1 of [26].

2.5. Theorem. *Let $P = MN$ be a parabolic subgroup of G and let τ be a smooth representation of M of finite length. Consider the set $U(\tau)$ consisting of all unramified characters ψ of M such that* $\mathrm{Ind}_P^G(\psi\tau)$ *has irreducible unitarizable subquotient. Then the set $U(\tau)$ is a compact subset of* $\mathrm{Unr}(M)$.

Proof. We shall consider first the case when τ is an irreducible cuspidal representation. Let $\Omega \subseteq \Omega(G)$ be a connected component such that $(M, \tau) \in \Omega$. We consider

$$\nu_\tau : \mathrm{Unr}(M) \to \Omega, \qquad \psi \to (M, \psi\tau).$$

With $\hat{G}_\Omega = \tilde{G}_\Omega \cap \hat{G}$ note that

$$U(\tau) = \nu_\tau^{-1}(\nu^u(\hat{G}_\Omega)).$$

Choose an open compact subgroup K of G such that all representations in $\hat{G}_\Omega$ have non-zero vectors invariant for K. Let ch_K be the characteristic function of K. For $\pi \in \hat{G}$, $\| \pi(f) \|$ will denote the operator norm of $\pi(f)$, $f \in H(G)$. Note that $\| \pi(\mathrm{ch}_K) \| = 0$ or 1 for every $\pi \in \hat{G}$. By Proposition 3.3.7 of [9], the set

$$\{\pi \in \hat{G};\ \| \pi(\mathrm{ch}_K) \| \geqq 1\}$$

is quasicompact. Since $\hat{G}_\Omega$ is closed ((ii) of Theorem 2.2) and contained in the above set by the choice of K, $\hat{G}_\Omega$ is quasicompact. Since ν^u is continuous by (i) of Theorem 2.2, $\nu^u(\hat{G}_\Omega)$ is a compact subset of Ω. To finish the proof one needs to show that $\nu_\tau^{-1}(X)$ is a compact set whenever $X \subseteq \Omega$ is compact. This is proved in the first section.

Suppose now that τ is irreducible but not necessarily cuspidal. Choose a parabolic subgroup $\tilde{P} = \tilde{M}\tilde{N}$ of M so that τ is a subrepresentation of $\mathrm{Ind}_{\tilde{P}}^M(\rho)$ where ρ is an irreducible cuspidal representation and that $\tilde{P}N$ is a parabolic subgroup of G with Levi decomposition $\tilde{P}N = \tilde{M}(\tilde{N}N)$. Now $\psi\tau$ is a subrepresentation of $\psi \cdot \mathrm{Ind}_{\tilde{P}}^M(\rho) \cong \mathrm{Ind}_{\tilde{P}}^M((\psi \mid \tilde{M})\rho)$. Note that the restriction map

$$r : \mathrm{Unr}(M) \to \mathrm{Unr}(\tilde{M})$$

has a finite kernel. It is clearly continuous. Let X be a compact subset of $\mathrm{Unr}(\tilde{M})$. Considering sequences in $r^{-1}(X)$ one obtains in a simple manner that $r^{-1}(X)$ is compact. Combining the fact that $\mathrm{Ind}_P^G(\psi\tau)$ is a subrepresentation of

$\mathrm{Ind}_P^G(\mathrm{Ind}_{\tilde{P}}^M((\psi \mid \tilde{M})\rho)) \cong \mathrm{Ind}_P^G((\psi \mid M)\rho)$, the first part of the proof and the above-mentioned property of r about compactness, we obtain the proof of the theorem for irreducible τ.

At the end we consider the case of smooth representation of finite length. The proof goes by induction on the length of τ, using exactness of the induction function and the fact that the finite union of compact sets is a compact set.

3. Another finiteness argument in the proof of Theorem 2.5

Note that the statement of Theorem 2.5 is not directly related to the topology of $\hat{G}$. The theorem has interest independent of the topology of $\hat{G}$. The main finiteness argument in the proof of Theorem 2.5 comes from the locally quasi-compactness of $\hat{G}$, so the proof involves the topology of $\hat{G}$ in a substantial way.

The above discussion implies that it could be interesting to have a proof of Theorem 2.5 which does not depend on the facts about topology of $\hat{G}$. Such proof exists. It is the proof of Theorem 3.1 of [26]. One disadvantage of that proof is that it is rather long and technically complicated.

The aim of this section is to give a direct proof of the following important part of Theorem 2.5: Let τ be a finite length smooth representation of M where $P = MN$ is a parabolic subgroup of G. The set $U(\tau)$ of all $\psi \in \mathrm{Unr}(M)$ such that $\mathrm{Ind}_P^G(\psi\tau)$ has an irreducible unitarizable subquotient is a relatively compact subset of $\mathrm{Unr}(M)$.

Before we go to the proof of the above fact, we shall prove the following simple lemma:

3.1. Lemma. *Let K be an open compact subgroup of G, let U be a finite dimensional complex vector space and σ_n a sequence of representations of the algebra $H(G, K)$ on U such that for any $f \in H(G, K)$ the set*

$$\{\sigma_n(f) : n \geqq 1\}$$

is a bounded subset of $\mathrm{End}_{\mathbb{C}}(U)$. *Then there exists a representation σ of $H(G, K)$ on U and a subsequence $(\sigma_{n(k)})_k$ of (σ_n) such that*

$$\lim_k \sigma_{n(k)}(f) = \sigma(f),$$

for all $f \in H(G)$.

Proof. Let $f_1, f_2, \dots$ be a basis of $H(G, K)$. Passing to a subsequence (σ_n^1) of (σ_n), we can suppose that $\sigma_n^1(f_1)$ converges. We construct (σ_n^{k+1}) recursively for $k \geqq 1$: we choose (σ_n^{k+1}) to be a subsequence of (σ_n^k) such that $\sigma_n^{k+1}(f_{k+1})$ converges. Set $\sigma_{n(k)} = \sigma_k^k$.

Each sequence $(\sigma_{n(k)}(f))_k$, $f \in H(G, K)$, converges. We denote the limit by $\sigma(f)$. Since the addition and the multiplication in $\mathrm{End}_{\mathbf{C}}(U)$ are continuous, σ is a representation of $H(G, K)$ and the lemma is proved.

We return to the proof that $U(\tau)$ is relatively compact. First of all, the reduction to the case when τ is an irreducible cuspidal representation is same as in the proof of Theorem 2.5, so we will assume that τ is an irreducible cuspidal representation of M. Let (M, τ) lie in the connected component Ω of $\Omega(G)$. Choose an open compact subgroup K such that each representation of $\tilde{G}_\Omega$ has a non-zero vector invariant for K (Lemma 2.4). Let (ψ_n) be a sequence in $U(\tau)$. For a proof it is enough to show that (ψ_n) has a convergent subsequence.

Let (π_n, V_n) be an irreducible unitarizable subquotient of $\mathrm{Ind}_P^G(\psi_n \tau)$. First we fix a G-invariant inner product on each V_n. By Theorem 1 of [2], the set $\{\dim_{\mathbf{C}} V_n^K; n \geqq 1\}$ is finite. Passing to a subsequence, we can assume that all $\dim_{\mathbf{C}} V_n^K$ are the same and equal to some d. By the choice of K, $d \geqq 1$.

Let U be a complex unitary space of dimension d. For each n fix a unitary isomorphism

$$I_n : V_n^K \to U.$$

Let π_n^K be the natural representation of $H(G, K)$ on V_n^K. There is a unique representation $\hat{\pi}_n^K$ of $H(G, K)$ on U such that I_n is an isomorphism of $H(G, K)$-representations.

Let $f \in H(G, K)$. Since (π_n, V_n) is unitary, the operator norm of $\pi_n^K(f)$ is bounded by $\int_G |f(g)| \, dg$. By the construction of $\hat{\pi}_n^K$, we can apply the previous lemma to the sequence $(\hat{\pi}_n^K)$. Passing to a subsequence, we may assume that there is a representation σ of $H(G, K)$ on U and that $(\hat{\pi}_n^K(f))$ converges to $\sigma(f)$ for each $f \in H(G, K)$. Thus, for the characteristic function ch_K of K and any $T \in \mathscr{Z}(G)$ we have

$$\lim_n (\hat{\pi}_n^K)(T * \mathrm{ch}_K) = \sigma(T * \mathrm{ch}_K)$$

and further

$$\lim_n \chi_{\pi_n}(T) \cdot \hat{\pi}_n^K(\mathrm{ch}_K) = \left(\lim_n \chi_{\pi_n}(T)\right) \mathrm{id}_U = \sigma(T * \mathrm{ch}_K)$$

since $T * \mathrm{ch}_K \in H(G, K)$. This implies that $(\chi_{\pi_n}(T))$ converges for all $T \in \mathscr{Z}(G)$. By Remarks 1.1, $\nu(\pi_n)$ converges in Ω. Thus we have proved that $\nu_\tau(U(\tau))$ is relatively compact (i.e. $\mathrm{Cl}\, \nu_\tau(U(\tau))$ is compact). By the first section this implies that $\nu_\tau^{-1}(\mathrm{Cl}\, \nu_\tau(U(\tau)))$ is compact. Since

$$\nu_\tau^{-1}(\mathrm{Cl}\, \nu_\tau(U(\tau))) \supseteq \nu_\tau^{-1}(\nu_\tau(\mathrm{Cl}(U(\tau)))),$$

$\nu_\tau^{-1}(\nu_\tau(\mathrm{Cl}(U(\tau))))$ is relatively compact. Thus $\nu_\tau^{-1}(\nu_\tau(U(\tau)))$ is relatively compact. But $\nu_\tau^{-1}(\nu_\tau(U(\tau))) = U(\tau)$. This finishes the proof.

4. Topology on the non-unitary dual

4.1. Lemma. *Let X be a set of smooth representations of G and $(\pi_0, V_0) \in \tilde{G}$. If*

$$(\mathscr{F}(\pi_0) \setminus \{0\}) \cap \mathrm{Cl}\, \mathscr{F}(X) \neq \varnothing$$

then $\mathscr{F}(\pi_0) \subseteq \mathrm{Cl}\, \mathscr{F}(X)$.

Proof. Let $v_0 \in V_0$, $\tilde{v}_0 \in \tilde{V}_0$ such that $c_{v_0,\tilde{v}_0} \in \mathrm{Cl}\, \mathscr{F}(X)$ where $v_0 \neq 0$ and $\tilde{v}_0 \neq 0$. Let $Y \subseteq G$ be compact, $\alpha > 0$, $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ and $g_1, \ldots, g_n \in G$. Take $r > 0$ so that $|\lambda_i| < r$ for all $1 \leqq i \leqq n$. By assumption there exist $(\pi, V) \in X$, $v \in V$, $\tilde{v} \in \tilde{V}$ so that

$$|c_{v,\tilde{v}}(g) - c_{v_0,\tilde{v}_0}(g)| < \alpha/(rn)$$

for all $g \in \bigcup_{i=1}^{n} Yg_i$. Set $v_1 = \sum_{i=1}^{n} \lambda_i \pi_0(g_i) v_0$. Then for $g \in Y$

$$\begin{aligned}
|c_{\sum_{i=1}^{n} \lambda_i \pi(g_i)v,\tilde{v}}(g) - c_{v_1,\tilde{v}_0}(g)| &= \left| \sum_{i=1}^{n} \lambda_i (c_{\pi(g_i)v,\tilde{v}}(g) - c_{\pi_0(g_i)v_0,\tilde{v}_0}(g)) \right| \\
&\leqq r \sum_{i=1}^{n} |c_{\pi(g_i)v,\tilde{v}}(g) - c_{\pi_0(g_i)v_0,\tilde{v}_0}(g)| \\
&= r \sum_{i=1}^{n} |c_{v,\tilde{v}}(gg_i) - c_{v_0,\tilde{v}_0}(gg_i)| \\
&< rn \cdot [\alpha/(rn)] \\
&= \alpha.
\end{aligned}$$

Since (π_0, V_0) is irreducible $c_{v_1,\tilde{v}_0} \in \mathrm{Cl}\, \mathscr{F}(X)$ for all $v_1 \in V_0$.

Now fix $v_1 \in V_0$, $v_1 \neq 0$. By the same arguments as above one shows that $c_{v_1,\tilde{v}_1} \in \mathrm{Cl}\, \mathscr{F}(X)$ for all $\tilde{v}_1 \in \tilde{V}_0$ because $(\tilde{\pi}_0, \tilde{V}_0)$ is irreducible. This finishes the proof of the lemma.

Let us define the closure operator on $\tilde{G}$. Let $X \subseteq \tilde{G}$. Set

$$\mathrm{Cl}(X) = \{\pi \in \tilde{G};\, \mathscr{F}(\pi) \subseteq \mathrm{Cl}\, \mathscr{F}(X)\}.$$

Note that $\mathscr{F}(\mathrm{Cl}(X)) \subseteq \mathrm{Cl}(\mathscr{F}(X))$ by definition. Evidently $X \subseteq \mathrm{Cl}\, X$. Suppose that $\pi \in \mathrm{Cl}(\mathrm{Cl}(X))$. Then $\mathscr{F}(\pi) \subseteq \mathrm{Cl}(\mathscr{F}(\mathrm{Cl}(X)))$. Since $\mathscr{F}(\mathrm{Cl}(X)) \subseteq \mathrm{Cl}\, \mathscr{F}(X)$ we have

$$\mathrm{Cl}(\mathscr{F}(\mathrm{Cl}(X))) \subseteq \mathrm{Cl}(\mathrm{Cl}(\mathscr{F}(X))) = \mathrm{Cl}(\mathscr{F}(X)).$$

Thus $\mathscr{F}(\pi) \subseteq \mathrm{Cl}(\mathscr{F}(X))$ which implies $\pi \in \mathrm{Cl}\, X$. This proves $\mathrm{Cl}(\mathrm{Cl}(X)) = \mathrm{Cl}(X)$. Let $X, Y \subseteq \tilde{G}$. Clearly $\mathrm{Cl}(X) \cup \mathrm{Cl}(Y) \subseteq \mathrm{Cl}(X \cup Y)$. Suppose that $\pi \in \mathrm{Cl}(X \cup Y)$. Then

$$\mathscr{F}(\pi) \subseteq \mathrm{Cl}(\mathscr{F}(X \cup Y)) = \mathrm{Cl}(\mathscr{F}(X) \cup \mathscr{F}(Y)) = \mathrm{Cl}(\mathscr{F}(X)) \cup \mathrm{Cl}(\mathscr{F}(Y)).$$

Now Lemma 4.1 implies $\mathscr{F}(\pi) \subseteq \mathrm{Cl}(\mathscr{F}(X))$ or $\mathscr{F}(\pi) \subseteq \mathrm{Cl}(\mathscr{F}(Y))$. Thus $\pi \in \mathrm{Cl}(X) \cup \mathrm{Cl}(Y)$ and so $\mathrm{Cl}(X \cup Y) = \mathrm{Cl}(X) \cup \mathrm{Cl}(y)$. At the end, since $\mathrm{Cl}(\varnothing) = \varnothing$, the operator Cl defined above on subsets of $\tilde{G}$ determines a topology on $\tilde{G}$.

We have two topologies on $\hat{G}$, the standard one (defined in the second section) and the one induced by the topology from $\tilde{G}$. Lemma 4.1 implies that the standard topology on $\hat{G}$ is finer then the second one. From [9] it is possible to obtain that these topologies are the same. This is just Theorem 1 in Section 7.3 of [17]. A proof that these two topologies coincide is also implicit in the fifth section of this paper using [18] (it is possible also not to use [18] but the proof of Theorem 2.7 of [28], or [31]).

The proof of the following theorem is a simple modification of the proof of Theorem 2.2.

4.2. Theorem. (i) *The mapping*

$$\nu : \tilde{G} \to \Omega(G)$$

is continuous.

(ii) *For a connected component $\Omega \subseteq \Omega(G)$ the connected component $\tilde{G}_\Omega$ is an open and closed subset of $\tilde{G}$.*

4.3. Remarks. (i) The statement (iii) of Theorem 2.2 holds if one replaces $\hat{G}$ by $\tilde{G}$ there.

(ii) For $T \in \mathscr{Z}(G)$, $\hat{T} : \tilde{G} \to \mathbf{C}$ is continuous ($\hat{T}(\pi) = \chi_\pi(T)$).

In the rest of this section we shall prove that connected components $\tilde{G}_\Omega$ of $\tilde{G}$ are actually connected components of the topological space $\tilde{G}$.

We shall fix a connected component Ω of $\Omega(G)$ and a cuspidal pair (M, ρ) of G which determines Ω. We shall denote by I_ρ the subset of all $\psi \in \mathrm{Unr}(M)$ such that $\mathrm{Ind}_{P(M)}^G(\psi\rho)$ is irreducible.

First we have

4.4. Lemma. (i) *The mapping*

$$\mu_\rho : \psi \mapsto \mathrm{Ind}_{P(M)}^G(\psi\rho), \qquad I_\rho \to \tilde{G}_\Omega \subseteq \tilde{G}$$

is continuous.

(ii) *Let $X \subseteq I_\rho$. If π is an irreducible subquotient of* $\mathrm{Ind}_{P(M)}^G(\psi\rho)$ *with $\psi \in \mathrm{Cl}\, X$, then $\pi \in \mathrm{Cl}(\mu_\rho(X))$.*

Proof. We shall denote the action of G in the representation $\mathrm{Ind}_{P(M)}^G(\psi\rho)$ by R_ψ.

Let K_0 be a maximal compact subgroup such that $K_0 P(M) = G$. Let (σ, V) be a representation of K_0 smoothly induced by ρ restricted to $K_0 \cap P(M)$. Let $(\tilde{\sigma}, \tilde{V})$ be the smooth contragredient of (σ, V). Now

$$r_\psi : f \to f\,|\,K_0, \qquad \mathrm{Ind}^G_{P(M)}(\psi\rho) \to V$$

is an isomorphism of representations of K_0. Set

$$R^0_\psi(g) = r_\psi \circ R_\psi(g) \circ r_\psi^{-1}$$

for $g \in G$. Then (R^0_ψ, V) is a representation of G isomorphic to R_ψ. Also the space of the contragredient representation of R^0_ψ is equal to the above defined $\tilde{V}$.

For an open subgroup K of K_0 let $(R^0_\psi)^K$ be the representation of $H(G, K)$ on the space V^K. Then for $f \in H(G, K)$

$$\psi \to (R^0_\psi)^K(f)$$

is analytic (see Lemma 3.5 of [26] and Lemma 7.2 of [24]).

Let (ψ_n) be a convergent sequence in $\mathrm{Unr}(M)$ and ψ_0 the limit. Let $v \in V^K$, $\tilde{v} \in \tilde{V}^K$ and let $Y \subseteq G$ be compact. We are going to prove that the functions $g \to \langle R^0_{\psi_n}(g)v, \tilde{v}\rangle$ converge uniformly on Y to $\langle R^0_{\psi_0}(g)v, \tilde{v}\rangle$. Choose a finite set $g_1, \ldots, g_n \in G$ so that $Y \subseteq \bigcup_{i=1}^m Kg_iK$. Denote $\int_G \mathrm{ch}_{KgK}(g)dg$ by c_g. We have $\langle R^0_\psi(g)v, \tilde{v}\rangle = c_g^{-1}\langle R^0_\psi(\mathrm{ch}_{KgK})v, \tilde{v}\rangle$. This and the continuity of $\psi \to (R^0_\psi)^K(f)$ implies that $\langle R^0_{\psi_n}(g)v, \tilde{v}\rangle$ converges pointwise to $\langle R^0_{\psi_0}(g)v, \tilde{v}\rangle$. Since for $g \in Kg_iK$

$$\langle R^0_{\psi_n}(g)v, \tilde{v}\rangle = \langle R^0_{\psi_n}(g_i)v, \tilde{v}\rangle \qquad (n \geqq 0),$$

from the pointwise convergence we obtain the uniform convergence on Y.

Let $X \subseteq I_\rho$ and $\psi_0 \in \mathrm{Cl}\, X \subseteq \mathrm{Unr}(M)$. Choose (ψ_n) in X which converges to ψ_0. Let π be an irreducible subquotient of $\mathrm{Ind}(\psi_0\rho)$. Since $\mathcal{F}(\pi) \subseteq \mathcal{F}(R_{\psi_0})$, the above section implies $\pi \in \mathrm{Cl}(\mu_\rho(X))$. This proves (i) and (ii) of the lemma.

Now we shall prove

4.5. Theorem. *The sets $\tilde{G}_\Omega$ are connected components of $\tilde{G}$ as topological space.*

Proof. By Theorem 4.2 it is enough to prove that $\tilde{G}_\Omega$ are connected subsets of $\tilde{G}$.

Fix Ω and (M, ρ) as before. To prove that $\tilde{G}_\Omega$ is connected it is enough to show that there exists a connected subset X of I_ρ which is dense in $\mathrm{Unr}(M)$. To see this, suppose that we have such X. Now $\mu_\rho(X)$ is connected, being the image of a connected set under a continuous map. Since the closure of a connected set is connected, $\mathrm{Cl}(\mu_\rho(X))$ is connected. By our assumption $\mathrm{Cl}\, X = \mathrm{Unr}(M)$. Now the definition of $\tilde{G}_\Omega$ and (ii) of Lemma 4.4 implies $\tilde{G}_\Omega \subseteq \mathrm{Cl}(\mu_\rho(X))$. Since $\mathrm{Cl}(\mu_\rho(X))$ is connected and $\tilde{G}_\Omega$ is open and closed (and non-empty) we have $\tilde{G}_\Omega = \mathrm{Cl}(\mu_\rho(X))$. Thus $\tilde{G}_\Omega$ is connected.

Now we shall construct X as above. Let $D = \mathrm{Unr}(M)\rho$. Denote by J_ρ the set of all $\psi \in \mathrm{Unr}(M)$ for which all $w(\psi\rho)$, $w \in W^D$ are different (in $\tilde{M}$). Then J_ρ is a non-empty Zariski open subset of $\mathrm{Unr}(M)$. Since $\mathrm{Unr}(M)$ is Zariski connected, J_ρ

is a dense subset of $\mathrm{Unr}(M)$ for the standard topology of $\mathrm{Unr}(M)$ (note that $\mathrm{Unr}(M)$ is irreducible). Set

$$X = I_\rho \cap J_\rho.$$

We have observed that $\mathrm{Unr}(M)$ is isomorphic to some $(\mathbf{C}^\times)^n$. We shall fix such an isomorphism. Now we identify $\mathrm{Unr}(M)$ with the subset of $\mathbf{C}^n$. Let $x, y \in J_\rho \subseteq \mathbf{C}^n$. Set

$$Y_1 = \{\lambda x + (1-\lambda)y : \lambda \in \mathbf{C}\}.$$

We identify Y_1 with $\mathbf{C}$. Note that $Y_1 \setminus \mathrm{Unr}(M)$ is finite. Set $Y_2 = Y_1 \cap \mathrm{Unr}(M)$. Further $Y_2 \setminus J_\rho$ is a countable set (this is a set of common zeros of finitely many non-zero analytic functions on $Y_2 \subseteq \mathbf{C}$). Set $Y_3 = Y_2 \cap J_\rho$. Note that Y_3 is connected (since if we remove countable many points from $\mathbf{C}$, the resulting space is connected). This implies that J_ρ is connected because $x, y \in J_\rho$ were arbitrary, $x, y \in Y_3$ and $Y_3 \subseteq J_\rho$.

There exists a meromorphic function on J_ρ such that $J_\rho \setminus I_\rho$ is just the set of non-analytic points of this function (Theorem 5.4.3.6 of [20]). From this we see that $I_\rho \cap J_\rho$ is a dense subset of J_ρ, so it is dense in $\mathrm{Unr}(M)$. Suppose that x, y are from $J_\rho \cap I_\rho$. The above interpretation of $J_\rho \setminus I_\rho$ implies that $Y_3 \setminus I_\rho$ is countable. Set $Y_4 = Y_3 \cap J_\rho$. Again Y_4 is connected, $Y_4 \subseteq I_\rho \cap J_\rho$ and $x, y \in Y_4$. So, $I_\rho \cap J_\rho$ is connected. This finishes the proof of theorem.

The category of all smooth representations of G is denoted by $\mathrm{Alg}(G)$. Let us recall that $\mathscr{Z}(G)$ is isomorphic to the center of the category $\mathrm{Alg}(G)$, i.e. to the ring of endomorphisms (natural transforms) of the identity functor (see [3]).

For $X \subseteq \tilde{G}$ we shall denote by $\mathrm{Alg}(G)_X$ the full subcategory of $\mathrm{Alg}(G)$ whose objects are representations all of whose irreducible subquotients lie in X. We will say that a representation $\pi \in \mathrm{Alg}(G)$ has support in X if $\pi \in \mathrm{Alg}(G)_X$. For a representation $(\pi, V) \in \mathrm{Alg}(G)$ we shall denote by (π_X, V_X) the sum of all subrepresentations of V supported in X. This is again in $\mathrm{Alg}(G)_X$. We shall call (π_X, V_X) an X-component of (π, V).

The first statement of the following proposition is a reformulation of Proposition 2.10 of [3].

4.6. Proposition. (i) *The category* $\mathrm{Alg}(G)$ *is a direct sum of categories* $\mathrm{Alg}(G)_{\tilde{G}_\Omega}$ *when* $\tilde{G}_\Omega$ *runs over all connected components of* $\tilde{G}$.

(ii) *The decomposition* $\mathrm{Alg}(G) = \oplus \mathrm{Alg}(G)_{\tilde{G}_\Omega}$ *is the finest possible in the following sense: if we have any decomposition of* $\mathrm{Alg}(G)$ *into a direct sum* $\oplus C_i$ *of subcategories and if* $\pi \in \tilde{G}_\Omega$ *is an object in* C_{i_0}, *then the whole category* $\mathrm{Alg}(G)_{\tilde{G}_\Omega}$ *is a subcategory of* C_{i_0}.

Proof. (i) Let X be a union of some connected components of $\tilde{G}$ (or $\Omega(G)$). We shall denote by $T(X) \in \mathscr{Z}(G)$ an element such that $(T(X))^{\wedge}$ equals 1 on X and 0 on the rest.

Let $(\pi, V) \in \mathrm{Alg}\, G$. First we have that $\pi(T(X))\pi(T(X)) = \pi(T(X))$, i.e. $\pi(T(X))$ is a projector. Set $V_0 = \pi(T(X))V$. Clearly, V_0 is a subrepresentation of V. Also $\pi(T(X))$ acts as identity on V_0. This implies $V_0 \subseteq V_X$. Suppose that $\pi(T(X))V_X \neq V_X$. One can find subrepresentations V_1 and V_2 such that

$$\pi(T(X))V_X \subseteq V_1 \subsetneq V_2 \subseteq V_X$$

and that V_2/V_1 is irreducible. Now $\pi(T(X))$ acts as 0 on V_2/V_1 so V_2/V_1 does not belong to X which is a contradiction. Thus we have proved that $\pi(T(X))V = V_X$.

Let K_0 be a maximal compact subgroup such that $K_0P_0 = G$. Let K be an open normal subgroup of K_0. First we shall prove that the number of components $\tilde{G}_\Omega$ where there exists $(\sigma, W) \in \tilde{G}_\Omega$ with $W^K \neq 0$, is finite. Let (ρ, U) be an irreducible cuspidal representation of $M \in \mathscr{M}$. If $\mathrm{Ind}_{P(M)}^G(\psi\rho)$, $\psi \in \mathrm{Unr}(M)$, have a subquotient with a non-zero K-invariant vector, then $U^{K \cap M} \neq 0$ (one looks at the restriction of functions in $\mathrm{Ind}_{P(M)}^G(\psi\rho)$ to K_0). Let us recall that in each $\mathrm{Unr}(M)\rho$ there is a unitarizable representation. Using the facts that there are finitely many standard parabolic subgroups, that connected components of $\tilde{M}$ intersected with $\hat{M}$ are open and closed, and that the set of all $\tau \in \hat{M}$ possessing a non-zero vector invariant for $K \cap M$ is a quasi-compact subset of $\hat{M}$, one obtains that the number of components $\tilde{G}_\Omega$ where there exists $(\sigma, W) \in \tilde{G}_\Omega$ with $W^K \neq 0$ is finite.

Now we shall prove that the sum of all $V_{\tilde{G}_\Omega}$ is V. Let $v \in V$. Choose an open normal subgroup K of K_0 such that v is fixed by K. Let $\Omega_1, \ldots, \Omega_m$ be all the connected components Ω such that in $\tilde{G}_\Omega$ is (σ, W) with $W^K \neq 0$. Let X be the complement of $\Omega_1 \cup \cdots \cup \Omega_m$. First we shall show that $\pi(T(X))v = 0$. Since $\pi(T(X))v \in V^K$ we have

$$\pi(T(X))v \in V^K \cap V_X.$$

Suppose that $\pi(T(X))v \neq 0$. Let V_1 be a subrepresentation of V generated by $\pi(T(X))v$ and V_2 a subrepresentation of V_1 such that V_1/V_2 is irreducible. Now $V_1/V_2 \in \mathrm{Alg}(G)_X$ and $(V_1/V_2)^K \neq 0$ which is a contradiction. Thus $\pi(T(X))v = 0$ which implies

$$v = \pi(T(X) + T(\Omega_1) + \cdots + T(\Omega_m))v$$

$$= \sum_{i=1}^{m} \pi(T(\Omega_i))v \in \sum_{i=1}^{m} V_{\tilde{G}_{\Omega_i}}.$$

Note that the sum

$$V = \sum_{\Omega \subseteq \Omega(G)} V_{\tilde{G}_\Omega}$$

is direct since the projectors $\pi(T(\Omega))$ are mutually orthogonal.

To finish (i) one needs to show that for (π, V), $(\sigma, W) \in \mathrm{Alg}\, G$ and $\Omega_1 \neq \Omega_2$

$$\mathrm{Hom}_G(V_{\Omega_1}, W_{\Omega_2}) = 0.$$

One obtains this directly from the fact that $\pi(T(\Omega_1))$ acts as identity on V_{Ω_1} and $\sigma(T(\Omega_1)) = 0$.

(ii) Let $\mathrm{Alg}\, G = \bigoplus C_i$.

Let $(\pi, V) \in \tilde{G}_\Omega$ and let (π, V) be an object in C_{i_0}. The identity endomorphism of the identity functor of C_{i_0} and zero endomorphism of identity functor of $\bigoplus_{i \neq i_0} C_i$ defines an endomorphism $T \in \mathscr{Z}(G)$. Note that

$$\hat{T}: \tilde{G} \to C, \qquad \hat{T}(\tilde{G}) \subseteq \{0, 1\}.$$

Since $\tilde{G}_\Omega$ is connected (here it is enough to know that Ω is connected) and $\hat{T}$ is continuous, $\hat{T}(\tilde{G}_\Omega) = 1$. Thus $\tilde{G}_\Omega$ are objects in C_{i_0}.

Since the sum $\mathrm{Alg}\, G = \bigoplus C_i$ is direct, if W is an object of C_i, then all subquotients of V are in C_i. This and the fact $\tilde{G}_\Omega \subseteq C_{i_0}$ implies $\mathrm{Alg}(G)_{\tilde{G}_\Omega} \subseteq C_{i_0}$.

5. Characters and the topology of the non-unitary dual

Let $H(G)'$ be the vector space of all linear forms on $H(G)$. We supply $H(G)'$ with the topology of pointwise convergence.

For an admissible smooth representation (π, V) we shall denote by Θ_π (or sometime by Θ_V) its character.

Denote by $H(G)'_+$ the set of all finite sums $\Theta_{\pi_1} + \cdots + \Theta_{\pi_m}$ where $\pi_1, \ldots, \pi_m \in \tilde{G}$ (possibility $m = 0$ is included so $0 \in H(G)'_+$). Clearly,

$$H(G)'_+ + H(G)'_+ = H(G)'_+.$$

If $\varphi_1, \varphi_2 \in H(G)'$, then we set

$$\varphi_1 \leqq \varphi_2$$

if $\varphi_2 - \varphi_1 \in H(G)'_+$. Since the characters of smooth irreducible representations of G are linearly independent, $\leqq$ is a partial order on $H(G)'$.

We shall need the following lemma, a proof of which is essentially contained in the proof of Lemma 3.6 of [26].

5.1. Lemma. *Let $M \in \mathscr{M}$ and let ρ be an irreducible cuspidal representation of M. Let (ψ_n) be a convergent sequence in* $\mathrm{Unr}(M)$ *and let ψ_0 be the limit.*

Let (π_n, V_n) be a subrepresentation of $\mathrm{Ind}_{P(M)}^G(\psi_n \rho)$ *for each $n \geqq 1$. Let $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$ be a basis of neighborhoods of the identity of G consisting of open* *compact subgroups and suppose that* $K_0P(M)=G$. *There exist a subsequence* $(\pi_{n_k}, V_{n_k})_k$ *of* (π_n, V_n) *and a subrepresentation* (π_0, V_0) *of* $\mathrm{Ind}_{P(M)}^G(\psi_0\rho)$ *satisfying the following condition*:

For each $m \geqq 0$ *there exist a finite dimensional vector space* W_m *and* $\sigma_0^{(m)}$, $\sigma_i^{(m)}$ $(i \geqq m)$ *representations of* $H(G, K_m)$ *on* W_m *such that* $\sigma_0^{(m)} \cong \pi_0^{K_m}$, $\sigma_i^{(m)} \cong \pi_{n_i}^{K_m}$ *for* $i \geqq m$ *and that*

$$\lim_i \sigma_i^{(m)}(f) = \sigma_0^{(m)}(f)$$

for each $f \in H(G, K_m)$.

Proof. We start similarly as in the proof of Lemma 4.4. Denote by (σ, V) a representation of K_0 smoothly induced by ρ restricted to $K_0 \cap P(M)$, denote

$$r_\psi : f \mapsto f\,|\,K_0, \qquad \mathrm{Ind}_{P(M)}^G(\psi\rho) \to V$$

and

$$R_\psi^0(g) = r_\psi \circ R_\psi(g) \circ r_\psi^{-1}.$$

Recall that for any K_m, $f \in H(G, K_m)$

$$\psi \to (R_\psi^0)^{K_m}(f)$$

is analytic. We supply (σ, V) with an inner product invariant for K_0. We identify (π_n, V_n) by r_ψ with a subrepresentation of $(R_{\psi_n}^0, V)$.

We construct sequences $(\pi_n^{(m)}, V_n^{(m)})_n$, $m \geqq -1$, recursively. If $m = -1$ then we take $(\pi_n^{(-1)}, V_n^{(-1)}) = (\pi_n, V_n)$. Suppose that we have constructed $(\pi_n^{(m)}, V_n^{(m)})_n$ up to some $m \geqq -1$. Passing to a subsequence (π_n^*, V_n^*) we can suppose that all $\dim_{\mathbf{C}}(V_n^*)^{K_{m+1}}$ are the same, say d. In each $(V_n^*)^{K_{m+1}}$ choose an orthonormal basis $v_1^n, \dots, v_d^n$. Passing to a subsequence which we again denote by (π_n^*, V_n^*) we can assume that all (v_i^n) converge. Denote the limit by v_i^0. Note that

$$\lim_n (\pi_n^*)^{K_{m+1}}(f)v_i^n = R_{\psi_0}^0(f)v_i^0, \qquad f \in H(G, K_{m+1}).$$

Set $V_0^{K_{m+1}} = \mathrm{span}_{\mathbf{C}}\{v_1^0, \dots, v_d^0\}$. Let W be a d-dimensional unitary space with a fixed orthonormal basis $w_1, \dots, w_d$. Let

$$\varphi_n : (V_n^*)^{K_{m+1}} \to W$$

be a linear isomorphism defined by $\varphi_n(v_i^n) = w_i$, $1 \leqq i \leqq d$, and let $\varphi_0 : V_0^{K_{m+1}} \to W$ be given by $\varphi_0(v_i^0) = w_i$, $1 \leqq i \leqq d$. Set

$$(\pi_n^{(m+1)}, V_n^{(m+1)}) = (\pi_n^*, V_n^*), \qquad n \geqq 1,$$

$$\sigma_n^{(m+1)}(f) = \varphi_n \circ (\pi_n^*)^{K_{m+1}}(f) \circ \varphi_n^{-1}, \qquad n \geqq 1, \quad f \in H(G, K_{m+1}).$$

Note that $V_0^{K_{m+1}}$ is invariant for $H(G, K_{m+1})$ and we shall denote this representation by $(\pi_0)^{K_{m+1}}$. Set

$$\sigma_0^{(m+1)}(f) = \varphi_0 \circ (\pi_0)^{K_{m+1}}(f) \circ \varphi_0^{-1} \qquad \text{for } f \in H(G, K_{m+1}).$$

At the end we take

$$V_0 = \bigcup_{n=0}^{\infty} (V_0)^{K_m},$$

$$(\pi_{n_k}, V_{n_k}) = (\pi_k^{(k)}, V_k^{(k)}).$$

By construction the condition in the lemma is satisfied.

Now we have a direct consequence of the previous lemma.

5.2. Proposition. *Let ρ be an irreducible cuspidal representation of $M \in \mathcal{M}$. Let $\psi_0 \in \mathrm{Unr}(M)$ and let (ψ_n) be sequence in $\mathrm{Unr}(M)$ which converges to ψ_0. Let π_n be an irreducible subquotient of $\mathrm{Ind}_{P(M)}^G(\psi_n \rho)$, $n \geqq 1$. There exist a subsequence $(\pi_{n_k})_k$ of (π_n) and a subquotient (π_0, V_0) of $\mathrm{Ind}_{P(M)}^G(\psi_0 \rho)$ such that $(\pi_{n_k})_k$ converges to π_0.*

Proof. Choosing a suitable $w \in W$ such that $wMw^{-1} = M$ and passing to a subsequence one can assume that π_n are subrepresentations of $\mathrm{Ind}_{P(M)}^G(\psi_n \rho)$. Now Lemma 5.1 implies the proposition.

5.3. Lemma. *Let (π_n, V_n) be a sequence in $\tilde{G}$, $(\pi_0, V_0) \in \tilde{G}$ and $\varphi \in H(G)'$. Suppose that $\lim_n \Theta_{\pi_n} = \varphi$ in $H(G)'$ and $\Theta_{\pi_0} \leqq \varphi$. Then $\lim_n \chi_{\pi_n}(T) = \chi_{\pi_0}(T)$ for any $T \in \mathscr{Z}(G)$, and $\lim_n \nu(\pi_n) = \nu(\pi_0)$ in $\Omega(G)$.*

Proof. Choose an open compact subgroup K of G such that $V_0^K \neq 0$. Let $\varphi = m_{\pi_0}\Theta_{\pi_0} + m_{\sigma_1}\Theta_{\sigma_1} + \cdots + m_{\sigma_k}\Theta_{\sigma_k}$ with $\sigma_i \in \tilde{G}$ mutually different, $m_{\pi_0}, m_{\sigma_i} \geqq 1$. Since the set

$$\{\pi_0\} \cup \{\sigma_i; \sigma_i^K \neq 0\}$$

is a finite set of finite-dimensional irreducible representations of $H(G, K)$, there exist $f \in H(G, K)$ such that $\pi_0^k(f)$ is the identity on V_0^K and $\sigma_i^K(f) = 0$ for $i = 1, \ldots, k$. Now for $T \in \mathscr{Z}(G)$ we have

$$\lim_n \Theta_{\pi_n}(T * f) = m_{\pi_0}\Theta_{\pi_0}(T * f) + \sum_{i=1}^{k} m_{\sigma_i}\Theta_{\sigma_i}(T * f),$$

i.e.

$$\lim_n \chi_{\pi_n}(T)\Theta_{\pi_n}(f) = \chi_{\pi_0}(T) m_{\pi_0}(\dim_{\mathbb{C}} V_0^K)$$

since $T * f \in H(G, K)$. But $\lim_n \Theta_{\pi_n}(f) = m_{\pi_0}(\dim_{\mathbb{C}} V_0^K)$ implies

$$\lim_n \chi_{\pi_n}(T) = \chi_{\pi_0}(T).$$

For $X \subseteq \tilde{G}$ set

$$\Theta_X = \{\Theta_\pi ; \pi \in X\}.$$

Now we have a description of the topology of $\tilde{G}$ in terms of characters. This description is analogous to the Miličić description for the unitary dual in [18].

5.4. Theorem. *Let $X \subseteq \tilde{G}$ and $\pi \in \tilde{G}$. Then $\pi \in \mathrm{Cl}\, X$ if and only if there exists $\varphi \in \mathrm{Cl}(\Theta_X)$ such that $\Theta_\pi \leqq \varphi$.*

Proof. Suppose that $\varphi \in \mathrm{Cl}(\Theta_X)$ and $\Theta_\pi \leqq \varphi$. Let $((\pi_n, V_n)) \in X$ such that $\lim_n \Theta_{\pi_n} = \varphi$. Then $\nu(\pi_n)$ converges to $\nu(\pi)$ by the previous lemma. Choose cuspidal pairs (M_n, ρ_n) with $M_n \in \mathcal{M}$ and $\psi_n \in \mathrm{Unr}(M_n)$ such that (π_n, V_n) is isomorphic to a subquotient of $\mathrm{Ind}^G_{P(M)}(\psi_n \rho_n)$. Removing a finite number of representations, we can assume that $M_n = M$ for all n and that all $\psi_n \rho_n$ are in one connected component. Thus we can choose an irreducible cuspidal representation ρ of M and a sequence (ψ_n) in $\mathrm{Unr}(M)$ such that (π_n) is a subquotient of $\mathrm{Ind}^G_{P(M)}(\psi_n \rho)$. For each n we can choose $w \in W$ such that $wMw^{-1} = M$ and that π_n is a subrepresentation of $\mathrm{Ind}^G_{P(M)}(w(\psi_n \rho))$. Therefore, passing to a subsequence, we can assume that π_n is a subrepresentation of $\mathrm{Ind}^G_{P(M)}(\psi_n \rho)$. Since $\nu(\psi_n \rho)$ converges, passing to a subsequence we can assume that (ψ_n) converges to some ψ_0.

Let (π_0, V_0) be a subrepresentation of $\mathrm{Ind}^G_{P(M)}(\psi_0 \rho)$ constructed in Lemma 5.1. Then $\Theta_{\pi_0} = \varphi$ by construction. Thus π is a subquotient of π_0 which implies $\mathcal{F}(\pi) \subseteq \mathcal{F}(\pi_0)$. Now we shall prove that each matrix coefficient of π_0 is in the closure of $\bigcup_{n \geq 1} \mathcal{F}(\pi_n)$ which implies $\pi \in \mathrm{Cl}\, X$ since $\mathcal{F}(\pi) \subseteq \mathcal{F}(\pi_0)$.

Let $v_0 \in V_0$ and $\tilde{v}_0 \in \tilde{V}_0$. Let $K_0 \supseteq K_1 \supseteq \cdots$ be a basis as in Lemma 5.1. Choose K_i so that $v_0 \in V_0^{K_i}$ and $\tilde{v}_0 \in \tilde{V}_0^{K_i}$. By Lemma 5.1 there exist isomorphisms

$$\alpha_j : V_{n_j}^{K_i} \to W, \qquad j \geqq 0$$

such that

$$\lim_j \alpha_j \pi_{n_j}^{K_i}(f) \alpha_j^{-1} = \alpha_0 \pi_0^{K_i}(f) \alpha_0^{-1}$$

for all $f \in H(G, K_i)$. Note that we have dual maps

$$\tilde{\alpha}_{n_j} : W' \to (V_{n_j}^{K_i})' \cong (\tilde{V}_{n_j})^{K_i}.$$

Now (similarly as in the proof of Lemma 4.4)

$$\langle \pi_0(g) v_0, \tilde{v}_0 \rangle = \lim_j \langle \pi_{n_j}(g)(\alpha_j^{-1}\alpha_0)(v_0), \tilde{\alpha}_j \tilde{\alpha}_0^{-1}(\tilde{v}_0) \rangle$$

for all $g \in G$. So we have obtained pointwise convergence of

$$g \mapsto \langle \pi_{n_j}(g)(\alpha_j^{-1}\alpha_0)(v_0), \tilde{\alpha}_j \tilde{\alpha}_0(\tilde{v}_0) \rangle$$

to $g \to \langle \pi_0(g)v_0, \tilde{v}_0\rangle$. Analogously as in the proof of Lemma 4.4 we obtain that the convergence is uniform over compacts.

For the proof of the second implication in the theorem we suppose that $(\pi, V) \in \mathrm{Cl}\, X$. Let K be an open compact subgroup such that $V^K \neq 0$. Let $v \in V^K$, $\tilde{v} \in \tilde{V}^K$ be non-zero vectors. Choose a sequence $((\pi_n, V_n))_{n\geq 1}$ in X and $v_n \in V_n$, $\tilde{v}_n \in \tilde{V}_n$ such that $c_{v_n, \tilde{v}_n}$ converges to $c_{v,\tilde{v}}$ uniformly on compacts. Since v and $\tilde{v}$ are K-invariant, $(c_{\pi_n(\mathrm{ch}_K)v_n, \tilde{\pi}_n(\mathrm{ch}_K)\tilde{v}_n})_n$ converges to $(\int_G \mathrm{ch}_K(g)dg)^2 c_{v,\tilde{v}}$ pointwise which implies a convergence over compacts. Thus we may suppose that all v_n and $\tilde{v}_n$ are K-invariant.

Lemma 2.1 implies that $(\nu(\pi_n))$ converges in $\Omega(G)$ to $\nu(\pi)$. As in the first part of the proof, passing to a subsequence, we can assume that there exist a cuspidal pair (M, ρ) of G and a convergent sequence (ψ_n) in $\mathrm{Unr}(M)$ such that (π_n, V_n) is a subrepresentation of $\mathrm{Ind}_{P(M)}^G(\psi_n \rho)$ for all $n \geqq 1$. Let $\psi_0 = \lim_n \psi_n$. Passing to a subsequence we may assume that there exist a space W as in Lemma 5.1, a subrepresentation (π_0, V_0) of $\mathrm{Ind}_{P(M)}^G(\psi_0\rho)$, and a linear isomorphism

$$\alpha_n : V_n^K \to W, \qquad n \geqq 0$$

such that

$$\lim_n \alpha_n \pi_n^K(f)\alpha_n^{-1} = \alpha_0 \pi_0^K(f)\alpha_0^{-1}$$

for all $f \in H(G, K)$.

Note that for $f \in H(G, K)$

$$\lim_n \langle \pi_n(f)v_n, \tilde{v}_n\rangle = \langle \pi(f)v, \tilde{v}\rangle$$

and that

$$\langle \pi_n(f)v_n, \tilde{v}_n\rangle = \langle (\alpha_n \pi_n^K(f)\alpha_n^{-1})(\alpha_n(v_n)), \tilde{\alpha}_n^{-1}\tilde{v}_n\rangle.$$

The rest of the proof one obtains using the proof of implication (iii)$\Rightarrow$(b) of Theorems 3 and 4 of [11]. For the sake of completeness we shall write the rest of the proof.

First note that it is enough to prove that π^K is a composition factor of π_0^K since $\Theta_{\pi_0} = \lim_n \Theta_{\pi_n}$.

Suppose that π^K is not a composition factor of π_0^K. Then one can find $f_1 \in H(G, K)$ so that $\pi^K(f_1)$ is the identity on V^K and that $\sigma(f_1) = 0$ for all composition factors σ of π_0^K. Set $f = f_1^{\dim_{\mathbb{C}} W} = f_1 * \cdots * f_1$. Thus $\pi_0^K(f) = 0$ and $\pi^K(f)$ is the identity. Since $\mathrm{End}_{\mathbb{C}}(V^K)$ is generated by idempotents, there exists $e_1 \in H(G, K)$ so that $\langle \pi^K(e_1)v, \tilde{v}\rangle \neq 0$ and that $\pi^K(e_1)$ is an idempotent. Thus for $e = e_1 * f$ we have $\pi^K(e) = \pi^K(e_1)$ is a non-zero idempotent, $\pi_0^K(e) = 0$ and $\langle \pi^K(e)v, \tilde{v}\rangle \neq 0$.

There exists $\eta_n \in \mathrm{End}_{\mathbb{C}}(W)'$ so that

$$\langle \pi_n^K(f)v_n, \tilde{v}_n \rangle = \eta_n(\alpha_n \pi_n^K(f)\alpha_n^{-1})$$

for all $f \in H(G, K)$. Now for $k \geqq 1$

$$\begin{aligned}\lim_n \eta_n[(\alpha_n \pi_n^K(e)\alpha_n^{-1})^k] &= \lim_n \eta_n(\alpha_n \pi_n^K(e^k)\alpha_n^{-1}) \\ &= \lim_n \langle \pi_n^K(e^k)v_n, \tilde{v}_n \rangle \\ &= \langle \pi^K(e^k)v, \tilde{v} \rangle \\ &= \langle \pi^K(e)^k v, \tilde{v} \rangle \\ &= \langle \pi^K(e)v, \tilde{v} \rangle \\ &\neq 0.\end{aligned}$$

Set $a_n = \alpha_n \pi_n^K(e)\alpha_n^{-1}$. Now we know

$$\lim_n \eta_n(a_n^k) = \langle \pi^K(e)v, \tilde{v} \rangle \neq 0, \qquad k \geqq 1,$$

$$\lim_n a_n = \alpha_0 \pi_0^K(e)\alpha_0^{-1} = 0.$$

Let

$$X^d + \sum_{i=0}^{d-1} c_i(n)X^i$$

be the characteristic polynomial of a_n. Then

$$a_n^d + \sum_{i=0}^{d-1} c_i(n)a_n^i = 0$$

and thus

$$\eta_n(a_n^d) = -\sum_{i=0}^{d-1} c_i(n)\eta_n(a_n^i)$$

which implies

$$\lim_n \eta_n(a_n^d) = -\sum_{i=0}^{d-1} \left(\lim_n c_i(n)\right)(\lim \eta_n(a_n^i)) = 0$$

since $\lim_n a_n = 0$ and thus $\lim_n c_i(n) = 0$. The contradiction is obtained and this finishes the proof of the theorem.

Now we shall give one more description of the topology of $\tilde{G}$.

For $\pi \in \hat{G}$ let $\Phi(\pi)$ be the vector subspace of $C(G)$ spanned by $\mathscr{F}(\pi)$. For $X \subseteq G$ set

$$\Phi(X) = \bigcup_{\pi \in X} \Phi(\pi).$$

Let $(\pi, V) \in \tilde{G}$. We shall denote the set of all linear forms $f \to \langle \pi(f)v, \tilde{v} \rangle$, $v \in V$, $\tilde{v} \in \tilde{V}$ on $H(G)$ by $\mathscr{L}(\pi)$. We denote the vector subspace spanned by $\mathscr{L}(\pi)$ by $\Lambda(\pi)$. For $X \subseteq \tilde{G}$ we denote

$$\mathscr{L}(X) = \bigcup_{\pi \in X} \mathscr{L}(\pi),$$

$$\Lambda(X) = \bigcup_{\pi \in X} \Lambda(\pi).$$

5.5. Theorem. *Let $(\pi, V) \in \tilde{G}$ and $X \subseteq \tilde{G}$. Then the following conditions are equivalent*:

(i) $\pi \in \mathrm{Cl}(X)$ (*i.e.* $\mathscr{F}(\pi) \subseteq \mathrm{Cl}(\mathscr{F}(X))$).

(ii) $\Phi(\pi) \subseteq \mathrm{Cl}(\Phi(X))$.

(iii) $\mathscr{L}(\pi) \subseteq \mathrm{Cl}(\mathscr{L}(X))$.

(iv) $\Lambda(\pi) \subseteq \mathrm{Cl}(\Lambda(X))$.

Proof. Suppose that $\mathscr{L}(\pi) \subseteq \mathrm{Cl}(\mathscr{L}(X))$. Let $v \in V$, $\tilde{v} \in \tilde{V}$. Choose a sequence $((\pi_n, V_n))$ in X and $v_n \in V_n$, $\tilde{v}_n \in \tilde{V}_n$ such that

$$\lim_n \langle \pi_n(f)v_n, \tilde{v}_n \rangle = \langle \pi(f)v, \tilde{v} \rangle$$

for all $f \in H(G)$. Choose an open compact subgroup K so that $v \in V^K$ and $\tilde{v} \in \tilde{V}^K$. Then we may suppose that all $v_n \in V_n^K$ and $\tilde{v}_n \in \tilde{V}_n^K$. Thus

$$\lim_n \langle \pi_n(\mathrm{ch}_{KgK})v_n, \tilde{v}_n \rangle = \langle \pi(\mathrm{ch}_{KgK})v, \tilde{v} \rangle$$

for all $g \in G$. This implies $\lim_n \langle \pi_n(g)v_n, \tilde{v}_n \rangle = \langle \pi(g)v, \tilde{v} \rangle$ for all $g \in G$. Using the fact that $v, \tilde{v}, v_n, \tilde{v}_n$ are K-invariant we obtain that $c_{v_n, \tilde{v}_n}$ converges to $c_{v, \tilde{v}}$ uniformly over compacts (as in the proof of Lemma 4.4). This proves (iii) $\Rightarrow$ (i).

Let $(\pi, V) \in \mathrm{Cl}(X)$. Let $v \in V$, $\tilde{v} \in \tilde{V}$, $v \neq 0$, $\tilde{v} \neq 0$. Choose a sequence $((\pi_n, V_n))$ in X and $v_n \in V_n$, $\tilde{v}_n \in V_n$ such that $(c_{v_n, \tilde{v}_n})$ converges to $c_{v, \tilde{v}}$ uniformly on compacts. Now $(\langle \pi(f)v_n, \tilde{v}_n \rangle)$ converges to $\langle \pi(f)v, \tilde{v} \rangle$ for any $f \in H(G)$. This is the implication (i) $\Rightarrow$ (iii).

Let $(\pi, V) \in \mathrm{Cl}(X)$ and $\varphi \in \Phi(\pi)$, $\varphi \neq 0$. Then

$$\varphi(g) = \sum_{i=1}^{k} \langle \pi(g)v^i, \tilde{v}^i \rangle, \qquad v^i \in V, \quad \tilde{v}^i \in \tilde{V}.$$

We can suppose that $v^1 \neq 0$ and $\tilde{v}^1 \neq 0$. Choose a sequence $((\pi_n, V_n))$ in X and $v_n^1 \in V_n$, $\tilde{v}_n^1 \in \tilde{V}_n$ such that $(c_{v_n^1, \tilde{v}_n^1})$ converges to $c_{v^1, \tilde{v}^1}$ uniformly on compacts. By Lemma 4.1

$$\pi \in \mathrm{Cl}(\{\pi_n : n = 1, 2, \ldots\}).$$

Thus passing to a subsequence we may suppose that there exist $v_n^2 \in V_n$, $\tilde{v}_n^2 \in \tilde{V}$ such that $(c_{v_n^2,\tilde{v}_n^2})$ converges to $(c_{v^2,\tilde{v}^2})$ uniformly on compacts. Continuing this procedure by passing to a subsequence we may suppose that there exist $v_n^1, \ldots, v_n^k \in V_n$, $\tilde{v}_n^1, \ldots, \tilde{v}_n^k \in \tilde{V}_n$ such that $(c_{v_n^i,\tilde{v}_n^i})$ converges to $c_{v^i,\tilde{v}^i}$ uniformly on compacts for $i = 1, \ldots, k$. Now $\Sigma_{i=1}^k c_{v_n^i,\tilde{v}_n^i} \in \Phi(X)$ and $(\Sigma_{i=1}^k c_{v_n^i,\tilde{v}_n^i})$ converges to $\Sigma_{i=1}^k c_{v^i,\tilde{v}^i} = \varphi$ uniformly on compacts. Thus $\varphi \in \mathrm{Cl}(\Phi(X))$. So we have proved the implication (i)$\Rightarrow$(ii).

Suppose that $\Phi(\pi) \subseteq \mathrm{Cl}(\Phi(X))$. Let $\lambda \in \Lambda(\pi)$. Then there exist $v^1, \ldots, v^k \in V$, $\tilde{v}^1, \ldots, \tilde{v}^k \in \tilde{V}$ such that

$$\lambda(f) = \sum_{i=1}^{k} \langle \pi(f)v^i, \tilde{v}^i \rangle$$

for all $f \in H(G)$. Set $\varphi(g) = \Sigma_{i=1}^k \langle \pi(g)v^i, \tilde{v}^i \rangle$. Choose a sequence (φ_n) in $\Phi(X)$ converging to φ uniformly over compacts. Set

$$\lambda_n(f) = \int_G f(g)\varphi_n(g)dg.$$

Then (λ_n) is a sequence in $\Lambda(X) \subseteq H(G)'$ and (λ_n) converges to λ pointwise. Thus $\Lambda(\pi) \subseteq \mathrm{Cl}(\Lambda(X))$. Therefore we have the implication (ii)$\Rightarrow$(iv).

Now we shall prove (iv)$\Rightarrow$(i).

Let $\Lambda(\pi) \subseteq \mathrm{Cl}(\Lambda(X))$. Choose $v \in V$, $\tilde{v} \in \tilde{V}$, $v \neq 0$, $\tilde{v} \neq 0$, and an open compact subgroup K such that $v \in V^K$, $\tilde{v} \in \tilde{V}^K$. By assumption there exist a sequence $((\pi_n, V_n))$ in X and $\lambda_n \in \Lambda(\pi_n)$ such that (λ_n) converges to $\lambda: f \rightarrow \langle \pi(f)v, \tilde{v} \rangle$ pointwise. Since

$$\lambda_n(T * f) = \chi_{\pi_n}(T)\lambda_n(f),$$

$$\lambda(T * f) = \chi_\pi(T)\lambda(f)$$

we obtain that $(\nu(\pi_n))$ converges to π. The rest of the proof of the implication (iv)$\Rightarrow$(i) is a slight modification of the second half of the proof of Theorem 5.4 and we will outline it. Again we use the proof of implication (iii)$\Rightarrow$(b) in the proof of Theorems 3 and 4 of [11].

Passing to a subsequence we may assume that there exist a vector space W, a representation (π_0, V_0) of finite length, and linear isomorphisms

$$\alpha_n: V_n^K \rightarrow W, \qquad n \geqq 0$$

such that

$$\lim \alpha_n \pi_n^K(f) \alpha_n^{-1} = \alpha_0 \pi_0^K(f) \alpha_0^{-1}$$

for all $f \in H(G, K)$. By Theorem 5.4 it is enough to prove that π^K is a composition factor of π_0^K. Suppose that this is not the case. Find $e \in H(G, K)$ such that $\pi^K(e)$ is an idempotent, that

$$\langle \pi^K(e)v, \tilde{v}\rangle \neq 0$$

and $\pi_0^K(e) = 0$.

Choose $\eta_n \in \mathrm{End}_{\mathbf{C}}(W)'$ such that

$$\lambda_n(f) = \eta_n(\alpha_n \pi_n^K(f)\alpha_n^{-1})$$

for all $f \in H(G, K)$. Set $a_n = \alpha_n \pi_n^K(e)\alpha_n^{-1}$. Then

$$\lim_n \eta_n(a_n^k) = \lim_n \eta_n[(\alpha_n \pi_n^K(e)\alpha_n^{-1})^k] = \lim_n \lambda_n(e^k)$$

$$= \lambda(e^k) = \langle \pi^K(e^k)v, \tilde{v}\rangle = \langle \pi^K(e)^k v, \tilde{v}\rangle = \langle \pi^K(e)v, \tilde{v}\rangle \neq 0$$

for all $k \geqq 1$. Also we know

$$\lim_n a_n = \lim_n \alpha_n \pi_n^K(e)\alpha_n^{-1} = \alpha_0 \pi_0^K(e)\alpha_0^{-1} = 0.$$

Again we look at the characteristic polynomial

$$X^d + \sum_{i=0}^{d-1} c_i(n)X^i$$

of a_n and we obtain

$$\lim_n \eta_n(a_n^d) = 0$$

which is a contradiction. This completes the proof of the implication.

5.6. Proposition. *The mapping*

$$\nu : \tilde{G} \to \Omega(G)$$

is closed.

Proof. Let $X \subseteq \tilde{G}$ be closed. Let $x_0 \in \mathrm{Cl}(\nu(X))$. Suppose that $x_0 \in \Omega$, where each element of Ω is represented by some $\omega\rho$ with $\omega \in \mathrm{Unr}(M)$. Since $x_0 \in \mathrm{Cl}(\nu(X))$, there exists a convergent sequence $(\psi_n)_{n \geqq 1}$ in $\mathrm{Unr}(M)$; set $\psi_0 = \lim_n \psi_n$, so that x_0 is represented by $\psi_0\rho$, and that for each $n \geqq 1$ there exists $(\pi_n, V_n) \in X$ such that π_n is a subquotient of $\mathrm{Ind}_{P(M)}^G(\psi_n\rho)$. Proposition 5.2 implies that there exists an irreducible subquotient π of $\mathrm{Ind}_{P(M)}^G(\psi_0\rho)$ so that $\pi \in \mathrm{Cl}\, X = X$. Now $\nu(\pi) = x_0$.

5.7. Theorem. *The triple $(\tilde{G}, \Omega(G), \nu)$ has the following universal property. Suppose that Y is a Hausdorff topological space and*

$$f: \tilde{G} \to Y$$

a continuous mapping. Then there exists a unique mapping

$$\varphi : \Omega(G) \to Y$$

such that $f = \varphi \circ \nu$. The mapping φ is continuous.

Proof. Let us have f and Y as above.

Suppose that $\nu(\pi_1) = \nu(\pi_2)$ for $\pi_1, \pi_2 \in \tilde{G}$. Let $\nu(\pi_1)$ be represented by (M, ρ). There exists a sequence (ψ_n) in $\mathrm{Unr}(M)$ converging to the identity and such that $\mathrm{Ind}_{P(M)}^G(\psi_n \rho)$ is irreducible (see the proof of Theorem 4.5). Since Y is Hausdorff and f is continuous it must be $f(\pi_1) = f(\pi_2)$. Thus there exist $\varphi : \Omega(G) \to Y$ so that $f = \varphi \circ \nu$. The uniqueness of φ follows from the surjectivity of ν.

It remains to show that φ is continuous. Let $X \subseteq Y$ be a closed subset. Then

$$f^{-1}(X) = \nu^{-1}(\varphi^{-1}(X))$$

implies $\nu(f^{-1}(X)) = \nu(\nu^{-1}(\varphi^{-1}(X))) = \varphi^{-1}(X)$ since ν is surjective. Since f is continuous and ν closed, $\varphi^{-1}(X)$ is closed. This proves the continuity of φ.

Before we proceed to describe further the topology in terms of Hecke algebras $H(G, K)$ we have a simple lemma.

5.8. Lemma. *For $f \in H(G)$ set*

$$\tilde{G}^f = \{\pi \in \tilde{G}; \pi(f) \neq 0\}.$$

The sets $\tilde{G}^f$ are open subsets of $\tilde{G}$.

Proof. Let $(\pi, V) \in \mathrm{Cl}(\tilde{G} \setminus \tilde{G}^f)$. Then for each $v \in V$, $\tilde{v} \in \tilde{V}$ there exist a sequence $((\pi_n, V_n))$ in $\tilde{G} \setminus \tilde{G}^f$ and $v_n \in V_n$, $\tilde{v}_n \in \tilde{V}_n$ so that $c_{v_n, \tilde{v}_n}$ converges uniformly on compacts to $c_{v,\tilde{v}}$. Thus $\langle \pi_n(f)v_n, \tilde{v}_n \rangle$ converges to $\langle \pi(f)v, \tilde{v} \rangle$. This implies $\langle \pi(f)v, \tilde{v} \rangle = 0$. As v and $\tilde{v}$ were arbitrary one has $\pi(f) = 0$. Thus $\tilde{G} \setminus \tilde{G}^f$ is closed.

For an open compact subgroup $K \subseteq G$ we shall write $\tilde{G}^K$ for $\tilde{G}^{\mathrm{ch}_K}$.

Denote by $H(G, K)^\sim$ the set of all equivalence classes of irreducible representations of $H(G, K)$. They are all finite dimensional. The mapping

$$(\pi, V) \to (\pi^K, V^K), \qquad \tilde{G}^K \to H(G, K)^\sim$$

is a bijection.

Now we shall describe the topology of open sets $\tilde{G}^K$ in terms of $H(G, K)^\sim$.

In the following proposition, (i) is essentially Lemma 4.1 and (ii) is proved in the course of proving Theorem 5.4.

5.9. Proposition. *Let $X \subseteq \tilde{G}^K$ and $(\pi, V) \in \tilde{G}^K$. Then*

(i) $(\pi, V) \in \mathrm{Cl}\, X$ *if and only if there exist $v \in V^K$, $\tilde{v} \in \tilde{V}^K$, $v \neq 0$, $\tilde{v} \neq 0$ and there exist a sequence $((\pi_n, V_n))$ in G^K and $v_n \in V_n^K$, $\tilde{v}_n \in \tilde{V}_n^K$ so that*

$$\lim_n \langle \pi_n^K(f)v_n, \tilde{v}_n \rangle = \langle \pi^K(f)v, \tilde{v} \rangle \qquad \textit{for all } f \in H(G, K).$$

(ii) $(\pi, V) \in \mathrm{Cl}\, X$ *if and only if there exists a sequence* $((\pi_n, V_n))$ *in* $\tilde{G}^K$ *and a finite sequence* $\sigma_1, \ldots, \sigma_k \in \tilde{G}^K$ *(possibly an empty sequence) so that*

$$\lim_n \operatorname{tr} \pi_n^K(f) = \pi^K(f) + \sum_{i=1}^{k} \sigma_i^K(f)$$

for all $f \in H(G, K)$.

For two points x, y of a topological space X we shall say that they are inseparable if each neighborhood of x has a non-empty intersection with each neighborhood of y. Two points $x, y \in X$ will be called Hausdorff-equivalent if there exist $x_1, x_2, \ldots, x_m \in X$ so that x_i and x_{i+1} are inseparable for $i = 1, \ldots, m-1$ and $x_1 = x$, $x_m = y$.

Now we have directly

5.10. Proposition. *Let* $\pi_1, \pi_2 \in \tilde{G}$. *Then the following properties are equivalent*:

(i) π_1 *and* π_2 *are inseparable.*

(ii) π_1 *and* π_2 *are Hausdorff-equivalent.*

(iii) $\nu(\pi_1) = \nu(\pi_2)$.

(iv) π_1 *and* π_2 *have the same infinitesimal characters.*

We will see now a relation between cohomological properties of representations and of the topology of $\tilde{G}$ of the type expected in [12] (we consider $\tilde{G}$ instead of $\hat{G}$).

We shall work in the abelian category $\mathrm{Alg}(G)$. For $\pi_1, \pi_2 \in \mathrm{Alg}(G)$ let $\mathrm{Ext}^n(\pi_1, \pi_2)$ be the group of all classes of n-extensions of π_1 and π_2 (see 3.1 of [23]). This is really a group by 3.1.24 of [23] and A.3 of [8]. Let $\mathrm{Ext}^*(\pi_1, \pi_2)$ be the graded group $(\mathrm{Ext}^n(\pi_1, \pi_2))_{n \geq 0}$. Suppose that there exists an endomorphism of the category $\mathrm{Alg}(G)$ which acts on π_1 (resp. π_2) as identity and on π_2 (resp. π_1) as zero. Now in the same manner as in the proof of Theorem 4.1 of [6] we obtain that $\mathrm{Ext}^n(\pi_1, \pi_2) = 0$ (using 3.1 of [23]). Now we can obtain a p-adic variant of Theorem 4.1 of [6].

5.11. Theorem. *Let* $\pi_1, \pi_2 \in \tilde{G}$. *Suppose that* $\mathrm{Ext}^*(\pi_1, \pi_2) \neq 0$. *Then* π_1 *and* π_2 *have the same infinitesimal character and* $\nu(\pi_1) = \nu(\pi_2)$.

Proposition 5.10 implies

5.12. Theorem. *Let* $\pi_1, \pi_2 \in \tilde{G}$. *Suppose that* $\mathrm{Ext}^*(\pi_1, \pi_2) \neq 0$. *Then* π_1 *and* π_2 *are inseparable.*

An interesting consequence of Theorem 5.4 is

5.13. Theorem. *The set* $\hat{G}$ *is a closed subset of* $\tilde{G}$.

Proof. If $\pi \in \mathrm{Cl}(\hat{G})$ then Θ_π is a limit of irreducible unitary characters. Now [18] implies $\pi \in \hat{G}$. For a proof in terms of p-adic groups that $\pi \in \hat{G}$, one can consult the proof of Theorem 2.7 of [28] (see also [31]).

6. Isolated representations modulo center

First we have

6.1. Lemma. (i) *Let* $\pi \in \tilde{G}$. *Then* $\mathrm{Unr}(G)\pi$ *is a closed connected subset of* $\tilde{G}$.
(ii) *Let* $\pi \in \hat{G}$. *Then* $\mathrm{Unr}^u(G)\pi$ *is a closed connected subset of* $\hat{G}$.

Proof. (i) Observe that $\psi \to \psi\pi$, $\mathrm{Unr}(G) \to \tilde{G}$ is continuous by Theorem 5.4. Thus $\mathrm{Unr}(G)\pi$ is connected. Let $\pi_0 \in \mathrm{Cl}(\mathrm{Unr}(G)\pi)$. Then we can choose a sequence (ψ_n) in $\mathrm{Unr}(G)$ such that $(\psi_n\pi)$ converges to π_0. The fact that $(\nu(\psi_n\pi))$ converges implies that by passing to a subsequence we can assume that (ψ_n) converges to some ψ_0. Now $\pi_0 = \psi_0\pi$ and thus $\pi_0 \in \mathrm{Unr}(G)\pi$.
The proof of (ii) is analogous.

Let Z be the center of G and ω a quasi-character of Z. For an irreducible representation π of G, ω_π will denote the central character of π. Set

$$\tilde{G}_\omega = \{\pi \in \tilde{G};\, \omega_\pi = \omega\} \qquad \text{and} \qquad \hat{G}_\omega = \{\pi \in \hat{G};\, \omega_\pi = \omega\}.$$

Similarly as for the infinitesimal characters one obtains that the mappings

$$\pi \mapsto \omega_\pi, \qquad \tilde{G} \to \tilde{Z}, \qquad \text{and} \qquad \hat{G} \to \hat{Z}$$

are continuous. Thus $\tilde{G}_\omega$ is a closed subset of $\tilde{G}$ and $\hat{G}_\omega$ is a closed subset of $\hat{G}$.
Motivated by Lemma 5.1, we shall say that $\pi \in \tilde{G}$ (resp. $\pi \in \hat{G}$) is isolated modulo unramified characters in $\tilde{G}$ (resp. in $\hat{G}$) if $\mathrm{Unr}(G)\pi$ (resp. $\mathrm{Unr}^u(G)\pi$) is an open subset of $\tilde{G}$ (resp. $\hat{G}$).
We shall say that $\pi \in \tilde{G}$ (resp. $\pi \in \hat{G}$) is isolated modulo center if $\{\pi\}$ is an open subset of $\tilde{G}_{\omega_\pi}$ (resp. of $\hat{G}_{\omega_\pi}$).

6.2. Lemma. *Let* $\pi \in \tilde{G}$ (*resp.* $\pi \in \hat{G}$). *Then* π *is isolated modulo unramified characters in* $\tilde{G}$ (*resp. in* $\hat{G}$) *if and only if* π *is isolated modulo center.*

Proof. Let $\pi \in \tilde{G}$ (resp. $\pi \in \hat{G}$) be isolated modulo unramified characters in $\tilde{G}$ (resp. in $\hat{G}$). Then $(\mathrm{Unr}(G)\pi) \cap \tilde{G}_{\omega_\pi}$ is open in $\tilde{G}_{\omega_\pi}$ (resp. $\mathrm{Unr}^u(G)\pi \cap \hat{G}_{\omega_\pi}$ is open in $\hat{G}_{\omega_\pi}$). But $(\mathrm{Unr}(G)\pi) \cap \tilde{G}_{\omega_\pi}$ and $\mathrm{Unr}^u(G)\pi \cap \hat{G}_{\omega_\pi}$ are finite subsets (since $G/{}^\circ GZ$ is finite where ${}^\circ G$ is the set of all $g \in G$ such that $|\chi(g)|_F = 1$ for all rational characters χ of G). Since points are closed by Theorem 5.4 we have that π is open in $\tilde{G}_{\omega_\pi}$ (resp. in $\hat{G}_{\omega_\pi}$).
Suppose that $\pi \in \tilde{G}$ (resp. $\pi \in \hat{G}$) is not isolated modulo unramified characters in $\tilde{G}$ (resp. in $\hat{G}$). Find a sequence (π_n) in $\tilde{G} \setminus \mathrm{Unr}(G)\pi$ (resp. in $\hat{G} \setminus \mathrm{Unr}^u(G)\pi$)

which converges to π. Therefore ω_{π_n} converges to ω_π. Since $Z \cap {}^\circ G$ is compact, one obtains that the relation

$$\omega_{\pi_n} \mid Z \cap {}^\circ G = \omega_\pi \mid Z \cap {}^\circ G$$

holds with finite exceptions. Passing to a subsequence we may assume that the above relation holds for all n. We can consider $\omega_{\pi_n}^{-1}\omega_\pi$ as a character of $Z/(Z \cap {}^\circ G)$. From the inclusion of lattices $Z/(Z \cap {}^\circ G) \subseteq G/{}^\circ G$ on obtains that there exist unramified characters ψ_n of G such that

$$\omega_\pi = (\psi_n \mid Z)\omega_{\pi_n}$$

for all n. Now $(\psi_n \mid Z)$ converges in $\tilde{Z}$ (resp. $\hat{Z}$). Since $Z/(Z \cap {}^\circ G)$ is of finite index in $G/{}^\circ G$, one sees directly that by passing to a subsequence one may assume that ψ_n is a convergent sequence. Denote $\psi = \lim_n \psi_n$. Now the sequence $(\psi^{-1}\psi_n\pi_n)$ is in $\tilde{G} \setminus \mathrm{Unr}(G)\pi$ (resp. $\hat{G} \setminus \mathrm{Unr}^u(G)\pi$) and

$$\omega_{\psi^{-1}\psi_n\pi_n} = ((\psi^{-1}\psi_n) \mid Z)\omega_{\pi_n} = (\psi^{-1} \mid Z)\omega_\pi = \omega_\pi$$

since $\psi \mid Z = \lim_n(\psi_n \mid Z) = \lim_n \omega_\pi\omega_{\pi_n}^{-1}$. Also $(\psi^{-1}\psi_n\pi_n)$ converges to π by Theorem 5.4. Therefore, we have obtained that π is not isolated in $\tilde{G}_{\omega_\pi}$ (resp. $\hat{G}_{\omega_\pi}$). This proves another implication.

Now a characterization of isolated points in $\tilde{G}$ follows directly from Theorem 4.5.

6.3. Proposition. *A representation $\pi \in \tilde{G}$ is isolated modulo center in $\tilde{G}$ if and only if it is cuspidal.*

Regarding the isolated points it is interesting to remark:

6.4. Proposition. *Let $\pi \in \tilde{G}$. Then π is a projective and an injective object in* Alg G *if and only if π is an isolated point of $\tilde{G}$ (i.e. $\{\pi\}$ is open in $\tilde{G}$). In particular, if there exists such π, the center of G is compact.*

Proof. Suppose that $\{\pi\}$ is open in $\tilde{G}$. Then $\mathrm{Unr}(G)\pi = \{\pi\}$ so G has no non-trivial split torus in the center. Also π must be cuspidal by the preceding proposition. Now Theorem 2.44 of [5] implies that π is injective and projective in Alg G.

Suppose that $\pi \in \tilde{G}$ is projective and injective in Alg G. This implies a decomposition of Alg G into the direct sum of $(\mathrm{Alg}\, G)_\pi$ and $(\mathrm{Alg}\, G)_\pi^\perp$ where objects of $(\mathrm{Alg}\, G)_\pi$ are direct sums of copies of π, and objects $(\mathrm{Alg}\, G)_\pi^\perp$ are all smooth representations of G with no subquotient isomorphic to π. Now (ii) of Proposition 4.6 implies that $\{\pi\}$ is a connected component of $\tilde{G}$. This implies that the center of G is compact and that π is cuspidal.

7. On representations with bounded matrix coefficients

Let $(\pi, V)\in\tilde{G}$. Suppose that there exist $v\in V$, $\tilde{v}\in\tilde{V}$, $v\neq 0$ and $\tilde{v}\neq 0$, so that the function $c_{v,\tilde{v}}$ is a bounded function on G. Then any other matrix coefficient is a bounded function on G. A representation which satisfies this condition will be called a representation with bounded matrix coefficients. Certainly, elements of $\hat{G}$ are representations with bounded matrix coefficients.

Now we shall improve Theorem 2.5.

7.1. Theorem. *Let $P=MN$ be a parabolic subgroup of G and τ a smooth representation of M of finite length. Let $B(\tau)$ be the set of all $\psi\in\mathrm{Unr}(M)$ so that* $\mathrm{Ind}_P^G(\psi\tau)$ *has an irreducible subquotient with bounded matrix coefficients. Then $B(\tau)$ is a relatively compact subset of* $\mathrm{Unr}(M)$.

Proof. A reduction to the case when τ is an irreducible cuspidal is the same as in the proof of Theorem 2.5. So we shall suppose that τ is an irreducible cuspidal representation of M.

Let Ω be a connected component in $\Omega(G)$ such that $(M,\tau)\in\Omega$. Choose K so that each representation in $\tilde{G}_\Omega$ has a non-trivial K-fixed vector.

We will assume that the measure of K is 1.

We introduce a norm on $H(G,K)$,

$$\|f\|_1=\int_G |f(g)|\,dg,$$

which makes $H(G,K)$ a normed algebra with identity.

Let $(\pi, V)\in\tilde{G}_\Omega$ be a representation with bounded matrix coefficients. Let $d=\dim_{\mathbf{C}}V^K$, $v_1,\dots,v_d$ a basis of V^K and $\tilde{v}_1,\dots,\tilde{v}_d\in\tilde{V}^K$ the biorthogonal basis. Then

$$|\langle\pi^K(f)v_i,\tilde{v}_j\rangle|=\left|\int_G f(g)\langle\pi(g)v_i,\tilde{v}_j\rangle dg\right|$$

$$\leqq \|f\|_1(\sup\{|\langle\pi(g)v_i,\tilde{v}_j\rangle|;g\in G\})$$

for all $f\in H(G,K)$, $1\leqq i,j\leqq d$.

Thus

$$\pi^K: H(G,K)\to\mathrm{End}_{\mathbf{C}}(V^K)$$

is a continuous linear map.

Now we introduce a norm on $\mathrm{End}_{\mathbf{C}}(V^K)$. Let $f\in\mathrm{End}_{\mathbf{C}}(V^K)$. Then there exist $F\in H(G,K)$ so that $\pi^K(F)=f$. Set

$$\|f\|=\inf\{\|F\|_1; F\in H(G,K)\text{ and }\pi^K(F)=f\}.$$

This makes $\mathrm{End}_{\mathbf{C}}(V^K)$ a normed algebra with identity (in fact, a Banach algebra). By Lemma 10 of [11] about norms on algebras of matrices, we can find a basis $v_1, \ldots, v_d$ of V^K and the biorthogonal basis $\tilde{v}_1, \ldots, \tilde{v}_d$ of $\tilde{V}^K$ so that

$$|\langle \pi^K(f)v_i, \tilde{v}_j\rangle| \leqq 2^{d-1}\, \|f\|_1$$

for all $f \in H(G, K)$, $1 \leqq i, j \leqq d$.

We are going to prove now that $B(\tau)$ is relatively compact. Let us suppose that (ψ_n) is a sequence in $\mathrm{Unr}(M)$ such that each $\mathrm{Ind}_P^G(\psi_n\tau)$ has an irreducible subquotient (π_n, V_n) whose matrix coefficients are bounded. For the proof it will be enough that (ψ_n) has a convergent subsequence.

Passing to a subsequence, we may assume that all V_n^K are of the same dimension, say d. Let W be a d-dimensional complex vector space. Let $w_1, \ldots, w_d$ be a basis of W and let $\tilde{w}_1, \ldots, \tilde{w}_d$ be the biorthogonal basis. Using the first part of the proof we can find a sequence (σ_n) of representations of $H(G, K)$ on W and linear isomorphisms

$$\varphi_n : V_n^K \to W, \qquad n \geqq 1,$$

such that φ_n are isomorphisms of $H(G, K)$-modules and that

$$|\langle \sigma_n(f)w_i, \tilde{w}_j\rangle| \leqq 2^{d-1}\, \|f\|_1$$

for all $f \in H(G, K)$, $1 \leqq i, j \leqq d$.

Using a diagonal procedure, passing to a subsequence we may assume that sequences $(\sigma_n(f))$ converge for all $f \in H(G, K)$. Set

$$\sigma(f) = \lim_n \sigma_n(f).$$

Then σ is a representation of $H(G, K)$. Thus $\lim_n \mathrm{tr}\, \sigma_n(f) = \mathrm{tr}\, \sigma(f)$ for all $f \in H(G, K)$. Now as in the proof of Lemma 5.3 one obtains that $(\nu(\pi_n))$ converges. Therefore, there exists a subsequence of (ψ_n) which is convergent.

8. Non-unitary dual of J. M. G. Fell

In [10] J. M. G. Fell introduced a space $\check{G}$ consisting of all functional equivalence classes of topologically completely irreducible linear system representations of G.† He supplied $\check{G}$ with a topology and called it a dual space or a functional dual space of G.

The purpose of this section is to show that there is a natural homeomorphism between $\check{G}$ and $\tilde{G}$.

First we shall briefly recall of some definitions from [10].

A pair of complex vector spaces H_1 and H_2 together with a non-degenerate bilinear form $(\ |\ )$ on $H_1 \times H_2$ is called a linear system $\langle H_1, H_2\rangle$. The isomor-

† J. M. G. Fell denoted that space by $\hat{G}$.

phism between two linear systems is defined in a natural way. For a linear system $H = \langle H_1, H_2\rangle$ one equips H_1 (resp. H_2) with a locally convex topology generated by the functionals $h_1 \mapsto (h_1 \mid h_2)$, $h_2 \in H_2$ (resp. $h_2 \mapsto (h_1 \mid h_2)$, $h_1 \in H_1$). These topologies are called $\sigma(H)$-topologies on H_1 and H_2.

Let $M_0(G)$ be the algebra of all compactly supported complex regular Borel measures on G. The algebra $H(G)$ is identified with the subalgebra of $M_0(G)$ using Haar measure.

A linear system representation T of G is a pair of a linear system $H(T) = \langle H_1, H_2\rangle$ and $\langle T_1, T_2\rangle$ where T_1 (resp. T_2) is a homomorphism (resp. anti-homomorphism) of G into the group of all linear bijections on H_1 (resp. H_2) satisfying

(i) $(T_1(g)h_1, h_2) = (h_1, T_2(g)h_2)$ for $h_1 \in H_1$, $h_2 \in H_2$, $g \in G$,

(ii) $g \mapsto (T_1(g)h_1, h_2)$ is continuous on G for $h_1 \in H_1$, $h_2 \in H_2$,

(iii) for $\mu \in M_0(G)$ there exist $T_1(\mu)$ and $T_2(\mu)$ endomorphisms of H_1 and H_2 such that for all $h_1 \in H_1$ and $h_2 \in H_2$ we have

$$(T_1(\mu)h_1, h_2) = \int_G (T_1(g)h_1, h_2)d\mu(g)$$

$$= (h_1, T_2(\mu)h_2).$$

We will also need a corresponding notion for algebras.

Let A be an associative complex algebra. A linear system representation of A in a linear system $\langle H_1, H_2\rangle$ is a pair $\langle T_1, T_2\rangle$ where T_1 is a representation of A on H_1, T_2 is an anti-representation of A on H_2 and

$$\langle T_1(a)h_1, h_2\rangle = \langle h_1, T_2(a)h_2\rangle$$

for all $h_1 \in H_1$, $h_2 \in H_2$ and $a \in A$. Equivalence of two system representations is defined in a natural way. A linear system representation $\langle T_1, T_2\rangle$ such that $T_1(a) \neq 0$ for some $a \in A$ is called algebraically irreducible if both T_1 and T_2 are algebraically irreducible. Analogously one defines algebraically completely irreducible linear system representation and topologically irreducible linear system representation (assuming $\sigma(H)$ topologies on H_1 and H_2). If for any linearly independent $x_1, \ldots, x_n \in H_1$, linearly independent $y_1, \ldots, y_m \in H_2$ and any $r_{ij} \in \mathbf{C}$, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$ there exists $a \in A$ so that

$$(T_1(a)x_i \mid y_j) = r_{ij}$$

for all $1 \leqq i \leqq n$, $1 \leqq j \leqq m$, then T is called topologically completely irreducible.

For a linear system representation $T = \langle T_1, T_2\rangle$ of A we have

$$\ker T_1 = \ker T_2$$

and this is called the kernel of T. The vector subspace of A' generated by all

$$a \mapsto (T_1(a)h_1 \mid h_2)$$

with $h_1 \in H_1$, $h_2 \in H_2$ will be denoted by $\Lambda(T)$. For X, a set of linear system representations, put $\Lambda(X) = \bigcup_{T \in X} \Lambda(T)$.

The space of all equivalence classes of topologically completely irreducible linear system representations of A will be denoted by $J(A)$.

The space A' is supplied with the topology of pointwise convergence. Topology on $J(A)$ is introduced by a closure operator: for $X \subseteq J(A)$ and $T \in J(A)$, $T \in \mathrm{Cl}(X)$ if and only if $\Lambda(T) \subseteq \mathrm{Cl}(\Lambda(X))$. For $S, T \in J(A)$ one says that they are functionally equivalent if $\mathrm{Cl}\,\Lambda(S) = \mathrm{Cl}\,\Lambda(T)$. This is equivalent to $\mathrm{Ker}\, S = \mathrm{Ker}\, T$. By $\hat{A}$ will be denoted the quotient topological space of $J(A)$ by the relation "being functional equivalent".

Let us return to linear system representations of G.

By definition, each linear system representation T of G is in a natural way a linear system representation of $M_0(G)$ which will be called an integrated form of T. Now one defines topological irreducibility, topological complete irreducibility, equivalence and functional equivalence for linear system representations of G, using corresponding notions for integrated forms.

The set of all equivalence classes (resp. functional equivalence classes) of all topologically completely irreducible linear system representations of G will be denoted by $J(G)$ (resp. by $\check{G}$). By definition we may consider

$$J(G) \subseteq J(M_0(G)), \qquad \check{G} \subseteq M_0(G)^{\cup}.$$

The sets $J(G)$ and $\check{G}$ are supplied with relative topologies.

J. M. G. Fell calls $\check{G}$ the (functional) dual space of G.

8.1. Lemma. *Let $T = \langle T_1, T_2 \rangle \in \check{G}$ be a linear system representation on $H = \langle H_1, H_2 \rangle$. The mapping*

$$T \to (T_1 \mid H(G)H_1, H(G)H_1)$$

is a homeomorphism of $\check{G}$ onto $\tilde{G}$.

Proof. Let $T^0 \in \check{G}$.

Note that $H(G)$ is a two-sided ideal in $M_0(G)$, and $H(G) \not\subseteq \ker T^0$. By Lemma 4 of [10] we have that $T^0 \mid H(G)$ is topologically completely irreducible. By Theorem 1 of [10]

$$r: T^0 \to T^0 \mid H(G)$$

is a homeomorphism onto the image

$$r: \check{G} \to H(G)^{\cup}.$$

Let $T^0 \in H(G)^\cup$, $T^0 = \langle T_1^0, T_2^0 \rangle$ be on $H^0 = \langle H_1^0, H_2^0 \rangle$. By definition of topologically completely irreducible system, there exist $f \in H(G)$ such that $T_1^0(f) \neq 0$. Thus there exists an open compact subgroup K such that $T_1^0(\mathrm{ch}_K) \neq 0$.

Let $H(G)_K^\cup$ be the set of all $T = \langle T_1, T_2 \rangle \in H(G)^\cup$ such that $T_1(\mathrm{ch}_K) \neq 0$. For $T \in H(G)_K^\cup$ let $T^K = \langle T_1^K, T_2^K \rangle$ be the linear system

$$\langle \text{range } T_1(\mathrm{ch}_K), \text{ range } T_2(\mathrm{ch}_K) \rangle$$

(see Section 4 of [10]). In this way we obtain a map

$$r_k : H(G)_K^\cup \to H(G,K)^\cup.$$

By Lemma 7 of [10], $H(G)_K^\cup$ is open in $H(G)^\cup$ and r_K is a homeomorphism.

By Theorem 1 of [2] and the implication (ii)$\Rightarrow$(iii) of Lemma 12 in [10], each $T = \langle T_1, T_2 \rangle \in H(G,K)^\cup$ is finite dimensional (i.e. T_1 and T_2 are finite dimensional). Since T_1 is topologically irreducible and finite dimensional, it is algebraically irreducible. Also a finite dimensional linear system representation $T = \langle T_1, T_2 \rangle$ is determined by its first term T_1.

Choose an irreducible smooth representation (π^0, V^0) of G such that $(\pi^0)^K \cong (T_1^0)^K$ ((v) of Proposition 2.10 of [5]). Let $\tilde{V}^0$ be the space of the contragradient representation. For $f \in H(G)$ let $(\pi^0)'(f) : \tilde{V}^0 \to \tilde{V}^0$,

$$[(\pi^0)'(f)](\tilde{v}) = \tilde{v} \circ \pi^0(f).$$

Now $\langle \pi^0, (\pi^0)' \rangle$ is a linear system representation of $H(G)$. It is topologically completely irreducible. Thus it is equivalent to T^0.

Up to now we have proved that we have a natural mapping of $\tilde{G}$ onto $H(G)^\cup$. Let us show that it is injective. Let $(\pi_i, V_i) \in \tilde{G}$, $i = 1, 2$, be non-equivalent. Choose an open compact subgroup K so that $V_1^K \neq 0$ and $V_2^K \neq 0$. Choose $f \in H(G,K)$ so that $\pi_1^K(f)$ is an identity on V_1^K and $\pi_2^K(f) = 0$. Now it is clear that (π_1^K, V_1^K) and (π_2^K, V_2^K) cannot be functionally equivalent (i.e. equal in $H(G,K)^\cup$). Thus, the map

$$\tilde{G} \to H(G)^\cup$$

is a bijection.

In this way we have obtained an injection

$$\check{G} \to \tilde{G},$$

$$\langle T_1, T_2 \rangle \to T_1 \,|\, H(G)H_1.$$

We see that it is surjective using Langlands classification ([6], [19]).

To prove the theorem it remains to show that the above mapping $\check{G} \to \tilde{G}$ is a homeomorphism. It is sufficient to show that the bijection

$$b : \tilde{G} \to H(G)^\cup$$

is a homeomorphism. This is a direct consequence of the first part of the proof and Theorem 5.5.

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