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REPRESENTATIONS OF p-ADIC SYMPLECTIC GROUPS

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Introduction

A non-archimedean field of a characteristic different from two is denoted by F. In this paper we consider the representation theory of groups $Sp(n,F)$ and $GSp(n,F)$. The inner geometry of these groups motivates us to consider the representations of these groups as modules over representations of general linear groups. Such an idea goes back to D. K. Faddeev in the finite field case ([F]). D. Barbasch had also such point of view in [Ba]. Besides the module structure, we also have a comodule structure.

Our motivation for such approach is to make symplectic case more close to the well understood theory of groups $GL(n)$, as it was developed by J. Bernstein and A.V. Zelevinsky ([BnZ1], [BnZ2], [Z1]), and to ideas developed in [T1]. The basic idea was to realize some of the properties of the representation theory of symplectic groups as a part of the structure theory of certain modules. This point of view is helpful in searching of new square integrable representations, and in examining reducibility of parabolically induced representations (see [T4]). In this paper, we are developing this approach in the symplectic case. Other classical groups can be treated in a similar way.

We describe now the content of the paper according to sections. Let us point out first that the parameter n in $Sp(n,F)$ or $GSp(n,F)$ denotes the semi simple rank of these groups. In the first section we collect some general facts about representations of finite length of reductive groups over F. Besides the group $R(G)$ of virtual characters of G, we introduce a group $\mathcal{R}(G)$ which is constructed from the representations of finite length of G. Two algebras of representations of $GL(n,F)$ are considered in the second section. The first algebra R was introduced by J. Bernstein and A.V. Zelevinsky ([BnZ2]). It is realized as the direct sum of $R(GL(n,F))$'s. The other algebra $\mathcal{R}$ is realized as the direct sum of $\mathcal{R}(GL(n,F))$'s. The multiplication in both cases is defined with the help of the parabolic induction from the maximal parabolic subgroups. It is well known that R is a Hopf algebra. The comultiplication is defined using Jacquet modules for maximal parabolic subgroups.

The direct sum of all $R(Sp(n,F))$ (resp. $\mathcal{R}(Sp(n,F))$) is denoted by $R[S]$ (resp. $\mathcal{R}[S]$). We introduce also corresponding groups for $GSp(n,F)$'s. They are denoted by $R[G]$ and $\mathcal{R}[G]$. In the fourth section, the groups $R[S]$ and $\mathcal{R}[S]$ are considered as modules over R and $\mathcal{R}$ respectively. Analogously, we consider $R[G]$ and $\mathcal{R}[G]$ as modules. We list then some important properties of these modules. A structure of a comodule on $R[S]$ and $R[G]$ over R, is introduced in the fifth section. The comultiplication is defined again using

Jacquet modules of maximal parabolic subgroups. We describe the Langlands classification for groups $Sp(n,F)$ and $GSp(n,F)$ more explicitly in the sixth section.

N. Winarsky obtained in [Wi] a necessary and sufficient conditions for reducibility of the unitary principal series representations of $Sp(n,F)$. D. Keys obtained exact number of irreducible pieces and he showed that these representations are multiplicity free ([Ke]). In the seventh section, using the above results we obtain corresponding results for $GSp(n,F)$. A necessary and sufficient conditions for reducibility of the unitary principal series and the length of these representations is obtained here. Then we derive a necessary and sufficient condition for the reducibility of the non-unitary principal series representations of $Sp(n,F)$ and $GSp(n,F)$ using the Langlands classification and above results.

Regular characters of a maximal torus in a split reductive group, for which corresponding non-unitary principal series representations have square integrable subquotients were characterized by F. Rodier in [R1]. In the eighth section we give an explicit characterization of regular characters for groups $Sp(n,F)$ and $GSp(n,F)$. All square integrable representations of $GSp(n,F)$ which may be obtained as subquotients of non-unitary principal series representations induced by regular characters, are described explicitly in this section. There is a considerable number of square integrable representations obtainable in this way. Let us describe the parameters of that representations in the case when the residual characteristic is odd. Such square integrable representations, up to a twist by a character of $GSp(n,F)$, are parameterized by all pairs $(k, \ell\psi_1 + m\psi_2)$ where k, ℓ, m are non-negative integers which satisfy $\ell, m \neq 1$, $k + \ell + m = n$ and ψ_1, ψ_2 are different characters of $F^\times$ of order two.

Restriction of the square integrable representations of the eighth section to the group $Sp(n,F)$, is studied in the ninth section. These representations split without multiplicities and we give a parameterization of the irreducible pieces. We get a considerable number of square integrable representations of $Sp(n,F)$ in this way. Let us mention that the only square integrable representation of $Sp(n,F)$ which corresponds to a regular character, is the Steinberg representation.

The Steinberg representation of $Sp(n,F)$ is a subquotient of a non-unitary principal series representation $\pi = \mathrm{Ind}_{P_\phi}^{Sp(n,F)}(\Delta_{P_\phi}^{1/2})$. It is well known that π is a multiplicity free representation of length 2^n. It has exactly one square integrable subquotient, the Steinberg representation. The Steinberg representation and the trivial representation are the only unitarizable subquotients of π. There exist very different examples of non-unitary principal series representations of $Sp(n,F)$ which possess square integrable subquotients. For suitable choice of F, there exists a non-unitary principal series representation π_1 of $Sp(2n,F)$ which has exactly 2^n irreducible square integrable subquotients. They are all of multiplicity one. There exists a subquotient of π_1 whose multiplicity is 2^n. For $n=1$ we have proved in [SaT] that all irreducible subquotients are unitarizable. In the last section, one such representation is analyzed in detail. Let us denote this representation of $Sp(4,F)$ by π_2 (there is no additional assumptions on the field F). The length of π_2 is 36. It has exactly 25 different irreducible subquotients. We find all multiplicities.

The last example is an illustration of application of some of the methods which were considered in the previous sections. Further development along these lines, and more advanced applications of these techniques and ideas, are announced in [T4]. The present paper should be considered as an introduction to this point of view on representations of p-adic symplectic groups. We apply this approach systematically to the representations of $GSp(2,F)$ (and also $Sp(2,F)$) in the joint paper [SaT] with P.J. Sally.

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The field of real numbers is denoted by $\mathbb{R}$ in the paper. The subring of integers is denoted by $\mathbb{Z}$, the non-negative integers are denoted by $\mathbb{Z}_+$, while the strictly positive integers ara denoted by $\mathbb{N}$.

One technical remark at the end. We have already mentioned algebra $\mathcal{R}$ and modules $\mathcal{R}[S]$ and $\mathcal{R}[G]$. Let me note that in this paper we could work simply with representations, instead of doing calculations in $\mathcal{R}, \mathcal{R}[S]$ or $\mathcal{R}[G]$ (this is equivalent). Regardless of this, we introduced this algebra and these modules because they arose naturally in our considerations.

1. Groups of representations

In this section we shall recall some well known facts from the representation theory of reductive p-adic groups.

We fix a local non-archimedean field F. The group of rational points of a reductive group defined over F is denoted by G. The category of all smooth representations of G and G-intertwinings among them, is denoted by $Alg(G)$. The full subcategory of representations of finite length in $Alg(G)$ is denoted by $Alg_{f.l.}(G)$. For each isomorphism class of representations in $Alg_{f.l.}(G)$, we fix a representative. The set of all such representatives is denoted by $\mathcal{R}^+(G)$. Note that $\mathcal{R}^+(G)$ is a set. The full subcategory of $Alg_{f.l.}(G)$ whose objects consist of $\mathcal{R}^+(G)$, is denoted by $\mathcal{R}^+(G)$ again.

We denote by $In(G)$ the set of all equivalence classes of all smooth indecomposable non-zero representations of finite length. The subset of all irreducible classes is denoted by $\tilde{G}$. The subset of all unitarizable classes in $\tilde{G}$ is denoted by $\hat{G}$. We shall consider

$$\hat{G} \subseteq \tilde{G} \subseteq In(G) \subseteq \mathcal{R}^+(G).$$

Let $\pi_1, \pi_2 \in \mathcal{R}^+(G)$. We denote by $\pi_2 + \pi_2$ a unique representation in $\mathcal{R}^+(G)$ which is equivalent to $\pi_1 \oplus \pi_2$. It is clear that the addition is associative and commutative. The representation on 0-dimensional space is the zero of the additive semigroup $\mathcal{R}^+(G)$. It is obvious that $In(G)$ generates $\mathcal{R}^+(G)$ as a semigroup with zero. In other words, each $\pi \in \mathcal{R}^+(G)$ may be written as $\pi = \pi_1 + \ldots + \pi_m$ with $\pi_i \in In(G)$. It is easy to see that $\pi_1 \ldots \pi_m \in In(G)$ are determined uniquely, up to a permutation, by $\pi \in \mathcal{R}^+(G)$ ([Bu], ch. 8, n^o 2, Theorem 1).

We denote by $\mathcal{R}(G)$ the free Abelian group over the basis $In(G)$. According to the above observation, we may identify $\mathcal{R}^+(G)$ with a subset of $\mathcal{R}(G)$ in a natural way. Also, $\mathcal{R}^+(G)$ is an additive subsemigroup of $\mathcal{R}(G)$.

The Grothendieck group of the category $Alg_{f.l.}(G)$ (or equivalently, of $\mathcal{R}^+(G)$) will be denoted by $R(G)$. The canonical mapping will be denoted by

$$\text{s.s.} : \mathcal{R}^+(G) \to R(G).$$

Recall that $R(G)$ may be identified with the free Abelian group over the basis $\tilde{G}$. We shall do so. Denote s.s.$(\mathcal{R}^+(G))$ by $R^+(G)$. There is a unique extension of s.s. to an additive homomorphism of $\mathcal{R}(G)$ into $R(G)$. This extension will be denoted again by s.s.. For $x_1, x_2 \in R(G)$ we shall write $x_1 \leq x_2$ if $x_2 - x_1 \in R^+(G)$.

Let χ be a character of G. Then $\pi \to \chi\pi$ induces automorphisms of $R(G)$ and $\mathcal{R}(G)$. They are positive, i.e. $\chi(R^+(G)) \subseteq R^+(G)$ and $\chi(\mathcal{R}^+(G)) \subseteq \mathcal{R}^+(G)$.

If π is a smooth representation of G, then $\tilde{\pi}$ denotes the smooth contragredient of π and $\bar{\pi}$ denotes the complex conjugate of π. One extends $\tilde{}$ and $\bar{}$ to $\mathcal{R}(G)$ and $R(G)$ additively. These are involutive automorphisms of $\mathcal{R}(G)$ and $R(G)$ and they commute.

Let $\mathcal{Z}(G)$ be the Bernstein center of G. It is the algebra of all invariant distributions T on G such that the convolution $T * f$ is compactly supported for any locally constant compactly supported function f on G. For each smooth representation (π, V) of $G, \mathcal{Z}(G)$ acts naturally on V. If π is irreducible, then $\mathcal{Z}(G)$ acts by scalars. The corresponding character of $\mathcal{Z}(G)$ is denoted by θ_π. It is called the infinitesimal character of π (here we follow mainly notation of $[BnDKa]$). The set of all infinitesimal characters of representations in $\tilde{G}$ is denoted by $\Theta(G)$. The set $\Theta(G)$ is in a natural one-to-one-correspondence with the set of all cuspidal pairs modulo conjugation (a cuspidal pair (M, ρ) consists of a Levi factor M of a parabolic subgroup $P = MN$ in G and of an irreducible cuspidal representation ρ of M). We identify these two sets. For $\theta \in \Theta(G)$, $\tilde{G}_\theta$ denotes the set of all $\pi \in \tilde{G}$ such that $\theta_\pi = \theta$. The set $\tilde{G}_\theta$ is finite. If $\theta = (M, \rho)$, then $\tilde{G}_\theta$ is just the set of all irreducible subquotients of $\operatorname{Ind}_P^G(\rho)$, i.e. $\pi \in \tilde{G}_\theta$ if and only if $\pi \in \tilde{G}$ and $\pi \leq s.s.(\operatorname{Ind}_P^G(\rho))$. Set

$$R_\theta(G) = \underset{\pi \in \tilde{G}_\theta}{\oplus} \mathbb{Z}\pi$$

Note that $R(G) = \oplus R_\theta(G), \theta \in \Theta(G)$, is a gradation of the group $R(G)$. This gradation is compatible with the order on $R(G)$, i.e. if $\pi^i = \sum \pi^i_\theta \in R(G) = \oplus R_\theta(G)$, $i = 1, 2$, then $\pi^1 \leq \pi^2$ if and only if $\pi^1_\theta \leq \pi^2_\theta$ for any $\theta \in \Theta(G)$.

Let $\theta \in \Theta(G)$. Denote by $\mathcal{R}^+_\theta(G)$ the set of all $\pi \in \mathcal{R}^+(G)$ such that each irreducible subquotient of π is in $\tilde{G}_\theta$. The additive subgroup of $\mathcal{R}(G)$ generated by $\mathcal{R}^+_\theta(G)$ is denoted by $\mathcal{R}_\theta(G)$. Set

$$In_\theta(G) = In(G) \cap \mathcal{R}^+_\theta(G).$$

Now $In_\theta(G)$ is a basis of $\mathcal{R}_\theta(G)$. Considering the action of the commutative algebra $\mathcal{Z}(G)$, one obtains in a standard way that we have the following disjoint union

$$In(G) = \underset{\Theta(G)}{\cup} In_\theta(G).$$

This was already proved without use of $\mathcal{Z}(G)$ by W. Casselman ([Cs], Theorem 7.3.1 and 7.3.2). Thus

$$\mathcal{R}(G) = \underset{\Theta(G)}{\oplus} \mathcal{R}_\theta(G).$$

is a gradation on $\mathcal{R}(G)$. Clearly

$$\text{s.s.}(\mathcal{R}_\theta(G)) = R_\theta(G).$$

For a smooth representation π of G and for an automorphism σ of $G, \sigma\pi$ denotes the representation $(\sigma\pi)(g) = \pi(\sigma^{-1}(g))$. In this way one obtains automorphisms $\sigma : \mathcal{R}(G) \to \mathcal{R}(G)$ and $\sigma : R(G) \to R(G)$.

Let P be a parabolic subgroup of G. Denote by N the unipotent radical of P. Let $P = MN$ be a Levi decomposition. The modular character of P is denoted by Δ_P. Using the normalized induction functor

$$\text{Ind}_P^G : \mathcal{R}^+(M) \to \mathcal{R}^+(G),$$

we define in a natural way homomorphisms

$$\text{Ind}_P^G : \mathcal{R}(M) \to \mathcal{R}(G)$$

$$\text{Ind}_P^G : R(M) \to R(G).$$

Recall now the notion of a Jacquet module. For a smooth representation (π, V) of G, we denote by $r_P^G(\pi)$ the representation of M on N-coinvariants twisted by $(\Delta_P^{-1/2})|M$. We have the Frobenius reciprocity for $\tau \in \mathcal{R}^+(M)$ and $\pi \in \mathcal{R}^+(G)$:

$$Hom_{\mathcal{R}^+(G)}(\pi, \text{Ind}_P^G(\tau)) \cong Hom_{\mathcal{R}^+(M)}(r_P^G(\pi), \tau).$$

In a natural way one obtains homomorphisms

$$r_P^G : \mathcal{R}(G) \to \mathcal{R}(M),$$

$$r_P^G : R(G) \to R(M).$$

Take the opposite parabolic subgroups $\bar{P} = M\bar{N}$ of P. By Corollary 4.2.5 of [Cs] we have

$$\left[r_P^G(\pi)\right]^\sim \cong r_{\bar{P}}^G(\tilde{\pi}).$$

for $\pi \in \mathcal{R}^+(G)$. We can reformulate the Frobenius reciprocity:

$$Hom_{\mathcal{R}^+(G)}\left(\text{Ind}_P^G(\tau), \pi\right) \cong Hom_{\mathcal{R}^+(M)}\left(\tau, r_{\bar{P}}^G(\pi)\right),$$

$\pi \in \mathcal{R}^+(G), \tau \in \mathcal{R}^+(M)$ ([Si2], Theorem 2.4.3).

The set of all irreducible essentially tempered representations of G (i.e. representations of G which become tempered after twisting by a suitable character of G) is denoted by $T(G)$. The essentially square integrable (resp. cuspidal) representations in $\tilde{G}$ are denoted by $D(G)$ (resp. $C(G)$). Let $T^u(G) = T(G) \cap \hat{G}$, $D^u(G) = D(G) \cap \hat{G}$ and $C^u(G) = C(G) \cap \hat{G}$.

For $\tau \in T(G)$ there exists a unique positive valued character χ of G such that $\chi^{-1}\tau \in T^u(G)$. Define $\nu(\tau) = \chi$ and $\tau^u = \chi^{-1}\tau^u$. Thus $\tau = \nu(\tau)\tau^u$.

2. Algebras of representations for $GL(n)$

Let

$$\begin{aligned}
\mathcal{R}_n^+ &= \mathcal{R}^+(GL(n,F)),\\
\mathcal{R}_n &= \mathcal{R}(GL(n,F)),\\
R_n^+ &= R^+(GL(n,F)),\\
R_n &= R(GL(n,F)).
\end{aligned}$$

We denote

$$\mathcal{R} = \underset{n\geq 0}{\oplus} \mathcal{R}_n,$$

$$R = \underset{n\geq 0}{\oplus} R_n.$$

The maps s.s.: $\mathcal{R}_n \to R_n$ naturally extend to s.s. : $\mathcal{R} \to R$. Let

$$\mathcal{R}^+ = \sum_{n\geq 0} \mathcal{R}_n^+,$$

$$R^+ = \sum_{n\geq 0} R_n^+.$$

For $\pi_i \in \mathcal{R}_{n_i}^+, i = 1,2$, we denote by $\pi_1 \times \pi_2$ (resp. $\pi_1 \underline{\times} \pi_2$) the unique representation in $\mathcal{R}_{n_1+n_2}^+$ which is isomorphic to the parabolically induced representation from the standard parabolic subgroup P (resp. $\bar{P}$) with respect to the upper (resp. lower) triangular matrices, whose Levi factor is naturally isomorphic to $GL(n_1,F)\times GL(n_2,F)$ by $\pi_1 \otimes \pi_2$ (see [BnZ]). The induction that we consider is normalized in such a way that it carries unitarizable representations to unitarizable ones. Conjugation by a suitable element of the Weyl group gives the following equality in $\mathcal{R}$

$$\pi_1 \times \pi_2 = \pi_2 \underline{\times} \pi_1.$$

We have also $(\pi_1 \times \pi_2) \times \pi_3 = \pi_1 \times (\pi_2 \times \pi_3)$, where $\pi_3 \in \mathcal{R}_{n_3}^+$. We extend $\times$ and $\underline{\times}$ to $\mathbb{Z}$-bilinear mappings on $\mathcal{R} \times \mathcal{R}$. In this way $(\mathcal{R}, +, \times)$ becomes a graded associative ring with identity. We extend $\pi \mapsto \tilde{\pi}, \pi \mapsto \bar{\pi}$ to $\mathcal{R}$. These are involutive automorphisms of the ring.

We shall define now a binary operation on R which will be denoted again by $\times$. Let $\pi_1, \pi_2 \in R$. We may consider $\pi_1, \pi_2 \in \mathcal{R}$ since $GL(n,F)^\sim \subseteq In(GL(n,F))$. Therefore, we have defined $\pi_1 \rtimes \pi_2 \in \mathcal{R}$. Now $\pi_1 \times \pi_2 \in R$ is defined to be s.s. $(\pi_1 \times \pi_2)$. In this way R becomes a graded associative commutative ring with identity. In a natural way one defines automorphisms $\pi \mapsto \tilde{\pi}$ and $\pi \mapsto \bar{\pi}$ on R.

A character χ of $F^\times = GL(1,F)$ is identified with a character of $GL(n,F)$ using the determinant homomorphism. We consider the map $\chi : \pi \mapsto \chi\pi$, $\pi \in \mathcal{R}_n^+$, and extend it

$\mathbb{Z}$-linearly to $\mathcal{R}$. In this way, χ is an automorphism of $\mathcal{R}$. One defines $\chi : R \to R$ in a natural way .

We shall denote by ${}^t g$ the transposed matrix of $g \in GL(n, F)$. The matrix transposed with respect to the second diagonal is denoted by ${}^\tau g$. The representations ${}^t\pi^{-1} : g \mapsto \pi({}^t g^{-1})$ and ${}^\tau\pi^{-1} : g \mapsto \pi({}^\tau g^{-1})$ are equivalent for $\pi \in \mathcal{R}_n^+$, i.e. ${}^\tau\pi^{-1} = {}^t\pi^{-1}$ in $\mathcal{R}^+$. We extend $\pi \mapsto {}^t\pi^{-1}$ $\mathbb{Z}$-linearly to $\mathcal{R}$. One has directly

$$ {}^t\left(\pi_1 \times \pi_2\right)^{-1} = {}^t\pi_1^{-1} \underline{\times} {}^t\pi_2^{-1} = {}^t\pi_2^{-1} \times {}^t\pi_1^{-1}. $$

Thus, $\pi \mapsto {}^t\pi^{-1}$ is an involutive antiautomorphism of $\mathcal{R}$. Observe that

$$ {}^t\pi^{-1} = \tilde{\pi} $$

for an irreducible representation π ([GfKa]). Thus

$$ \text{s.s.}\,(\tilde{\pi}) = \text{s.s.}\,\left({}^t\pi^{-1}\right) $$

for any $\pi \in \mathcal{R}$.

Let $\alpha = (n_1, \ldots, n_k)$ be an ordered partition of n and let $\pi \in \mathcal{R}_n^+$. We denote by $r_{\alpha,(n)}(\pi)$ the Jacquet module introduced in 1.1 of [Z1]. It is a representation of $GL(n_1, F) \times \ldots \times GL(n_k, F)$. Working with standard parabolics with respect to the lower triangular matrices, instead of the upper triangular ones, we introduce $\underline{r}_{\alpha,(n)}(\pi)$ in an analogous way. Now

$$ r_{\alpha,(n)}\left(\tilde{\pi}\right) = \left[\underline{r}_{\alpha,(n)}(\pi)\right]^{\sim}. $$

We consider s.s. $\left(r_{(k,n-k),(n)}(\pi)\right) \in R_k \otimes R_{n-k}$.

For $\pi \in GL(n, F)^{\sim}$ set

$$ m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}\,\left(r_{(k,n-k)(n)}(\pi)\right) \in R \otimes R, $$

$$ \underline{m}^*(\pi) = \sum_{k=0}^{n} \text{s.s.}\,\left(\underline{r}_{(k,n-k),(n)}(\pi)\right) \in R \otimes R. $$

With comultiplication m^*, $(R, +, \times)$ is a Hopf algebra (the similar statement holds for $\underline{m}^*$ and $(R, +, \underline{\times})$.We shall denote

$$ \begin{aligned} Irr &= \underset{n\geq 0}{\cup} GL(n, F)^{\sim}, \\ Irr^u &= \underset{n\geq 0}{\cup} GL(n, F)^{\wedge}, \\ In &= \underset{n\geq 0}{\cup} In(GL(n, F)), \\ D &= \underset{n\geq 1}{\cup} D(GL(n, F)), \\ C &= \underset{n\geq 1}{\cup} C(GL(n, F)), \end{aligned} $$

The modulus of F will be denoted by $|\ |_F$. We denote by ν_n, or simply by ν, the positive valued character $g \mapsto |det\, g|_F$ of $GL(n,F)$.

For each $\delta \in D$ there exists a unique $\alpha \in \mathbb{R}$ such that $\nu^{-\alpha}\delta$ is unitarizable. This α will be denoted by $e(\delta)$.

If X is a set,then we shall denote by $M(X)$ the set of all finite multisets in X. By definition, $M(X)$ is the set of all possible n-tuples of elements of X, with all possible $n \in \mathbb{Z}_+$. The set $M(X)$ is an additive semigroup for the operation

$$(x_1, \dots, x_n) + (y_1, \dots, y_m) = (x_1, \dots, x_n, y_1, \dots, y_m).$$

We shall now describe the gradation obtained on R and $\mathcal{R}$ from $\Theta(GL(n,F))$ (see the first section).

For $\pi \in Irr$ we shall say that $\omega = (\rho_1, \dots, \rho_n) \in M(C)$ is the support of π if

$$\pi \leq \rho_1 \times \dots \times \rho_n$$

in R. The support of π is denoted by $\operatorname{supp}\pi$. Let $Irr_\omega = \{\pi \in Irr;\ \operatorname{supp}\pi = \omega\}$. We denote by R_ω the subgroup of R generated by Irr_ω. Now

$$R = \underset{M(C)}{\oplus} R_\omega.$$

This is a gradation of the Hopf algebra R.

Put $In_\omega = \{\pi \in In;\ \text{s.s.}(\pi) \in R_\omega\}$. The subgroup of $\mathcal{R}$ generated by In_ω is denoted by $\mathcal{R}_\omega$. We have

$$\mathcal{R} = \underset{M(C)}{\oplus} \mathcal{R}_\omega.$$

This is a gradation of the ring $\mathcal{R}$.

We shall introduce a new gradation on R which may be be useful in the study of representations of p-adic symplectic groups. We shall write $\rho_1 \sim \rho_2$ if $\rho_1 \cong \rho_2$ or $\rho_1 \cong \bar{\rho}_2$ for $\rho_1, \rho_2 \in Irr$. The set of all equivalence classes in C for this relation, is denoted by $C_\sim$. The canonical projection is denoted by

$$\kappa : C \to C_\sim.$$

Now we define

$$\kappa : M(C) \to M(C_\sim)$$
$$(\rho_1, \dots, \rho_n) \mapsto (\kappa(\rho_1), \dots \kappa(\rho_n)).$$

Note that $\kappa(\omega_1) + \kappa(\omega_2) = \kappa(\omega_1 + \omega_2)$. For $\Omega \in M(C_\sim)$ set

$$R_\Omega = \sum R_\omega \ (\omega \in M(C), \kappa(\omega) = \Omega).$$

Again

$$R = \underset{M(C_\sim)}{\oplus} R_\Omega,$$

and this is another gradation of the Hopf algebra R.

3. Symplectic groups

In the rest of this paper we shall assume that char $F \neq 2$.

The vector space of all $n \times m$ matrices over F is denoted by $M_{(n,m)}(F)$. We denote $M_{(n,n)}(F)$ by $M_n(F)$.

Let J_n denotes the matrix

$$\begin{bmatrix} 00 & \dots & 01 \\ 00 & \dots & 10 \\ \vdots & & \\ 10 & \dots & 0 \end{bmatrix}$$

in $M_n(F)$. The identity $n \times n$ matrix is denoted by I_n.

For $S \in M_{2n}(F)$, according to [F], set

$$^{\times}S = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^{t}S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$$

Clearly $^{\times}(S_1S_2) = {}^{\times}S_2\, {}^{\times}S_1$. If

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A, B, C, D \in M_n(F),$$

then

$$^{\times}S = \begin{bmatrix} {}^{\tau}D & -{}^{\tau}B \\ -{}^{\tau}C & {}^{\tau}A \end{bmatrix}.$$

By definition,

$$Sp(n, F) = \{S \in M_{2n}(F);\ {}^{\times}S\, S = I_{2n}\}.$$

We may say also that $Sp(n, F)$ is the set of all matrices $S \in M_{2n}(F)$ which satisfy

$$^{t}S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} S = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$$

The third way to describe $Sp(n, F)$ is to say that $Sp(n, F)$ is the set of all matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $A, B, C, D \in M_n(F)$ which satisfy ${}^{\tau}DA - {}^{\tau}BC = I_n$, ${}^{\tau}DB = {}^{\tau}BD$ and ${}^{\tau}AC = {}^{\tau}CA$. Then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} {}^{\tau}D & -{}^{\tau}B \\ -{}^{\tau}C & {}^{\tau}A \end{bmatrix}.$$

Now

$$GSp(n, F) = \{S \in M_{2n}(F);\ {}^{\times}SS \in (F^{\times})I_{2n}\}.$$

We may describe $GSp(n, F)$ also as the set of all $S \in M_{2n}(F)$ which satisfy

$$^{t}S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} S \in (F^{\times}) \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}.$$

For $S \in GSp(n,F)$ there exists a unique $\psi(S) \in F^\times$ such that ${}^\times SS = \psi(S) I_{2n}$. It is easy to see that

$$Sp(n,F) \rtimes \left\{ \begin{bmatrix} I_n & 0 \\ 0 & \lambda I_n \end{bmatrix}; \lambda \in F^\times \right\} = GSp(n,F).$$

Note that $\psi\left(g \cdot \begin{bmatrix} I_n & 0 \\ 0 & \lambda I_n \end{bmatrix}\right) = \lambda$ for $g \in Sp(n,F)$. Clearly, ψ is multiplicative. Also, $Sp(n,F)$ is the derived subgroup of $GSp(n,F)$.

Take $Sp(0,F)$ to be the trivial group and take $GSp(0,F)$ to be $F^\times$. We consider $Sp(0,F)$ and $GSp(0,F)$ as 0×0 matrices formally.

The diagonal subgroup in $Sp(n,F)$ (resp. $GSp(n,F)$) will be taken for a maximal split torus. These maximal tori are denoted by A_0. We fix the Borel subgroup in $Sp(n,F)$ (resp. $GSp(n,F)$) which consists of all upper triangular matrices in $Sp(n,F)$ (resp. $GSp(n,F)$). These Borel subgroups are denoted by P_ϕ.

We parametrize A_0 in $Sp(n,F)$ in the following way

$$a : (F^\times)^n \;\to\; A_0\,,$$

$$(x_1,\dots,x_n) \;\mapsto\; \begin{bmatrix} \operatorname{diag}(x_1,\dots,x_n) & 0 \\ 0 & \operatorname{diag}(x_n^{-1},\dots,x_1^{-1}) \end{bmatrix}.$$

In $GSp(n,F)$ we do it as follows:

$$a : (F^\times)^n \times F^\times \to A_0$$

$$(x_1,\dots,x_n,x) \mapsto \begin{bmatrix} \operatorname{diag}(x_1,\dots,x_n) & 0 \\ 0 & x\operatorname{diag}(x_n^{-1},\dots,x_1^{-1}) \end{bmatrix}.$$

The Weyl groups defined by the above maximal tori in $Sp(n,F)$ and $GSp(n,F)$ are naturally isomorphic. These groups are denoted by W. The simple roots determined by the Borel subgroup in $Sp(n,F)$ are

$$\alpha_i(a(x_1,\dots,x_n)) \;=\; x_i x_{i+1}^{-1}, \quad 1 \le i \le n-1,$$

and

$$\alpha_n(a(x_1,\dots,x_n)) \;=\; x_n^2.$$

In $GSp(n,F)$ the simple roots are

$$\alpha_i(a(x_1,\dots,x_n,x)) \;=\; x_i x_{i+1}^{-1}\,, \quad 1 \le i \le n-1,$$

and

$$\alpha_n(a(x_1,\dots,x_n,x)) \;=\; x_n^2 x^{-1}.$$

The Weyl group W has $2^n n!$ elements. The action of W by conjugation on $A_0 \subseteq Sp(n,F)$ is generated by transformations

$$a(x_1,\dots,x_i,x_{i+1},\dots,x_n) \mapsto a(x_1,\dots,x_{i+1},x_i,\dots,x_n)\,,\ 1 \le i \le n-1$$

and

$$a(x_1, \dots, x_{n-1}, x_n) \mapsto a(x_1, \dots, x_{n-1}, x_n^{-1}).$$

In the case of $A_0 \subseteq GSp(n, F)$ generating transformations are

$$a(x_1, \dots, x_i, x_{i+1}, \dots, x_n, x) \mapsto a(x_1, \dots, x_{i+1}, x_i, \dots, x_n, x), \quad 1 \leq i \leq n-1,$$

and

$$a(x_1, \dots, x_{n-1}, x_n, x) \mapsto a(x_1, \dots, x_{n-1}, x x_n^{-1}, x).$$

The standard parabolic subgroups of $Sp(n, F)$, and also of $GSp(n, F)$, are parametrized by subsets of $\{\alpha_1, \dots, \alpha_n\}$. We shall use the following parametrization. First we consider the case of $Sp(n, F)$. Fix $n \in \mathbb{Z}_+$. Take an ordered partition $\alpha = (n_1, \dots, n_k)$ of m, where $m \leq n$.If $m = 0$, then the only partition is denoted by (0). Set

$$M_\alpha = \left\{ \begin{bmatrix} g_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & g_k & & & & \\ & & & h & & & \\ & & & & {}^\tau g_k^{-1} & & \\ & & & & & \ddots & \\ 0 & & & & & & {}^\tau g_1^{-1} \end{bmatrix} ; \ g_i \in GL(n_i, F), h \in Sp(n-m, F) \right\}$$

Further, $P_\alpha = M_\alpha P_\phi$ is a parabolic subgroup of $Sp(n, F)$. These parabolic subgroups correspond to the subset

$$\{\alpha_1, \dots, \alpha_n\} \backslash \{\alpha_{n_1}, \alpha_{n_1+n_2}, \dots, \alpha_{n_1+\ldots+n_k}\}.$$

The unipotent radical of P_α is denoted by N_α.

One obtains standard parobolic subgroups (resp. Levi factors) in $GSp(n, F)$ from the standard parabolic subgroups P_α (resp. Levi factors M_α) in $Sp(n, F)$ by multiplying them with the subgroup

$$\left\{ \begin{bmatrix} I_n & 0 \\ 0 & \lambda I_n \end{bmatrix} ; \lambda \in F^\times \right\}.$$

These subgroups in $GSp(n, F)$ are denoted again by P_α and M_α. Then

$$M_\alpha = \left\{ \begin{bmatrix} g_1 & & & & & & 0 \\ & \ddots & g_k & & & & \\ & & & h & & & \\ & & & & \psi(h)\, {}^\tau g_k^{-1} & & \\ & & & & & \ddots & \\ 0 & & & & & & \psi(h)\, {}^\tau g_1^{-1} \end{bmatrix} \right\}$$

where $g_i \in GL(n_i, F), h \in GSp(n-m, F)$.

Suppose that we have two standard parabolic subgroups P'_α and P''_α. Then they are associate if and only if α' and α'' are partitions of the same number, and if they are equal as unordered partitions.

The characters of $F^\times$will be identified with characters of $GSp(n,F)$ in the following way

$$\chi \mapsto (g \mapsto \chi(\psi(g))).$$

4. Modules of representations

Let

$$\begin{aligned}
\mathcal{R}_n^+[S] &= \mathcal{R}^+(Sp(n,F)),\\
\mathcal{R}_n[S] &= \mathcal{R}(Sp(n,F)),\\
R_n^+[S] &= R^+(Sp(n,F)),\\
R_n[S] &= R(Sp(n,F)),\\
\mathcal{R}[S] &= \underset{n\geq 0}{\oplus} \mathcal{R}_n[S],\\
R[S] &= \underset{n\geq 0}{\oplus} R_n[S],\\
\mathcal{R}^+[S] &= \sum_{n\geq 0} \mathcal{R}_n^+[S],\\
R^+[S] &= \sum_{n\geq 0} R_n^+[S],\\
Irr[S] &= \underset{n\geq 0}{\cup} Sp(n,F)^\sim,\\
In[S] &= \underset{n\geq 0}{\cup} In(Sp(n,F)),\\
C[S] &= \underset{n\geq 0}{\cup} C(Sp(n,F)),\\
T[S] &= \underset{n\geq 0}{\cup} T(Sp(n,F)).
\end{aligned}$$

One introduces analogous notation $\mathcal{R}_n^+[G]$, $\mathcal{R}_n[G], \dots$ for the groups $GSp(n,F)$.

Take $\pi \in \mathcal{R}_n^+$ and $\sigma \in \mathcal{R}_m^+[S]$. We take the maximal parabolic subgroup $P_{(n)}$ in $Sp(n+m,F)$. Using the identification

$$(g,h) \longleftrightarrow \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & {}^\tau g^{-1} \end{bmatrix} ; g \in GL(n,F), h \in Sp(m,F),$$

we identify $GL(n,F) \times Sp(m,F)$ with $M_{(n)}$. In this way we consider $\pi \otimes \sigma$ as a representation of $M_{(n)}$. Let

$$\pi \rtimes \sigma = \mathrm{Ind}_{P_{(n)}}^{Sp(n+m,F)}(\pi \otimes \sigma),$$

$$\pi \underline{\rtimes} \sigma \ = \ \mathrm{Ind}_{{}^tP_{(n)}}^{Sp(n+m,F)}(\pi \otimes \sigma).$$

We extend $\rtimes$ and $\underline{\rtimes}$ $\mathbb{Z}$-bilinearly to

$$\mathcal{R} \times \mathcal{R}[S] \to \mathcal{R}[S].$$

Now we have

Proposition 4.1.

(i) *With the action* $\rtimes : \mathcal{R} \times \mathcal{R}[S] \to \mathcal{R}[S]$, $\mathcal{R}[S]$ *is a* $\mathbb{Z}_+$*-graded module over* $(\mathcal{R}, +, \times)$

(ii) *With the action* $\underline{\rtimes} : \mathcal{R} \times \mathcal{R}[S] \to \mathcal{R}[S]$, $\mathcal{R}[S]$ *is a* $\mathbb{Z}_+$*-graded module over* $(\mathcal{R}, +, \underline{\times})$.

(iii) *For* $\pi \in \mathcal{R}_n^+$ *and* $\sigma \in \mathcal{R}_m^+[S]$ *we have*

$$\pi \underline{\rtimes} \sigma \ = \ {}^\tau\pi^{-1} \rtimes \sigma,$$

$$(\pi \rtimes \sigma)^\sim \ = \ \tilde{\pi} \rtimes \tilde{\sigma},$$

$$(\pi \rtimes \sigma)^- \ = \ \bar{\pi} \rtimes \bar{\sigma}.$$

Proof. Only the associativity $\pi_1 \rtimes (\pi_2 \rtimes \sigma) = (\pi_1 \times \pi_2) \rtimes \sigma$ is not evident in (i). This follows from the transitivity of the induction in stages ([BnZ2]). The same situation is with (ii). One obtains (iii) using conjugation with

$$\begin{bmatrix} 0 & J_{n+m} \\ -J_{n+m} & 0 \end{bmatrix}. \quad \square$$

Now we define $\rtimes : R \times R[S] \to R[S]$ by

$$\text{s.s.}(\pi) \rtimes \text{s.s.}(\sigma) = \text{s.s.}(\pi \rtimes \sigma),$$

$\pi \in \mathcal{R}$, $\sigma \in \mathcal{R}[S]$ (in the above formula we use the definition of $\pi \rtimes \sigma \in \mathcal{R}[S]$ which we introduced before). One has

Proposition 4.2. *The additive group* $R[S]$ *is a* $\mathbb{Z}_+$*-graded module over* R*. We have*

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma$$

for $\pi \in R$ *and* $\sigma \in R[S]$.

Proof. The relation $\pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma$ follows from part (iii) of the previous proposition since $P_{(n)}$ and ${}^tP_{(n)}$ are associate parabolics ([BnDKa]). $\square$

We shall describe now the gradations of $R[S]$ and $\mathcal{R}[S]$ by infinitesimal characters.

The disjoint union of the sets of all cuspidal pairs modulo conjugation, which correspond to $Sp(n,F)$, $n \geq 0$, may be identified with the set

$$M(C_\sim) \times C[S].$$

Let $\omega = (x, \sigma) \in M(C_\sim) \times C[S]$. Take $(\rho_1, \dots, \rho_n) \in M(C)$ such that $\kappa((\rho_1, \dots, \rho_n)) = x$. Let

$$Irr_\omega[S] = \{\tau \in Irr[S]; \tau \leq \rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma\}.$$

The subset of all $\pi \in In[S]$ all of whose irreducible subquotients are in $Irr_\omega[S]$ is denoted by $In_\omega[S]$. Let $R_\omega[S]$ be the subgroup of $R[S]$ generated by $Irr_\omega[S]$ and let $\mathcal{R}_\omega[S]$ be the subgroup of $\mathcal{R}[S]$ generated by $In_\omega[S]$. For $(x_1, \dots, x_n) \in M(C_\sim)$ and $((y_1, \dots y_m), \sigma) \in M(C_\sim) \times C[S]$ set

$$(x_1, \dots, x_n) + ((y_1, \dots, y_m), \sigma) = ((x_1, \dots, x_n, y_1, \dots, y_m), \sigma).$$

Now

$$\mathcal{R}[S] = \underset{M(C_\sim) \times C[S]}{\oplus} \mathcal{R}_\omega[S]$$

is a graded module over $\mathcal{R} = \underset{M(C_\sim)}{\oplus} \mathcal{R}_\omega$ for both structures. We have an analogous situation for $R[S]$.

We consider now the case of $GSp(n, F)$.

Let $\pi \in \mathcal{R}_n^+$ and $\sigma \in \mathcal{R}_m^+[G]$. Using the identification

$$(g, h) \longleftrightarrow \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \psi(h)\, {}^\tau g^{-1} \end{bmatrix}, g \in GL(n, F), h \in GSp(m, F),$$

we identify $GL(n, F) \times GSp(m, F)$ with $M_{(n)}$. Let

$$\pi \rtimes \sigma \; = \; \mathrm{Ind}_{P_{(n)}}^{GSp(n+m,F)}(\pi \otimes \sigma),$$

$$\pi \underline{\rtimes} \sigma \; = \; \mathrm{Ind}_{{}^tP_{(n)}}^{GSp(n+m,F)}(\pi \otimes \sigma).$$

We extend again $\rtimes$ and $\underline{\rtimes}$ $\mathbb{Z}$-bilinearly to

$$\mathcal{R} \times \mathcal{R}[G] \to \mathcal{R}[G].$$

We factor also $\rtimes$ to a $\mathbb{Z}$-bilinear map

$$R \times R[G] \to R[G].$$

Now we have an analogue of Propositions 4.1 and 4.2.

Proposition 4.3.

(i) *For the mapping $\rtimes$ (resp. $\underline{\rtimes}$): $\mathcal{R} \times \mathcal{R}[G] \to \mathcal{R}[G]$, the additive group $\mathcal{R}[G]$ is a $\mathbb{Z}_+$-graded module over $(\mathcal{R}, +, \times)$ (resp. $(\mathcal{R}, +, \underline{\rtimes})$).*

(ii) *Let $\pi \in \mathcal{R}_n^+$ and $\sigma \in \mathcal{R}_m^+[G]$. For a character χ of $F^\times$ we have*

$$\chi(\pi \rtimes \sigma) = \pi \rtimes \chi\sigma.$$

Suppose that π has a central character, say ω_π. Then

$$\pi \underline{\rtimes} \sigma = {}^t\pi^{-1} \rtimes \omega_\pi \sigma,$$

(iii) *The additive group $R[G]$ is a $\mathbb{Z}_+$-graded module over R. We have*

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes \omega_\pi \sigma$$

for $\pi \in Irr$ and $\sigma \in R[G]$.

(iv) *If $\pi \in \mathcal{R}_n^+$ and $\sigma \in \mathcal{R}_m^+[G]$, then for the restrictions to symplectic groups we have the following equality in $\mathcal{R}[S]$:*

$$(\pi \rtimes \sigma)|Sp(n+m, F) = \pi \rtimes (\sigma|Sp(m, F)).$$

(v) *We identify $F^\times$ with the center of $GL(n, F)$ using the homomorphism $\lambda \mapsto \lambda I_n$. Also, using the homomorphism $\lambda \mapsto \lambda I_{2n}$, we identify $F^\times$ with the center of $GSp(n, F)$. Let $\pi_i \in \mathcal{R}_{n_i}^+$, $n = 1, \ldots, k$, be representations which have central characters $\omega_{\pi_i}, i = 1, \ldots, k$. Let $\sigma \in \mathcal{R}_m^+[G]$ be a representation having a central character ω_σ. If $m > 0$, then the central character of $\pi_1 \times \pi_2 \times \cdots \times \pi_k \rtimes \sigma$ is*

$$\omega_{\pi_1} \omega_{\pi_2} \ldots \omega_{\pi_k} \omega_\sigma.$$

If $m = 0$, then the central character is

$$\omega_{\pi_1} \ldots \omega_{\pi_k} \omega_\sigma^2.$$

(vi) *Let $\sigma \in \mathcal{R}_m^+[G]$ be a representation with a central character, say ω_σ. Let χ be a character of $F^\times$. If $m \geq 1$, then the central character of $\chi\sigma$ is $\chi^2\omega_\sigma$.*

(vii) *We have for $\pi \in \mathcal{R}^+$, $\sigma \in \mathcal{R}^+[G]$*

$$(\pi \rtimes \sigma)^\sim = \tilde{\pi} \rtimes \tilde{\sigma},$$

$$(\pi \rtimes \sigma)^- = \bar{\pi} \rtimes \bar{\sigma}.$$

5. Comodules of representations

Take an ordered partition $\alpha = (n_1, \ldots, n_k)$ of m and take $n \geq m$. Identifying

$$(g_1, g_2, \ldots, g_k, h), g_i \in GL(n_i, F), h \in Sp(n-m, F),$$

with the matrix

$$\begin{bmatrix} g_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & g_k & & & & \\ & & & h & & & \\ & & & & {}^\tau g_k^{-1} & & \\ & & & & & \ddots & \\ 0 & & & & & & {}^\tau g_1^{-1} \end{bmatrix},$$

we identify $GL(n_1,F)\times\ldots\times GL(n_k,F)\times Sp(n-m,F)$ with $M_\alpha\subseteq Sp(n,F)$.

Let $\sigma\in\mathcal{R}_n^+[S]$. The Jacquet module for N_α is denoted by

$$s_{\alpha,(0)}(\sigma)=r_{P_\alpha}^{Sp(n,F)}(\sigma).$$

The $\mathbb{Z}$-linear extension to $\mathcal{R}_n[S]$ is denoted by $s_{\alpha,(0)}$ again.

Let $\beta=(n_1',\ldots,n_{k'}')$ be an ordered partition of $m'\le n$. We shall write $\beta\le\alpha$ if $m'\ge m$ and if there exists a subsequence $p(1)<p(2)<\ldots<p(k)$ of $\{1,2,\ldots,k'\}$ such that

$$\begin{aligned} n_1'+\ldots+n_{p(1)}' &= n_1 \\ n_{p(1)+1}'+\ldots+n_{p(2)}' &= n_2 \\ &\vdots \\ n_{p(k-1)+1}'+\ldots+n_{p(k)}' &= n_k \end{aligned}$$

(for $\alpha=(0)$ we assume that $k=0$). The relation $\le$ is transitive.

If $\beta\le\alpha$ and $\sigma\in\mathcal{R}(M_\alpha)$, then we set

$$s_{\beta,\alpha}(\sigma)=r_{M_\beta}^{M_\alpha}(\sigma).$$

Now, $s_{\beta,\alpha}$ are transitive. This means that $\alpha_1\le\alpha_2\le\alpha_3$ implies

$$s_{\alpha_1,\alpha_3}=s_{\alpha_1,\alpha_2}\circ s_{\alpha_2,\alpha_3}.$$

Note that we may identify

$$M_\alpha^\sim\longleftrightarrow GL(n_1,F)^\sim\times\ldots\times GL(n_k,F)^\sim\times Sp(n-m,F)^\sim,$$

where $\alpha=(n_1,\ldots,n_k)$ is a partition of $m\le n$. Thus, we have a natural identification

$$R(M_\alpha)\longleftrightarrow R_{n_1}\otimes\cdots\otimes R_{n_k}\otimes R_{n-m}[S].$$

We can lift also $s_{\beta,\alpha}$ to

$$\text{s.s.}(s_{\beta,\alpha}):R(M_\alpha)\to R(M_\beta).$$

These mappings are transitive again.

We define now a $\mathbb{Z}$-linear mapping

$$\mu^* : R[S] \to R \otimes R[S].$$

For $\sigma \in R_n[S]$, the formula is

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(s_{(k),(0)}(\sigma)) \in \sum_{k=0}^{n} R_k \otimes R_{n-k}[S].$$

Note that μ^* is additive and it is $\mathbb{Z}_+$-graded. From the transitivity of $s_{\alpha,\beta}$'s one obtains that μ^* is coassociative. This means that the following diagram commutes

$$\begin{array}{ccc} R[S] & \xrightarrow{\mu^*} & R \otimes R[S] \\ \mu^* \downarrow & & \downarrow id \otimes \mu^* \\ R \otimes R[S] & \xrightarrow{m^* \otimes id} & R \otimes R \otimes R[S]. \end{array}$$

The mapping μ^* is graded with respect to $M(C_\sim)$-gradation on R and $M(C_\sim) \times C[S]$ gradation on $R[S]$. In other words

$$\mu^*(R_\Omega[S]) \subseteq \sum_{\omega + \Omega' = \Omega} R_\omega \otimes R_{\Omega'}[S]$$

where $\Omega, \Omega' \in M(C_\sim) \times C[S], \omega \in M(C_\sim)$.

We make now necessary modifications for the case of $GSp(n, F)$. We identify first M_α with $GL(n_1, F) \times \ldots \times GL(n_k, F) \times GSp(n - m, F)$, when $\alpha = (n_1, \ldots, n_k)$ is a partition of $m \leq n$. The identification mapping is

$$(g_1, \ldots, g_k, h) \longleftrightarrow \begin{bmatrix} g_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & g_k & & & & \\ & & & h & & & \\ & & & & \psi(h)\,{}^\tau g_k^{-1} & & \\ & & & & & \ddots & \\ 0 & & & & & & \psi(h)\,{}^\tau g_1^{-1} \end{bmatrix}$$

In the same way as above, we introduce $s_{\alpha,(0)}$ and $s_{\beta,\alpha}$ for $GSp(n, F)$. Again, $s_{\beta,\alpha}$'s are transitive. The map

$$\mu^* : R[G] \to R \otimes R[G]$$

is defined sothat acts on $\sigma \in R_n[G]$

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(s_{(k),(0)})(\sigma).$$

This map is additive, $\mathbb{Z}_+$-graded and coassociative.

6. Langlands classification

Let $t = ((\delta_1, \dots, \delta_n), \tau) \in M(D) \times T[S]$. We shall write t simply as $(\delta_1, \dots, \delta_n, \tau)$.

Denote

$$D_+ = \{\delta \in D; e(\delta) > 0\}.$$

If $t = (\delta_1, \dots, \delta_n, \tau) \in M(D_+) \times T[S]$, then we say that t is written in a standard order if

$$e(\delta_1) \geq e(\delta_2) \geq \dots \geq e(\delta_n)$$

Let $t = (\delta_1, \dots, \delta_n, \tau) \in M(D_+) \times T[S]$. Suppose that it is written in a standard order. The representation

$$\delta_1 \times \delta_2 \times \dots \times \delta_n \rtimes \tau \in \mathcal{R}^+[S]$$

is uniquely determined by t. This is a consequence of irreducibility of tempered induction for $GL(n)$-groups ([Jc] or [Z1]). This representation will be denoted by $\lambda(t)$. Similarly, the representation

$$\delta_1 \underline{\times} \delta_2 \dots \underline{\times} \delta_n \underline{\rtimes} \tau$$

is uniquely determined by t. It will denoted by $\underline{\lambda}(t)$. Observe that the fourth section implies

$$\text{s.s.}(\lambda(t)) = \text{s.s.}(\underline{\lambda}(t)).$$

We also have

$$\underline{\lambda}(t) = \tilde{\delta}_1 \times \tilde{\delta}_2 \times \dots \times \tilde{\delta}_n \rtimes \tau.$$

We shall now describe the Langlands classification in the case of $Sp(n)$-groups.

Let $t \in M(D_+) \times T[S]$. The representation $\lambda(t)$ has a unique irreducible quotient which we denote by $L(t)$. The mapping

$$t \mapsto L(t)$$

is a one-to one mapping of $M(D_+) \times T[S]$ onto $Irr[S]$.

One can describe $L(t)$ in a few different ways. We recall now two such descriptions.

The representation $\underline{\lambda}(t)$ has a unique irreducible subrepresentation. This subrepresentation is isomorphic to $L(t)$.

There exists an integral intertwining operator from $\lambda(t)$ into $\underline{\lambda}(t)$ whose image is $L(t)$ (for the explicit formula one may consult [BlWh]).

The multiplicity of $L(t)$ in $\lambda(t)$ is one. Thus, the multiplicity of $L(t)$ in $\underline{\lambda}(t)$ is one. This implies that the intertwining space between $\lambda(t)$ and $\underline{\lambda}(t)$ is one-dimensional. Therefore, if we have an intertwining between $\lambda(t)$ and $\underline{\lambda}(t)$ which is injective or surjective, then $\lambda(t)$ is irreducible.

Let $t = (\delta_1, \dots, \delta_n, \tau) \in M(D_+) \times T[S]$. Suppose that it is written in a standard order and suppose that $\delta_i \in GL(k_i, F)^\sim$. Set

$$e_*(t) = (\underbrace{e(\delta_1), \dots, e(\delta_1)}_{k_1 \text{ times}}, \dots, \underbrace{e(\delta_n), \dots, e(\delta_n)}_{k_n \text{ times}}, \underbrace{0, 0, \dots, 0}_{m \text{ times}}),$$

where $\tau \in Sp(m, F)^{\sim}$.

We consider a partial order on $\mathbb{R}^k$ defined by

$$(x_1, \dots, x_k) \le (y_1, \dots, y_k) \Leftrightarrow$$

$$\begin{aligned} x_1 &\le y_1, \\ x_1 + x_2 &\le y_1 + y_2, \\ &\vdots \\ x_1 + \dots + x_k &\le y_1 + \dots + y_k. \end{aligned}$$

Let $t \in M(D_+) \times T[S]$ and let σ be a subquotient of $\lambda(t)$ different from $L(t)$. Then $\sigma = L(t_1)$ for some $t_1 \in M(D_+) \times T[S]$, $t_1 \neq t$. Moreover then it must hold

$$e_*(t_1) < e_*(t).$$

We could introduce the order on $\mathbb{R}^k$ in a different way. Let $\beta_1, \dots, \beta_n$ be the basis of $\mathbb{R}^k$ biorthogonal to the basis

$$\begin{gathered} (1,\ -1,\ 0,\ 0, \dots\dots, 0), \\ (0,\ 1,\ -1,\ 0, \dots\dots, 0), \\ \vdots \\ (0,\ 0, \dots, 0,\ 1,\ -1,\ 0), \\ (0,\ 0, \dots, 0,\ 0,\ 1,\ -1), \\ (0,\ 0, \dots\dots, 0,\ 0,\ 0,\ 2), \end{gathered}$$

where we consider the usual inner product on $\mathbb{R}^k$. Simple computation gives

$$\beta_i = (\underbrace{1, 1, \dots, 1,}_{i \text{ times}} 0, 0, \dots, 0), \qquad 1 \le i \le n-1,$$

and

$$\beta_n = \frac{1}{2}(1, 1, \dots, 1, 1).$$

It is easy to check that for $x, y \in \mathbb{R}^k$

$$x \le y \Leftrightarrow (x, \beta_i) \le (y, \beta_i), \qquad i = 1, \dots, k.$$

We shall write now the Langlands parameter of the contragredient representation. Let $t = (\delta_1, \dots, \delta_n, \tau) \in M(D_+) \times T[S]$. Suppose that it is written in a standard order. Then we have an injective intertwining operator

$$L(t) \to \delta_1 \underline{\times} \delta_2 \underline{\times} \dots \underline{\times} \delta_n \underline{\rtimes} \tau.$$

Applying Proposition 4.1, one obtains

$$\delta_1 \underline{\times} \delta_2 \underline{\times} \dots \underline{\times} \delta_n \underline{\rtimes} \tau = \tilde{\delta}_1 \times \tilde{\delta}_2 \times \dots \times \tilde{\delta}_n \rtimes \tau$$

Passing to contragredients, one obtains a surjective intertwining operator

$$\delta_1 \times \delta_2 \times \dots \times \delta_n \rtimes \tilde{\tau} \to L(t)^{\sim}.$$

Since $(\delta_1, \dots , \delta_n, \tilde{\tau}) \in M(D_+) \times T[S]$ and since it is written in a standard order, we obtain

$$L((\delta_1, \dots , \delta_n, \tau))^{\sim} = L((\delta_1, \dots , \delta_n, \tilde{\tau})).$$

We set $\bar{t} = (\bar{\delta}_1. \dots , \bar{\delta}_n, \bar{\tau})$ for $t = (\delta_1, \dots , \delta_n, \tau) \in M(D_+) \times T[S]$. Then

$$L(t)^{-} = L(\bar{t}).$$

We shall recall now of a criterion for square integrability of representations, according to [Cs]. Let $\pi \in Sp(n, F)^{\sim}$. Let P_α be any standard parabolic subgroup being minimal with the property that

$$s_{\alpha,(0)}(\pi) \neq 0$$

(all such P_α's are associate parabolic subgroups). Write $\alpha = (n_1, \dots n_k)$, where α is a partition of $m \leq n$. Let σ be any irreducible subquotient of $s_{\alpha,(0)}(\pi)$. Then we write.

$$\sigma = \rho_1 \otimes \cdots \otimes \rho_k \otimes \rho.$$

All representations $\rho_1, \dots , \rho_k$ and ρ must be cuspidal. Let

$$e_*(\sigma) = (\underbrace{e(\rho_1), \dots , e(\rho_1)}_{n_1 \text{ times}}, \dots , \underbrace{e(\rho_k), \dots , e(\rho_k)}_{n_k \text{ times}}, \underbrace{0, \dots , 0}_{(n-m) \text{ times}}).$$

We are able to present now the criteria of [Cs]:

(i) Suppose that the following conditions hold

$$(e_*(\sigma), \beta_{n_1}) > 0,$$

$$(e_*(\sigma), \beta_{n_1+n_2}) > 0,$$

$$\vdots$$

$$(e_*(\sigma), \beta_m) > 0,$$

for any α and σ as above. Then π is a square integrable representation.

(ii) If π is a square integrable representation, then all inequalities of (i) hold for any α and σ as above.

Note that the conditions in (i) are equivalent to the following conditions

$$\begin{aligned}(e_*(\sigma), \beta_1) &> 0,\\ (e_*(\sigma), \beta_2) &> 0,\\ &\vdots\\ (e_*(\sigma), \beta_n) &> 0,\end{aligned}$$

if $\alpha \neq (0)$.

Now we describe the case of $GSp(n, F)$-groups.

We write $t = ((\delta_1, \dots, \delta_n), \tau) \in M(D_+) \times T[G]$ again simply as $t = (\delta_1, \dots, \delta_n, \tau)$. We say that $t = (\delta_1, \dots, \delta_n, \tau) \in M(D_+) \times T[G]$ is written in a standard order if

$$e(\delta_1) \geq e(\delta_2) \geq \dots \geq e(\delta_n).$$

If $t = (\delta_1, \dots, \delta_n, \tau)$ is written in a standard order, then the representation

$$\lambda(t) = \delta_1 \times \cdots \times \delta_n \rtimes \tau \in \mathcal{R}^+[G]$$

is uniquely determined by t. The representation $\delta_1 \underline{\times} \dots \underline{\times} \delta_n \underline{\rtimes} \tau$ is denoted by $\underline{\lambda}(t)$. The representation $\lambda(t)$ has a unique irreducible quotient. Denote it by $L(t)$. The mapping

$$t \mapsto L(t)$$

is a one-to-one mapping of $M(D_+) \times T[G]$ onto $Irr[G]$. The representation $L(t)$ may be characterized as the unique irreducible subrepresentation of $\underline{\lambda}(t)$. The multiplicity of $L(t)$ in $\lambda(t)$ is one. There exists an integral intertwining operator from $\lambda(t)$ into $\underline{\lambda}(t)$ whose image is $L(t)$. The intertwining space between $\lambda(t)$ and $\underline{\lambda}(t)$ is one-dimensional. We have

$$L((\delta_1, \dots, \delta_n, \tau))^- = L((\bar{\delta}_1, \dots, \bar{\delta}_n, \bar{\tau})).$$

We compute the Langlands parameters of the contragredient representations in the same way as before. One gets

$$L((\delta_1, \dots, \delta_n, \tau))^\sim = L((\delta_1, \dots, \delta_n, \omega_{\delta_1}^{-1} \dots \omega_{\delta_n}^{-1} \tilde{\tau})).$$

If χ is a character of $F^\times$, then we have

$$\chi\lambda((\delta_1, \dots, \delta_n, \tau)) = \lambda((\delta_1, \dots, \delta_n, \chi\tau))$$

and

$$\chi L((\delta_1, \dots, \delta_n, \tau)) = L((\delta_1, \dots, \delta_n, \chi\tau)).$$

Remark 6.1. *Note that* $\chi L((\delta_1, \dots, \delta_n, \tau)) = L((\delta_1, \dots, \delta_n, \tau))$ *if and only if* $\chi\tau = \tau$.

We consider now the following inner product on $\mathbb{R}^{n+1}$

$$((x_i),(y_i)) = \sum_{i=1}^{n} x_i y_i + (2x_{n+1} + \sum_{i=1}^{n} x_i)(2y_{n+1} + \sum_{i=1}^{n} y_i)$$

Let $\beta_1', \dots, \beta_n'$ be the basis dual to the basis

$$\begin{gathered}(1,-1,0,\dots\dots\dots,0),\\ (0,1,-1,0,\dots\dots,0),\\ \vdots\\ (0,0,\dots,0,1,-1,0),\\ (0,0,\dots,0,0,2,-1),\end{gathered}$$

of the subspace

$$\{(x_i) \in \mathbb{R}^{n+1}; 2x_{n+1} + \sum_{i=1}^{n} x_i = 0\}.$$

Then

$$\begin{aligned}\beta_1' &= \left(1,0,\dots\dots\dots,0,-\frac{1}{2}\right),\\ \beta_2' &= (1,1,0,\dots\dots\dots,-1),\\ &\vdots\\ \beta_{n-1}' &= \left(1,1,\dots,1,0,-\frac{n-1}{2}\right),\\ \beta_n' &= \left(1,1,\dots\dots,1,1,-\frac{n}{2}\right).\end{aligned}$$

For $\tau \in T[G]$, there exists a unique $\gamma \in \mathbb{R}$ such that the representation $|\ \ |_F^{-\gamma}\ \tau$ is unitarizable. Set

$$e_0(\tau) = \gamma.$$

Let $\alpha = (n_1, \dots, n_k)$ be a partition of $m \le n$. Take $\delta_i \in D_+ \cap GL(n_i, F)^\sim$ and $\tau \in T(GSp(n-m, F))$. Suppose that $t = (\delta_1, \dots, \delta_k, \tau)$ is in a standard order. Set

$$e_*'(t) = (\underbrace{e(\delta_1), \dots, e(\delta_1)}_{n_1 \text{ times}}, \dots, \underbrace{e(\delta_k), \dots, e(\delta_k)}_{n_k \text{ times}}, \underbrace{0, \dots, 0}_{(n-m) \text{ times}}, e_0(\tau))$$

$$e_*(t) = (\underbrace{e(\delta_1), \dots, e(\delta_1)}_{n_1 \text{ times}}, \dots, \underbrace{e(\delta_k), \dots, e(\delta_k)}_{n_k \text{ times}}, \underbrace{0, \dots, 0}_{(n-m) \text{ times}})$$

Suppose that for $t, t_1 \in M(D_+) \times T[G]$, $L(t_1)$ is a subquotient of $\lambda(t)$ and suppose that $t \neq t_1$. Then

$$\begin{aligned} (e'_*(t_1), \beta'_1) &\leq (e'_*(t), \beta'_1), \\ &\vdots \\ (e'_*(t_1), \beta'_n) &\leq (e'_*(t), \beta'_n), \end{aligned}$$

and the strict inequality holds for at least one index $1 \leq i \leq n$. If $e_*(t) = (x_i)$, $e_*(t_1) = (y_i)$, then the above condition becomes

$$\begin{aligned} y_1 &\leq x_1, \\ y_1 + y_2 &\leq x_1 + x_2, \\ &\vdots \\ y_1 + \ldots + y_n &\leq x_1 + \ldots + x_n. \end{aligned}$$

The strict inequality holds again for at least one case.

We are going to repeat the criterion for square integrability, in the case of $GSp(n, F)$.

Fix $\pi \in GSp(n, F)$. Take a standard parabolic subgroup P_α such that it is minimal among standard parabolic subgroups which satisfy

$$s_{\alpha,(0)}(\pi) \neq 0.$$

Recall that all such P'_α's are associate. Assume that $\alpha = (n_1, \ldots, n_k)$ is a partition of $m \leq n$. Take any irreducible subquotient σ of $s_{\alpha,(0)}(\pi)$. Write it as $\sigma = \rho_1 \otimes \ldots \otimes \rho_k \otimes \rho$. Set

$$e'_*(\sigma) = (\underbrace{e(\rho_1), \ldots, e(\rho_1)}_{n_1 \text{ times}}, \ldots, \underbrace{e(\rho_k), \ldots, e(\rho_k)}_{n_k \text{ times}}, \underbrace{0, \ldots, 0}_{(n-m) \text{ times}}, e_0(\rho)),$$

$$e_*(\sigma) = (\underbrace{e(\rho_1), \ldots, e(\rho_1)}_{n_1 \text{ times}}, \ldots, \underbrace{e(\rho_k), \ldots, e(\rho_k)}_{n_k \text{ times}}, \underbrace{0, \ldots, 0}_{(n-m) \text{ times}})$$

Suppose that we have for any α and σ as above

$$\begin{aligned} (e'_*(\sigma), \beta'_{n_1}) &> 0, \\ (e'_*(\sigma), \beta'_{n_1+n_2}) &> 0, \\ &\vdots \\ (e'_*(\sigma), \beta'_m) &> 0. \end{aligned}$$

If the central character of π is unitary, then π is a square integrable representation. For an irreducible representation with a unitary central character to be square integrable the above conditions are also necessary .The above inequalities are equivalent to the following inequalities

$$\begin{aligned} (e_*(\sigma), \beta_{n_1}) &> 0, \\ (e_*(\sigma), \beta_{n_1+n_2}) &> 0, \\ &\vdots \\ (e_*(\sigma), \beta_m) &> 0 \end{aligned}$$

(we work in $\mathbb{R}^n$ with the standard inner product).

At the end of this section we relate Langlands classifications for GSp and Sp groups.

Lemma 6.2. *Let $t = (\delta_1, \dots, \delta_n, \tau) \in M(D_+) \times T[G]$. Suppose that $L(t)$ is a representation of $GSp(p, F)$ and suppose that τ is a representation of $GSp(q, F)$. We decompose*

$$\tau | Sp(q, F) = \tau_1 + \dots + \tau_k$$

into a direct sum of irreducible representations of $Sp(q, F)$. Then $\tau_1, \dots, \tau_k$ are tempered representations of $Sp(q, F)$. Denote

$$t_i = (\delta_1, \dots, \delta_n, \tau_i) \in M(D_+) \times T[S]$$

Suppose that $\tau_i \neq \tau_j$ for $i \neq j$. Then we have a direct sum decomposition

$$L(t) | Sp(p, F) = L(t_1) + \dots + L(t_k).$$

In particular, $L(t) | Sp(p, F)$ is a multiplicity one representation.

Proof. We may assume that $t = (\delta_1, \dots, \delta_n, \tau)$ is written in a standard order. Now $L(t)$ is the unique irreducible subrepresentation of $\underline{\lambda}(t)$. We have a direct sum

$$\underline{\lambda}(t) | Sp(p, F) = \underline{\lambda}(t_1) + \dots + \underline{\lambda}(t_k).$$

Let U be an irreducible subrepresentation of $\underline{\lambda}(t) | Sp(p, F)$. Then for each i we consider the composition

$$U \hookrightarrow \underline{\lambda}(t_1) + \dots + \underline{\lambda}(t_k) \rightarrow \underline{\lambda}(t_i).$$

The image of U is contained in $L(t_i)$. Thus $U \subseteq L(t_1) + L(t_2) + \dots + L(t_k)$. In particular, since $L(t) | Sp(p, F)$ is a sum of irreducible representations (see [Si1]), we have

$$L(t) | Sp(p, F) \subseteq L(t_1) + \dots + L(t_k).$$

Note that $L(t_i)$ are not equivalent for different i. Since U is irreducible, we see that $U = L(t_{i_0})$ for a unique $1 \leq i_0 \leq k$. Thus

$$L(t) | Sp(p, F) = \bigoplus_{i \in X} L(t_i)$$

for a unique $X \subseteq \{1, 2, \dots, k\}$. Note that $GSp(p, F)$ acts by conjugation on $L(t_i)$'s and $\{L(t_i), i \in X\}$ is invariant for this action. Now $\bigoplus_{i \notin X} L(t_i)$ is again invariant. Thus $\bigoplus_{i \notin X} L(t_i)$ is a $GSp(p, F)$-subrepresentation. Therefore if $X \neq \{1, \dots, k\}$, then $\bigoplus_{i \notin X} L(t_i)$ has an irreducible $GSp(p, F)$-subrepresentation $V \neq L(t)$. Both representations are irreducible subrepresentations of $\underline{\lambda}(t)$. We obtained a contradiction. Thus $X = \{1, 2, \dots, k\}$ and this completes the proof. $\square$

Lemma 6.3. *Let $t = (\delta_1, \dots, \delta_n, \tau) \in M(D_+) \times T[G]$. Suppose that $L(t)$ is a representation of $GSp(p, F)$ and that τ is a representation of $GSp(q, F)$. We always have a following decomposition*

$$\tau | Sp(q, F) = m \sum_{i=1}^{k} \tau_i$$

in $R[S]$, where each τ_i is irreducible, and, for different i's, they are inequivalent. Let

$$t_i = (\delta_1, \dots, \delta_n, \tau_i).$$

Then $L(t_i)$ are not equivalent for different i's and we have

$$L(t) | Sp(p, F) = m \sum_{i=1}^{k} L(t_i)$$

in $R[S]$.

Proof. Let $\tau | Sp(q, F) = \tau_1' + \dots + \tau_{mk}'$ be a decomposition into a direct sum of irreducible representations. Suppose that U is an irreducible subrepresentation of $\underline{\lambda}(t) | Sp(p, F)$. Let

$$t_i' = (\delta_1, \dots, \delta_n, \tau_i').$$

As in the proof of the preceding lemma, we find that

$$U \subseteq L(t_1') + \dots + L(t_{mk}').$$

Thus $U \cong L(t_i)$ for some $1 \leq i \leq k$.

Denote by $\underline{\pi}$ the action of $GSp(p, F)$ on $\underline{\lambda}(t)$. Note that $GSp(p, F)$ acts transitively on

$$\{\underline{\pi}(g) U; g \in GSp(p, F)\}.$$

Thus $V = \operatorname{span} \{\underline{\pi}(g) U; g \in GSp(p, F)\}$ is $GSp(p, F)$-invariant. It is of finite length. Also, as an $Sp(p, F)$-representation, it is completely reducible. There exists an irreducible $GSp(p, F)$-subrepresentation V_1 of V. Thus $V_1 = L(t)$. Now $U \cong L(t_i)$ is isomorphic to a subquotient of V_1 as a representation of $Sp(p, F)$. Since U was arbitrary (we could take any $L(t_i')$ for U, since $\lambda(t) | Sp(p, F) = \lambda(t_1') + \dots + \lambda(t_{mk}')$), we see that each $L(t_i)$ appears as a subrepresentation of $L(t) | Sp(p, F)$.

We have proved that in $R[S]$ we have

$$L(t) | Sp(p, F) \cong m' \sum_{i=1}^{k} L(t_i).$$

Now we have

$$\begin{aligned} m^2 k &= \operatorname{card} \{\chi \in (F^\times)^\sim; \chi\tau = \tau\} \\ &= \operatorname{card} \{\chi \in (F^\times)^\sim; \chi L(t) = L(t)\} \\ &= (m')^2 k. \end{aligned}$$

([GeKn], Lemma 2.1.). Thus $m = m'$. This finishes the proof of the lemma. □

7. Non-unitary principal series representations

The trivial one-dimensional representation of a group G will be denoted by 1_G. If G is the trivial group, then we shall denote 1_G simply by 1.

Let $\chi_1, \dots, \chi_n \in (F^\times)^\wedge$(resp. $(F^\times)^\sim$). The representations $\chi_1 \times \dots \times \chi_n \rtimes 1$ will be called the unitary principal series representations (resp. the non-unitary principal series representations) of $Sp(n, F)$.

By Theorem 1 of N. Winarsky's paper [Wi], a unitary principal series representation $\chi_1 \times \dots \times \chi_n \rtimes 1$ is reducible if and only if there exists a character χ_i, $1 \leq i \leq n$, whose order is two (if such character exists then it is clear from $SL(2)$-case that $\chi_1 \times \dots \times \chi_n \rtimes 1$ is reducible). One can obtain a more precise information about the reducible unitary principal series representations from paper [Ke] of D.Keys.

First, a unitary principal series representation $\chi_1 \times \dots \times \chi_n \rtimes 1$ is a multiplicity one representation, by Theorem C_n of [Ke]. The computation of R-group in the proof of Theorem C_n of [Ke] gives that the length of $\chi_1 \times \dots \chi_n \rtimes 1$ equals 2 to the number of different characters of order 2 among $\chi_1, \dots, \chi_n$.

We can easily determine the reducibility points for the non-unitary principal series representations of $Sp(n, F)$.

Theorem 7.1. *Let $\chi_1, \dots, \chi_n \in (F^\times)^\sim$. Consider the following three conditions:*

(i) *For any $1 \leq i \leq n$, χ_i is not of order 2.*
(ii) *For any $1 \leq i \leq n$, $\chi_i \neq \nu^{\pm 1}$.*
(iii) *For any $1 \leq i < j \leq n$, $\chi_i \neq \nu^{\pm 1}\chi_j^{\pm 1}$ (all possible combinations of two signs are allowed).*

The non-unitary principal series representation $\chi_1 \times \dots \times \chi_n \rtimes 1$ of $Sp(n, F)$ is irreducible if and only if conditions (i), (ii) and (iii) hold.

Proof. Suppose that condition (i) or (ii) is not satisfied for some χ_i. Then we know from the representation theory of $SL(2, F)$ that $\chi_i \rtimes 1$ is reducible. Write $\chi_i \rtimes 1 = \pi_1 + \pi_2$ in $R[S]$, where $\pi_1, \pi_2 \in R[S]$, $\pi_1 > 0$, $\pi_2 > 0$. By the fourth section we have the following equalities in $R[S]$:

$$\chi_1 \times \dots \times \chi_i \times \dots \times \chi_n \rtimes 1 =$$

$$\chi_1 \times \dots \times \chi_{i-1} \times \chi_{i+1} \times \dots \times \chi_n \times \chi_i \rtimes 1 =$$

$$\chi_1 \times \dots \times \chi_{i-1} \times \chi_{i+1} \times \dots \times \chi_n \rtimes \pi_1 +$$

$$\chi_1 \times \dots \times \chi_{i-1} \times \chi_{i+1} \times \dots \times \chi_n \rtimes \pi_2.$$

Thus, $\chi_1 \times \dots \times \chi_n \rtimes 1$ is reducible.

Suppose that (iii) does not hold for some $1 \leq i < j \leq n$. Write $\chi_i = \nu^{\epsilon_1}\chi_j^{\epsilon_2}$ where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. Note that in this situation we know that in R we have $\chi_i \times \chi_j^{\epsilon_2} = \pi_1 + \pi_2$ where $\pi_1, \pi_2 \in R$, $\pi_1 > 0$, $\pi_2 > 0$. We have the following equalities in $R[S]$:

$$\chi_1 \times \dots \times \chi_i \times \dots \times \chi_j \times \dots \times \chi_n \rtimes 1 =$$

$$\chi_1 \times \ldots \times \chi_i \times \ldots \times \chi_j^{\epsilon_2} \times \ldots \times \chi_n \rtimes 1 =$$

$$\chi_1 \times \ldots \times \chi_i \times \chi_j^{\epsilon_2} \times \chi_{i+1} \times \ldots \times \chi_{j-1} \times \chi_{j+1} \times \ldots \times \chi_n \rtimes 1 =$$

$$\chi_1 \times \ldots \times \chi_{i-1} \times \pi_1 \times \chi_{i+1} \times \ldots \times \chi_{j-1} \times \chi_{j+1} \times \ldots \times \chi_n \rtimes 1+$$

$$\chi_1 \times \ldots \times \chi_{i-1} \times \pi_2 \times \chi_{i+1} \times \ldots \times \chi_{j-1} \times \chi_{j+1} \times \ldots \times \chi_n \rtimes 1.$$

The final result shows that $\chi_1 \times \ldots \times \chi_n \rtimes 1$ is reducible.

We have proved that conditions (i), (ii) and (iii) are necessary for irreducibility.

Now we shall suppose that conditions (i), (ii) and (iii) hold. Note that condition (iii) implies the following condition which will be denoted (iii)$'$: for any $1 \leq i \neq j \leq n$, $\chi_i^{\pm 1} \neq \nu^{\pm 1}\chi_j^{\pm 1}$. We want to prove irreducibility of $\chi_1 \times \ldots \times \chi_n \rtimes 1$. For any $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ we have by the fourth section that

$$\chi_1 \times \chi_2 \times \ldots \chi_n \rtimes 1 = \chi_1^{\epsilon_1} \times \ldots \times \chi_n^{\epsilon_n} \rtimes 1$$

in $R[S]$. It is enough now to prove irreducibility of $\chi^{\epsilon_1} \times \ldots \times \chi_m^{\epsilon_m} \rtimes 1$ for some ϵ_i's as above. Note that $\chi_1^{\epsilon_1}, \ldots, \chi_n^{\epsilon_n}$ again satisfy conditions (i), (ii) and (iii)$'$. With a suitable choice of ϵ_i's, we can get that

$$e(\chi_i) \geq 0, \qquad 1 \leq i \leq n.$$

We shall assume this in the rest of the proof. For any permutation p of $\{1, 2, \ldots, n\}$ we have equality

$$\chi_1 \times \ldots \times \chi_n \rtimes 1 = \chi_{p(1)} \times \ldots \times \chi_{p(n)} \rtimes 1$$

in $R[S]$. Therefore, it is enough to prove irreducibility of $\chi_{p(1)} \times \ldots \times \chi_{p(n)} \rtimes 1$ for some p. With a suitable choice of p, we can assume that

$$e(\chi_1) \geq e(\chi_2) \geq \ldots \geq e(\chi_n) \geq 0.$$

We introduce $j \in \{0, 1, 2, \ldots.., n-1, n\}$ in the following way. If all $e(\chi_i) > 0$ (resp. all $e(\chi_i) = 0$), then we set $j = n$ (resp. $j = 0$). Suppose that there exist χ_k and χ_l such that $e(\chi_k) > 0$ and $e(\chi_l) = 0$. Then, one denotes by j the index which satisfies $e(\chi_j) > 0$ and $e(\chi_{j+1}) = 0$ (such an index must exist in this situation). Set

$$\tau = \chi_{j+1} \times \ldots \times \chi_n \rtimes 1$$

(if $j = n$, then we take $\tau = 1$). Since we do not have characters of order 2 and since τ is a unitary principal series representations, the representation τ is irreducible. Clearly, τ is a tempered representation. Set

$$t = (\chi_1, \ldots, \chi_j, \tau).$$

Then $t \in M(D_+) \times T[S]$. Also, $t = (\chi_1, \ldots, \chi_j, \tau)$ is written in a standard order. In $\mathcal{R}[S]$ we have

$$\chi_1 \times \ldots\ldots \times \chi_n \rtimes 1 = \chi_1 \times \ldots \times \chi_j \rtimes (\chi_{j+1} \times \ldots \times \chi_n \rtimes 1) =$$

$$\chi_1 \times \ldots \times \chi_j \rtimes \tau = \lambda(t).$$

Recall that $\mu_1 \times \mu_2$ is irreducible if and only if $\mu_1\mu_2^{-1} \neq \nu^{\pm 1}$,$\mu_1, \mu_2 \in (F^\times)^\sim$. If $\mu_1 \times \mu_2$ is irreducible, then we have in $\mathcal{R}$

$$\mu_1 \times \mu_2 = \mu_2 \times \mu_1.$$

Also, if $\mu_1 \rtimes 1$ is irreducible, then we have

$$\mu_1 \rtimes 1 = \mu_1^{-1} \rtimes 1 = \mu_1 \underline{\rtimes} 1.$$

in $\mathcal{R}[S]$. The condition (iii) implies that $\chi_i \times \chi_k$ is irreducible for all i, k. Therefore we have the following sequence of equalities in $\mathcal{R}[S]$:

$$\begin{aligned} \lambda(t) \;&=\; (\chi_1 \times \chi_2) \times \ldots \times \chi_n \rtimes 1 = (\chi_2 \times \chi_1) \times \chi_3 \times \ldots \times \chi_n \rtimes 1 = \\ \ldots \;&=\; \chi_2 \times \chi_3 \times \chi_4 \times \ldots \times \chi_n \times \chi_1 \rtimes 1 = \\ &=\; \chi_2 \times \ldots \times \chi_n \times (\chi_1^{-1} \rtimes 1) \end{aligned}$$

by (i) and (ii). Furthermore, we have in $\mathcal{R}[S]$

$$\begin{aligned} \lambda(t) \;&=\; \chi_2 \times \ldots \times \chi_{n-1} \times (\chi_1^{-1} \times \chi_n) \rtimes 1 = \ldots \\ &=\; (\chi_1^{-1} \times \chi_2) \times \ldots \times \chi_n \rtimes 1. \end{aligned}$$

Continuing the procedure with χ_2 and then with $\chi_3, \ldots, \chi_j$, we obtain that in $\mathcal{R}[S]$ we have

$$\lambda(t) = \chi_1^{-1} \times \ldots\ldots \times \chi_j^{-1} \times \chi_{j+1} \times \ldots \times \chi_n \rtimes 1 = \chi_1^{-1} \times \chi_2^{-1} \times \ldots \chi_j^{-1} \rtimes \tau = \underline{\lambda}(t)$$

By the sixth section $\lambda(t)$ is irreducible. This finishes the proof of the theorem. $\square$

We study now the case of $GSp(n, F)$.

Let $\chi_1, \ldots \chi_n, \chi \in (F^\times)^\wedge$ (resp. $(F^\times)^\sim$). The representation $\chi_1 \times \ldots \times \chi_n \rtimes \chi$ is called the unitary principal series representation (resp. the non-unitary principal series representation) of $GSp(n, F)$.

Recall that

$$(\chi_1 \times \chi_2 \times \ldots \times \chi_n \times \chi)|Sp(n, F) = \chi_1 \times \chi_2 \times \ldots \times \chi_n \rtimes 1$$

We shall consider now unitary principal series representations of $GSp(n, F)$.

Lemma 7.2. *Let $\chi_1, \ldots, \chi_n, \chi \in (F^\times)^\wedge$. Decompose*

$$\chi_1 \times \chi_2 \times \ldots \times \chi_n \rtimes \chi = \sigma_1 \oplus \ldots \oplus \sigma_m$$

into a direct sum of irreducible representations of $GSp(n, F)$. Then $\sigma_i \neq \psi\sigma_j$ for any $\psi \in (F^\times)^\sim$ and any $1 \leq i \neq j \leq n$. In particular, $\chi_1 \times \ldots \times \chi_n \times \chi$ is a multiplicity one representation.

Proof. Suppose that $\sigma_i = \psi\sigma_j$ for some ψ and $i \neq j$. Then $\sigma_i|Sp(n, F) = \sigma_j|Sp(n, F)$ since ψ is trivial on $Sp(n, F)$. This implies that $\chi_1 \times \ldots \times \chi_n \rtimes 1$ is not a multiplicity one representation of $Sp(n, F)$ which is a contradiction. $\square$

For $\sigma \in R^+[G]$ set

$$X_{Sp(n,F)}(\sigma) = \{\chi \in (F^\times)^\sim; \chi\sigma = \sigma\}$$

This is clearly a subgroup of $(F^\times)^\sim$. We shall study this group in more detail.

Lemma 7.3. *Let $\chi_1, \dots, \chi_n, \chi \in (F^\times)^\sim$. For an irreducible subquotient σ of $\chi_1 \times \dots \times \chi_n \rtimes \chi$ we have*

$$X_{Sp(n,F)}(\sigma) \subseteq X_{Sp(n,F)}(\chi_1 \times \chi_2 \times \dots \times \chi_n \rtimes \chi).$$

If $\chi_1 \times \chi_2 \times \dots \times \chi_n \rtimes 1$ is a multiplicity one representation of $Sp(n, F)$, then the equality holds. In particular, if $\chi_1, \dots, \chi_n, \chi$ are all unitary characters, then the equality holds.

Proof. Suppose that $\psi\sigma = \sigma$. Then $\sigma = \psi\sigma$ is a composition factor of $\psi(\chi_1 \times \dots \times \chi_n \rtimes \chi) = \chi_1 \times \dots \times \chi_n \rtimes \psi\chi$ which is again a non-unitary principal series representation. Since $\chi_1 \times \dots \times \chi_n \rtimes \chi$ and $\psi(\chi_1 \times \dots \times \chi_n \rtimes \chi)$ have a common composition factor, they have the same Jorden-Hölder sequences ([BnZ2], Theorem 2.9.). This implies that they are equal in $R[G]$. Thus

$$\psi \in X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes \chi).$$

Suppose now that $\psi \in X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes \chi)$. Decompose

$$\chi_1 \times \dots \times \chi_n \rtimes \chi = \sigma_1 \oplus \cdots \oplus \sigma_m$$

into a direct sum of irreducible representations. Suppose that $\chi_1 \times \chi_2 \times \dots \times \chi_n \rtimes 1$ is a multiplicity one representation. Consider

$$\sigma_1 \oplus \dots \oplus \sigma_m = \psi(\sigma_1 \oplus \cdots \oplus \sigma_m) = \psi\sigma_1 \oplus \cdots \oplus \psi\sigma_n.$$

Now the multiplicity one condition on $\chi_1 \times \dots \times \chi_n \rtimes 1$ and the simple argument from the proof of the preceding lemma, imply $\sigma_i = \psi\sigma_i$, for all $1 \le i \le m$. Thus, $\psi \in X_{Sp(n,F)}(\sigma_i)$ for any $1 \le i \le m$. $\square$

Lemma 7.4. *If $\chi_1, \dots, \chi_n, \chi \in (F^\times)^\sim$, then the group $X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes \chi)$ is equal to the subgroup of $(F^\times)^\wedge$ generated by all characters of order two in the set $\{\chi_1, \dots, \chi_n\}$.*

Proof. Since $\chi(\chi_1 \times \dots \times \chi_n \rtimes 1_{F^\times}) = \chi_1 \times \dots \times \chi_n \rtimes \chi$, we have

$$X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes \chi) = X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes 1_{F^\times}).$$

For any permutation p of $\{1, \dots, n\}$ and for $\epsilon_i \in \{\pm 1\}$ the following equalities hold in $R[G]$:

$$\chi_1 \times \dots \times \chi_n \rtimes \chi = \chi_{p(1)} \times \dots \times \chi_{p(n)} \rtimes \chi$$

and

$$\chi_1 \times \dots \times \chi_n \rtimes \chi = \chi_1^{\epsilon_1} \times \chi^{\epsilon_2} \times \dots \times \chi_n^{\epsilon_n} \rtimes \left(\prod_{i=1}^{n} \chi_i^{(1-\epsilon_i)/2} \right) \chi.$$

By the above remarks, it is enough to prove the lemma in the following situation: the sequence $\chi_1, \dots, \chi_n, \chi$ equals

$$\underbrace{\varphi_1, \dots, \varphi_1}_{n_1 \text{ times}}, \underbrace{\varphi_2, \dots, \varphi_2}_{n_2 \text{ times}}, \dots, \underbrace{\varphi_k, \dots, \varphi_k}_{n_k \text{ times}}, 1_{F^\times}$$

where $\varphi_i \neq \varphi_j$ and $\varphi_i \neq \varphi_j^{-1}$ for any $i \neq j$, $1 \leq i, j \leq k$.

Note that the Weyl group of $GSp(n, F)$ (and of $Sp(n, F)$) is isomorphic to the semi-direct product of the group of permutations of n elements and $\{\pm 1\}^n$. The second factor is normal. If p is a permutation of $\{1, \dots, n\}$, then p acts as

$$\chi_1 \otimes \dots \otimes \chi_n \otimes \chi \mapsto \chi_{p^{-1}(1)} \otimes \dots \otimes \chi_{p^{-1}(n)} \otimes \chi.$$

If $\epsilon = (\epsilon_i)_{1 \leq i \leq n}$ is a sequence in $\{\pm 1\}$, then ϵ acts in the following way

$$\chi_1 \otimes \dots \otimes \chi_n \otimes \chi \mapsto \chi_1^{\epsilon_1} \otimes \dots \otimes \chi_n^{\epsilon_n} \otimes \left(\chi \prod \chi_i^{(1-\epsilon_i)/2}\right).$$

Let X be the subgroup of $(F^\times)^\wedge$ generated by all characters of order two among $\varphi_1, \dots, \varphi_k$. To prove the lemma we need to prove that $X = X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes \chi)$. Suppose that $\varphi \in X$. Then $\varphi(\chi_1 \times \dots \times \chi_n \rtimes \chi) = \chi_1 \times \dots \times \chi_n \rtimes \varphi\chi$. By the Frobenius reciprocity the last representation has $\chi_1 \otimes \dots \otimes \chi_n \otimes \varphi\chi$ for a subquotient (in fact, for a quotient) of the Jacquet module for the standard minimal parabolic subgroup. Note that $\chi_1 \otimes \dots \otimes \chi_n \otimes \varphi\chi$ is in the same orbit of the Weyl group by the above remark about the action of the Weyl group. This implies that we have in $R[G]$

$$\chi_1 \times \dots \times \chi_n \rtimes \chi = \varphi(\chi_1 \times \dots \times \chi_n \rtimes \chi).$$

Thus $\varphi \in X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes \chi)$. This proves $X \subseteq X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes \chi)$.

Suppose that $\varphi \in X_{Sp(n,F)}(\chi_1 \times \dots \times \chi_n \rtimes \chi)$. Then

$$\varphi(\chi_1 \times \dots \times \chi_n \rtimes \chi) = \chi_1 \times \dots \times \chi_n \rtimes \chi,$$

i.e.

$$\chi_1 \times \dots \times \chi_n \rtimes \varphi\chi = \chi_1 \times \dots \times \chi_n \rtimes \chi.$$

Note again that $\chi_1 \otimes \dots \otimes \chi_n \otimes \varphi\chi$ is a subquotient of the Jacquet module of $\chi_1 \times \dots \times \chi_n \rtimes \varphi\chi$ for the standard minimal parabolic, while $\chi, \otimes \dots \otimes \chi_n \otimes \chi$ is a subquotient of the Jacquet module of $\chi_1 \times \dots \times \chi_n \rtimes \chi$ for the standard minimal parabolic also. The equality of representations in $R[G]$ implies that $\chi_1 \otimes \dots \otimes \chi_n \otimes \varphi\chi$ is in the Weyl group orbit of $\chi_1 \otimes \dots \otimes \chi_n \otimes \chi$. Take an element w of the Weyl group such that

$$\chi_1 \otimes \dots \otimes \chi_n \otimes \varphi\chi = w(\chi_1 \otimes \dots \otimes \chi_n \otimes \chi).$$

Write $w = \epsilon p^{-1}$ where $\epsilon = (\epsilon_i)_{1 \leq i \leq n}$ is a sequence in $\{\pm 1\}$, and p is a permutation of $\{1, 2, \dots, n\}$. Thus

$$\chi_1 \otimes \dots \otimes \chi_n \otimes \varphi\chi = \chi_{p(1)}^{\epsilon_1} \otimes \dots \otimes \chi_{p(n)}^{\epsilon_n} \otimes \chi \prod_{i=1}^{n} \chi_{p(i)}^{(1-\epsilon_i)/2}.$$

The condition that $\varphi_i \neq \varphi_j^{-1}$ and $\varphi_i \neq \varphi_j$ for any $i \neq j$, and

$$\chi_1 \otimes \dots \otimes \chi_n = \chi_{p(1)}^{\epsilon_1} \otimes \dots \otimes \chi_{p(n)}^{\epsilon_n},$$

impliy that if $\epsilon_i \neq 1$, then $(\chi_{p(i)})^2 = 1$. Therefore

$$\varphi = \prod_{i=1}^{n} \chi_{p(i)}^{(1-\epsilon_i)/2} \in X.$$

This proves $X_{Sp(n,F)}(\chi_1 \times \ldots \times \chi_n \rtimes \chi) \subseteq X$. The proof of the lemma is now complete. $\square$

Suppose that A is an Abelian group. Let A' be the subgroup of A which consists of all elements of order two or one. Then A' is a vector space over $\mathbb{Z}/2\mathbb{Z}$. Elements $a_1, \ldots, a_n \in A'$ will be called $(\mathbb{Z}/2\mathbb{Z})$-linearly independent if they are linearly independent in the above vector space. It is equivalent to the fact that they generate the subgroup of A (in fact of A') with 2^n elements. If Y is subset of A', then we shall denote the cardinal number of a maximal $(\mathbb{Z}/2\mathbb{Z})$-linearly independent subset of Y by

$$\dim_{\mathbb{Z}/2\mathbb{Z}} Y.$$

Theorem 7.5. *Suppose that $\chi_1, \ldots, \chi_n, \chi \in (F^\times)^\wedge$. Let d be the maximal number of distinct characters of order 2 among $\chi_1, \ldots, \chi_n$, and let ℓ be the maximal number of $(\mathbb{Z}/2\mathbb{Z})$-linearly independent elements among characters $\chi_1, \ldots, \chi_n$ which are of order two. Then the unitary principal series representation*

$$\chi_1 \times \ldots \times \chi_n \rtimes \chi$$

of $GSp(n, F)$ is a multiplicity one representation. Its length is $2^{d-\ell}$. In particular, a representation $\chi_1 \times \ldots \times \chi_n \rtimes \chi$ is irreducible if and only if the maximal subset of distinct elements of order two among $\chi_1, \ldots, \chi_n$ is $(\mathbb{Z}/2\mathbb{Z})$-linearly independent.

Proof. Denote $\sigma = \chi_1 \times \ldots \times \chi_n \rtimes \chi$. Decompose $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m$ into a direct sum of irreducible representations. We know that σ is a multiplicity one representation since $\sigma | Sp(n, F)$ is a such representation. We want to compute m.

The length of $\sigma | Sp(n, F)$ is 2^d. The length of $\sigma_i | Sp(n, F)$ is the dimension of the intertwining algebra of $\sigma_i | Sp(n, F)$ since $\sigma_i | Sp(n, F)$ is a multiplicity one representation. The dimension of the intertwining algebra is the cardinal number of $X_{Sp(n,F)}(\sigma_i)$ by Lemma 2.1 of [GeKn]. Lemma 7.3 says that $X_{Sp(n,F)}(\sigma_i) = X_{Sp(n,F)}(\sigma)$. Lemma 7.4 shows that the cardinal numbers of $X_{Sp(n,F)}(\sigma)$ is 2^ℓ.

We compare now the lengths of both sides of

$$\sigma | Sp(n, F) = (\sigma_1 | Sp(n, F)) \oplus \ldots \oplus (\sigma_m | Sp(n, F)).$$

This implies $2^d = m \cdot 2^\ell$. Therefore $m = 2^{d-\ell}$. $\square$

It is obvious that any two distinct characters of order two are $(\mathbb{Z}/2\mathbb{Z})$-linearly independent. Thus:

Corollary 7.6. *The unitary principal series representations of $GSp(2, F)$ are irreducible.*

The unitary principal series representations of $GSp(n, F)$ for $n \geq 3$ are not always irreducible. We present a simple example.

Example 7.7. *For any F there exist two distinct characters of order two, say χ_1 and χ_2. They are $(\mathbb{Z}/2\mathbb{Z})$-linearly independent. Let $\chi_3 = \chi_1\chi_2$. The length of $\chi_1 \times \chi_2 \times \chi_3 \rtimes 1_{F^\times}$ is two. It is interesting to note that when we induce parabolically $\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes 1_{F^\times}$ to any Levi factor containing the standard maximal torus, of any proper parabolic subgroup, then the induced representation is irreducible.*

This example is essentially the only example of reducibility for the unitary principal series of $GSp(3, F)$, when the residual characteristic of F is not two.

Corollary 7.8. *Suppose that the residual characteristic of F is different from two. Then $F^\times$ has exactly three characters of order two. The lengths of the unitary principal series representations of $GSp(n, F)$ are either one or two. A unitary principal series representation*

$$\chi_1 \times \ldots \times \chi_n \rtimes \chi$$

is reducible if and only if the set $\{\chi_1, \ldots, \chi_n\}$ contains three different characters of $F^\times$ of order two.

Theorem 7.9. *For $\chi_1, \ldots, \chi_n, \chi \in (F^\times)^\sim$ consider the following three conditions:*

(i) *card $\{\chi_i; \chi_i^2 = 1_{F^\times}$ and $\chi_i \neq 1_{F^\times}\} = dim_{\mathbb{Z}/2\mathbb{Z}}\{\chi_i; \chi_i^2 = 1_{F^\times}\}$.*
(ii) *$\chi_i \neq \nu^{\pm 1}, 1 \leq i \leq n$.*
(iii) *$\chi_i \neq \nu^{\pm 1}\chi_j^{\pm 1}, \quad 1 \leq i < j \leq n$.*

The non-unitary principal series representation $\chi_1 \times \ldots \times \chi_n \rtimes \chi$ of $GSp(n, F)$ is irreducible if and only if the conditions (i), (ii) and (iii) hold.

Proof. The proof is very similar to the proof of Theorem 7.1. Therefore we shall only outline it.

Theorem 7.5 implies that (i) is necessary for the irreducibility of $\chi_1 \times \ldots \times \chi_n \rtimes \chi$. The representation theory of $GL(2, F)$ implies that the conditions (ii) and (iii) are necessary for the irreducibility of $\chi_1 \times \ldots \times \chi_n \rtimes \chi$.

Note that $\chi_1 \times \ldots \times \chi_n \rtimes \chi = \chi(\chi_1 \times \ldots \times \chi_n \rtimes 1_{F^\times})$ is irreducible if and only if $\chi_1 \times \ldots \times \chi_n \rtimes 1_{F^\times}$ is irreducible. Let p be a permutation of $\{1, 2, \ldots, n\}$ and $(\epsilon_i)_{1 \leq i \leq n}$ a sequence in $\{\pm 1\}$. Now $\chi_1 \times \ldots \times \chi_n \rtimes \chi$ is irreducible if and only if $\chi_{p(1)} \times \ldots \times \chi_{p(n)} \rtimes 1_{F^\times}$ is irreducible, and furthermore, if and only if $\chi_1^{\epsilon_1} \times \ldots \times \chi_n^{\epsilon_n} \rtimes 1_{F^\times}$ is irreducible. Note that if $\chi_1, \ldots, \chi_n, \chi$ satisfy (i), (ii) and (iii), then $\chi_{p(1)}, \ldots, \chi_{p(n)}, \chi'$ satisfy (i), (ii) and (iii) and also $\chi_1^{\epsilon_1}, \ldots, \chi_n^{\epsilon_n}, \chi'$ satisfy (i), (ii) and (iii), for any $\chi' \in (F^\times)^\sim$. Thus, we may suppose that $e(\chi_1) \geq e(\chi_2) \geq \ldots \geq e(\chi_n) \geq 0$ and $\chi = 1_{F^\times}$. Choose $0 \leq j \leq n$ such that $e(\chi_j) > 0$ and $e(\chi_{j+1}) = 1$ (more precisely, choose j in the same way as in the proof of Theorem 7.1). Let

$$\tau = \chi_{j+1} \times \ldots \times \chi_n \rtimes 1_{F^\times}.$$

By Theorem 7.5, τ is irreducible (and tempered) since $\chi_1, \ldots, \chi_n$ satisfy (i). Thus

$$t = (\chi_1, \ldots, \chi_j, \tau) \in M(D_+) \times T[G].$$

It is written in a standard order. Consider

$$\chi_1 \times \ldots \times \chi_n \rtimes 1_{F^\times} = \chi_1 \times \ldots \times \chi_j \rtimes \tau = \lambda(t).$$

We have in $\mathcal{R}[G]$:

$$\lambda(t) = \chi_1 \times \ldots \times \chi_n \rtimes 1_{F^\times} = \chi_2 \times \ldots \times \chi_n \times \chi_1 \rtimes 1_{F^\times} =$$
$$\chi_2 \times \ldots \times \chi_n \times \chi_1^{-1} \rtimes \chi_1 = \chi_1^{-1} \times \chi_2 \times \ldots \times \chi_n \rtimes \chi_1 =$$
$$= \chi_1^{-1} \times \chi_2^{-1} \times \chi_3 \times \ldots \times \chi_n \rtimes \chi_1\chi_2 =$$
$$= \chi_1^{-1} \times \ldots \times \chi_j^{-1} \rtimes (\chi_1\chi_2 \ldots \chi_j)\tau = \underline{\lambda}(t).$$

Thus $\lambda(t) = \chi_1 \times \ldots \times \chi_n \rtimes 1_{F^\times}$ is irreducible. $\square$

Remarks 7.10.

(i) *If $n \leq 2$, then the condition (i) of the preceding theorem is always satisfied.*

(ii) *If the residual characteristic of F is different from two then the condition (i) of the preceding theorem has a very simple form:*

$$\textit{card}\ \{\chi_i, \chi_i^2 = 1_{F^\times}, \chi_i \neq 1_{F^\times}\} \leq 2.$$

8. On square integrable representations of $GSp(n)$

Let φ be a character of a maximal split torus in a reductive group over a p-adic field. It is called regular if only the identity element of the Weyl group fixes it. Clearly, if one character in an orbit of the Weyl group is regular, then all the others are regular. We are going now to consider regular characters for groups $Sp(n, F)$ and $GSp(n, F)$. We shall always consider standard maximal tori which consist of diagonal elements in these groups. They are denoted by A_0. The Weyl groups are identified in a natural way. This group is denoted by W.

We have a few simple observations at the beginning. Let φ be a character of the standard maximal torus in $GSp(n, F)$. Suppose that the restriction φ' of φ to the standard maximal torus in $Sp(n, F)$, is regular. Then it is obvious that then φ is regular as well. This holds because the restriction commutes with the action of the Weyl group.

Let φ be a character of the standard maximal torus A_0 in $GSp(n, F)$. We may write

$$\varphi = \chi_1 \otimes \ldots \otimes \chi_n \otimes \chi,$$

where $\chi_1, \ldots, \chi_n, \chi$ are characters of $F^\times$. Note that the restriction to the standard maximal torus in $Sp(n, F)$ is

$$\chi_1 \otimes \ldots \otimes \chi_n \otimes 1.$$

Take a character ψ of $GSp(n, F)$. Note that $(\psi|A_0)w\varphi = w((\psi|A_0)\varphi)$. This implies that $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ is regular if and only if $\chi_1 \otimes \ldots \otimes \chi_n \otimes 1_{F^\times}$ is regular. We can also see this from the following proposition which characterizes regular characters.

Proposition 8.1.

(a) *Consider the case of $Sp(n,F)$. A character*

$$\chi_1 \otimes \chi_2 \otimes \ldots \otimes \chi_n \otimes 1$$

is regular if and only if the following two conditions are satisfied

(i) $\chi_i \neq \chi_j^{\pm 1}, \quad 1 \leq i < j \leq n.$

(ii) $\chi_i^2 \neq 1_{F^\times}, \quad 1 \leq i \leq n.$

(b) *Consider the case of $GSp(n,F)$. A character*

$$\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$$

is regular if and only if the following three conditions are satisfied

(i) $\chi_i \neq \chi_j^{\pm 1}, \quad 1 \leq i < j \leq n.$

(ii) $\chi_i \neq 1_{F^\times}, \quad 1 \leq i \leq n.$

(iii) *card* $\{\chi_i; \chi_i^2 = 1_{F^\times}, \chi_i \neq 1_{F^\times}\} = dim_{\mathbb{Z}/2\mathbb{Z}}\ \{\chi_i;\ \chi_i^2 = 1_{F^\times}\}.$

If the residual characteristic of F is different from two, then the condition (iii) is equivalent to the following condition

(iii)' *card* $\{\chi_i; \chi_i^2 = 1_{F^\times}, \chi_i \neq 1_{F^\times}\} \leq 2.$

Proof. (a) Denote by $Sym(n)$ the group of permutations of $\{1, \ldots, n\}$. We may identify $W = \{\pm 1\}^n \rtimes Sym(n)$. Now, $((\epsilon_i), p^{-1})$ transforms $\chi_1 \otimes \ldots \otimes \chi_n \otimes 1$ into

$$\chi_{p(1)}^{\epsilon_1} \otimes \ldots \otimes \chi_{p(n)}^{\epsilon_n} \otimes 1.$$

Suppose that $((\epsilon_i),\, p^{-1})$ is not identity, i.e. $\epsilon_i = -1$ or $p(i) \neq i$ for some i. Suppose

$$\chi_1 \otimes \ldots \otimes \chi_n \otimes 1 = \chi_{p(1)}^{\epsilon_1} \otimes \ldots \otimes \chi_{p(n)}^{\epsilon_n} \otimes 1.$$

If $p(i) \neq i$ then $\chi_i = \chi_{p(i)}^{\epsilon_i}$ with $i \neq p(i)$, and therefore condition (i) of (a) is not satisfied. Suppose that $p(i) = 1$ for all i. Then, there exists i with $\epsilon_i = -1$. Now $\chi_i = \chi_i^{-1}$. Thus, condition (ii) is not satisfied.

We have proved that (i) and (ii) are sufficient conditions for the regularity. We shall see now that they are necessary.

Suppose $\chi_i = \chi_j = \chi$ for some $1 \leq i < j \leq n$. Then

$$\chi_1 \otimes \ldots \otimes \chi_i \otimes \ldots \otimes \chi_j \otimes \ldots \otimes \chi_n \otimes 1$$

is in the same orbit as

$$\chi_1 \otimes \ldots \otimes \chi_{i-1} \otimes \chi \otimes \chi \otimes \chi_{i+1} \otimes \ldots \otimes \chi_{j-1} \otimes \chi_{j+1} \otimes \ldots \otimes \chi_n \otimes 1.$$

It is easy to see that one of the generators of W described in the third section, acts trivially on the above element. Thus $\chi_1 \otimes \ldots \otimes \chi_n \otimes 1$ is not regular.

Suppose $\chi_i = \chi_j^{-1}$ for some $1 \leq i < j \leq n$. Then $\chi_1 \otimes \ldots \otimes \chi_i \otimes \ldots \otimes \chi_j \otimes \ldots \otimes \chi_n \otimes 1$ is in the same orbit as

$$\chi_1 \otimes \ldots \otimes \chi_i \otimes \ldots \otimes \chi_j^{-1} \otimes \ldots \otimes \chi_n \otimes 1.$$

It is not regular by the previous case.

Let $\chi_i^2 = 1_{F^\times}$. Then $\chi_1 \otimes \ldots \otimes \chi_i \otimes \ldots \otimes \chi_n \otimes 1$ is in the same orbit as

$$\chi_1 \otimes \ldots \otimes \chi_{i-1} \otimes \chi_{i+1} \otimes \ldots \otimes \chi_n \otimes \chi_i \otimes 1.$$

One of the generators of W described in the third section acts trivially on the above element. Thus, it is not a regular character.

This proves that (i) and (ii) are also necessary for the regularity.

(b) An element $((\epsilon_i), p^{-1}) \in W$ transforms

$$\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi \mapsto \chi_{p(1)}^{\epsilon_1} \otimes \ldots \otimes \chi_{p(n)}^{\epsilon_n} \otimes \chi \prod_{i=1}^{n} \chi_{p(i)}^{(1-\epsilon_i)/2}.$$

Assume that $((\epsilon_i), p^{-1}) \neq 1$ acts trivially on $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$. If there exists i with $p(i) \neq i$, then $\chi_i = \chi_{p(i)}^{\epsilon_i}$. Thus, the condition (i) is not satisfied. If $p(i) = i$ for all $1 \leq i \leq n$, then there exists i with $\epsilon_i = -1$. From $\chi_j = \chi_j^{\epsilon_j}$, $1 \leq j \leq n$, one obtains that $\epsilon_j = -1$ implies $\chi_j^2 = 1_{F^\times}$. If $\chi_i = 1_{F^\times}$ for some i, then condition (ii) is not satisfied. Suppose that (i) and (ii) are satisfied. Then $p(i) = i$ for all $1 \leq i \leq n$, and there is i such that $\epsilon_i = -1$, $\chi_i^2 = 1_{F^\times}$ and $\chi_i \neq 1_{F^\times}$. Furthermore

$$\prod_{\epsilon_j = -1} \chi_j = 1_{F^\times}.$$

Therefore $\{\chi_k, \chi_k^2 = 1_{F^\times}, \chi_k \neq 1_{F^\times}\}$ is $(\mathbb{Z}/2\mathbb{Z})$-linearly dependent. Thus the condition (iii) is not satisfied.

We have proved that the conditions (i), (ii) and (iii) are sufficient for the regularity of $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$.

Suppose that we have a character $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ which does not satisfy (i) or (ii). Then, in the same way as in (a), it is easy to see that it is not regular. Suppose now that (iii) does not hold, but (i) and (ii) hold. Then there exists a sequence $\epsilon_i \in \{\pm 1\}$, $1 \leq i \leq n$, such that ϵ_i can be -1 only when $\chi_i^2 = 1_{F^\times}$, $\epsilon_i = -1$ for at least one $1 \leq i \leq n$, and

$$\prod_{i=1}^{n} \chi_i^{(1-\epsilon_i)/2} = 1_{F^\times}.$$

This gives that $(\epsilon_i)_{1 \leq i \leq 1} \neq 1$ acts trivially on $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$. $\square$

We shall now consider the case of $GSp(n,F)$. The set of all roots is denoted by Σ. We introduce the characters

$$a_i^* : a(x_1, \dots, x_n, x) \mapsto x_i \ , \qquad 1 \le i \le n$$

$$a_{n+1}^* : a(x_1, \dots, x_n, x) \mapsto x.$$

These are the characters of the standard maximal torus A_0 in $GSp(n,F)$. The simple roots are

$$a_i^*(a_{i+1}^*)^{-1}, \quad 1 \le i \le n-1,$$

and

$$(a_n^*)^2(a_{n+1}^*)^{-1} \ .$$

The positive roots are

$$a_i^*(a_j^*)^{-1} \ , \qquad 1 \le i < j \le n,$$

$$a_i^* a_j^*(a_{n+1}^*)^{-1}, \ 1 \le i < j \le n,$$

$$(a_i^*)^2(a_{n+1}^*)^{-1} \ , \qquad 1 \le i \le n.$$

One gets the negative roots by taking inverses of the positive roots.

One associates to each $\alpha \in \Sigma$ a coroot $\alpha^\vee$ and a one-parameter subgroup $t_{\alpha^\vee}$ in the same way as F. Rodier did it in [R1]. A short computation gives

$$t_{(a_i^*(a_j^*)^{-1})^\vee}(x) = a(1, \dots, x, 1, \dots, 1, x^{-1}, 1, \dots, 1, 1), \quad 1 \le i < j \le n,$$

where x is at the i^{th} place, while x^{-1} is at the j^{th} place,

$$t_{(a_i^* a_j^*(a_{n+1}^*)^{-1})^\vee} = a(1, \dots, 1, x, 1, \dots, 1, x, 1, \dots, 1, 1), \quad 1 \le i < j \le n,$$

where x's are at the i^{th} and j^{th} place, and

$$t_{((a_i^*)^2(a_{n+1}^*)^{-1})^\vee}(x) = a(1, \dots, 1, x, 1, \dots 1, 1), \quad 1 \le i \le n,$$

where x is at the i^{th} place.

Note that

$$t_{(\alpha^{-1})^\vee}(x) = t_{\alpha^\vee}(x^{-1}) \ , \quad \alpha \in \Sigma.$$

Let $\varphi = \chi_1 \otimes \dots \otimes \chi_n \otimes \chi$ be a regular character of A_0. We denote by $x \mapsto |x|_F$ the topological modules of F and we denote by $\nu : x \mapsto |x|_F$ the restriction to $F^\times$. Denote by $S(\varphi)$ the set of all $\alpha \in \Sigma$ such that

$$\varphi t_{\alpha^\vee} = \nu.$$

Let

$$s(\varphi) = \text{card } S(\varphi).$$

We are now going to compute $s(\varphi)$. We know from [R1]

$$s(\varphi) \le n.$$

It is clear that $S(\varphi)$ does not depend on χ because always $t_{\alpha^\vee}(x) \in Sp(n,F)$. It is simple to see that $s(\varphi)$ is constant on the orbits of the action of the Weyl group.

Each character is associated to some character $\varphi = \chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ satisfying $e(\chi_i) \geq 0, \quad 1 \leq i \leq n$. We shall assume this. We shall call the sets

$$[\chi, \chi\nu^k] = \{\chi\nu^i \ , i \in \mathbb{Z} \text{ and } 0 \leq i \leq k\},$$

$\chi \in (F^\times)^\sim$, $k \in \mathbb{Z}$, $k \geq 0$, segments in $(F^\times)^\sim$. Then χ is called the beginning of the segment $[\chi, \chi\nu^k]$. Decompose

$$\{\chi_1, \ldots, \chi_n\} = \Delta_1 \cup \ldots \cup \Delta_p$$

into a minimal possible number of disjoint segments in $(F^\times)^\sim$. Recall that the character is regular, so this is possible. Such decomposition is unique. Denote

$$\Delta_i = \left[\psi_i\nu^{\gamma_i}, \psi_i\nu^{\gamma_i+k_i}\right], \psi_i \in (F^\times)^\wedge.$$

Obviously $\sum_{i=1}^{p}(k_i+1) = n$, i.e. $\sum_{i=1}^{p} k_i = n-p$. Furthermore, φ is associate to

$$\psi_1\nu^{\gamma_1+k_1} \otimes \psi_1\nu^{\gamma_1+k_1-1} \otimes \ldots \otimes \psi_1\nu^{\gamma_1} \otimes \psi_2\nu^{\gamma_2+k_2} \otimes \ldots \otimes \psi_p\nu^{\gamma_p} \otimes \chi.$$

We shall assume that $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ is equal to the above character.

Note that

$$\varphi t_{(a_i^*(a_j^*)^{-1})^\vee} = \chi_i\chi_j^{-1} \ , \quad 1 \leq i < j \leq n$$

This gives a root in $S(\varphi)$ if and only if χ_i and χ_j are in the same segment and if they are consecutive elements there. Note that a root $(a_i^*)^{-1}a_j^*$, $1 \leq i < j \leq n$, cannot give an element in $S(\varphi)$. Thus, this type of roots gives $\sum_{i=1}^{p} k_i = n-p$ roots in $S(\varphi)$.

Furthermore

$$\varphi t_{((a_i^*)^2(a_{n+1}^*)^{-1})^\vee} = \chi_i \ , \quad 1 \leq i \leq n.$$

Thus $(a_i^*)^2(a_{n+1}^*)^{-1} \in S(\varphi)$ if and only if $\chi_i = \nu$. Note that if this is the case, then χ_i is the beginning of some segment Δ_i since $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ is regular. Obviously, $(a_i^*)^{-2}(a_{n+1}^*)^{-1} \notin S(\varphi)$ for all $1 \leq i \leq n$. Therefore, this type of roots gives at most one root in $S(\varphi)$. It gives a root if and only if $\chi_i = \nu$ for some i.

We consider now

$$\varphi t_{(a_i^* a_j^*(a_{n+1}^*)^{-1})^\vee} = \chi_i\chi_j, \qquad 1 \leq i < j \leq n.$$

Note that $(a_i^*)^{-1}(a_j^*)^{-1}a_{n+1}^* \notin S(\varphi)$. Let $\chi_i = \psi_u\nu^{\gamma_u+s}, \chi_j = \psi_v\nu^{\gamma_v+t}$. Then

$$\chi_i\chi_j = \psi_u\psi_v\nu^{\gamma_u+\gamma_v+s+t}.$$

We consider the case when $a_i^* a_j^* (a_{n+1}^*)^{-1} \in S(\varphi)$. This is equivalent to

$$\psi_u = \psi_v^{-1},$$

$$\gamma_u + \gamma_v + s + t = 1.$$

Clearly $s, t \in \{0, 1\}$ and $s \neq 1$ or $t \neq 1$. Let us suppose that $\psi_u \neq \psi_v$. Then χ_i and χ_j are in different segments. If $s \neq 0$ or $t \neq 0$, then $\gamma_u = \gamma_v = 0$. This is impossible since $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ is regular. Thus, χ_i and χ_j are beginnings of different segments. Now we consider $\psi_u = \psi_v$. If $\gamma_u \neq 0$ or $\gamma_v \neq 0$ then $s = t = 0$. Thus, χ_i and χ_j are beginnings of different segments (and the regularity conditions tells $\gamma_u \neq 1/2$). Suppose $\gamma_u = \gamma_v = 0$. The case of $s = t = 0$ is not possible since $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ is regular. Thus $s = 0$, $t = 1$, or $t = 0$, $s = 1$. This implies that χ_i and χ_j are in the same segment, one character is the beginning of the segment, say χ_i. Then $\chi_j = \chi_{i-1}$. Therefore $s = 0$, $t = 1$ and

$$\chi_i = \psi_u \qquad , \chi_j = \psi_u \, \nu,$$

where

$$\psi_u^2 = 1_{F^\times}, \qquad \psi_u \neq 1_{F^\times}.$$

We have proved the following

Proposition 8.2.

(i) *Let φ be a character of the standard maximal torus in $GSp(n, F)$. Then φ is associated to a character $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ satisfying*

$$e(\chi_i) \geq 0 \, , \qquad 1 \leq i \leq n.$$

We have

$$s(\varphi) = s(\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi).$$

(ii) *Let $\varphi = \chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ be a regular character satisfying*

$$e(\chi_i) \geq 0 \, , \qquad 1 \leq i \leq n.$$

Decompose

$$\{\chi_1, \ldots, \chi_n\} = \Delta_i \cup \ldots \cup \Delta_s$$

into a minimal number of disjoint segments in $(F^\times)^\sim$. Denote by p the number of pairs (i, j), $i < j$, such that the beginnings β_i of Δ_i and β_j of Δ_j satisfy

$$\beta_i \beta_j = \nu.$$

Denote by t the number of characters ψ of order two of $F^\times$ such that

$$\{\psi, \nu\psi\} \subseteq \{\chi_1, \ldots, \chi_n\}.$$

Let ϵ be a 1 if $\nu \in \{\chi_1, \ldots, \chi_n\}$ and 0 otherwise. Then

$$s(\varphi) = n - s + p + t + \epsilon.$$

Remark 8.3. *Let $\varphi = \chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ be a regular character. Then by [R1]*

$$\chi_1 \times \ldots \times \chi_n \rtimes \chi$$

is a multiplicity one representation of length $2^{s(\varphi)}$. The above representation contains a unique irreducible subrepresentation and a unique irreducible quotient.

We denote by $\mathcal{F}$ the space of all functions

$$f : \{\chi \in (F^\times)^\wedge; \chi^2 = 1_{F^\times} \quad \text{and} \quad \chi \neq 1_{F^\times}\} \to \mathbb{Z}_+ \backslash \{1\}$$

such that the support

$$\{\chi \in (F^\times)^\wedge; \chi^2 = 1_{F^\times}, \chi \neq 1_{F^\times} \quad \text{and} \quad f(\chi) \neq 0\}$$

is $(\mathbb{Z}/2\mathbb{Z})$-linearly independent (the support of f is finite since the characteristic of F is not two). For $f \in \mathcal{F}$ set

$$\text{ord}\ (f) = \sum_{\chi} f(\chi).$$

Remark 8.4. *Suppose that the residual characteristic is different from two. Then there are exactly three characters of order two and $\mathcal{F}$ consists of all functions f from them into $\mathbb{Z}_+ \backslash \{1\}$ which have no more than two characters in support.*

Let ψ be an enumeration of $\{\chi \in (F^\times)^\wedge;\ \chi^2 = 1_{F^\times},\ \chi \neq 1_{F^\times}\}$, by an initial segment of positive integers, say $\{1, 2, \ldots, r\}$. For

$$(k, f, \chi) \in \mathbb{Z}_+ \times \mathcal{F} \times (F^\times)^\sim$$

set

$$\varphi(k, f, \chi)_\psi = \nu^k \otimes \nu^{k-1} \otimes \ldots \otimes \nu^1 \otimes \nu^{f(\psi(1))-1}\psi(1) \otimes \nu^{f(\psi(1))-2}\psi(1) \otimes \ldots \otimes \psi(1) \otimes$$

$$\cdots \otimes \nu^{f(\psi(r))-1}\psi(r) \otimes \ldots \otimes \psi(r) \otimes \chi$$

(note that $\nu^{f(\psi(i))-1}\psi(i) \otimes \nu^{f(\psi(i))-2}\psi(i) \otimes \ldots \otimes \psi(i)$ shows up in the above formula if and only if $f(\psi(i)) > 0$, i.e $f(\psi(i)) \geq 2$). For different ψ's, the characters $\varphi(k, f, \chi)_\psi$ are associate. The above characters are regular. Denote

$$\sigma(k, f, \chi) = \nu^k \times \nu^{k-1} \times \ldots \times \nu^1 \times \nu^{f(\psi(1))-1}\psi(1) \times \ldots \times \psi(1) \times$$

$$\ldots \times \nu^{f(\psi(r))-1}\psi(r) \times \ldots \times \psi(r) \rtimes \chi.$$

This is an element of $\mathcal{R}[G]$ (the notation is correct since $\sigma(k, f, \chi)$ does not depend on ψ).

From Proposition 8.2 we get

$$s(\varphi(k,f,\chi)_\psi) = k + ord(f).$$

Therefore, $\sigma(k,f,\chi)$ has a unique irreducible essentially square integrable subquotient which will be denoted $\delta(k,f,\chi)$ ([R1], Proposition 5.). Denote by X_f the subgroup of characters generated by the support of f. Now if $\chi' \in \chi X_f$, then $\varphi(k,f,\chi)_\psi$ and $\varphi(k,f,\chi')_\psi$ are associate. Thus

$$\text{s.s.}(\sigma(k,f,\chi)) = \text{s.s.}(\sigma(k,f,\chi'))$$

and

$$\delta(k,f,\chi) = \delta(k,f,\chi').$$

Therefore, we define

$$\delta(k,f,\chi X_f),$$

as $\delta(k,f,\chi')$ for any $\chi' \in \chi X_f$.

Theorem 8.5.

(i) *For* $(k,f,\chi),(k',f',\chi') \in \mathbb{Z}_+ \times \mathcal{F} \times (F^\times)^\sim$ *we have*

$$\delta(k,f,\chi X_f) = \delta(k',f',\chi' X_{f'})$$

if and only if $k = k', f = f'$ *and* $\chi X_f = \chi' X_{f'}$.

(ii) *If* δ *is an essentially square integrable subquotient of some*

$$\chi_1 \times \ldots \times \chi_n \rtimes \chi, \quad \chi_i, \chi \in (F^\times)^\sim,$$

such that $\chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ *is a regular character, then there exists* $(k,f,\chi) \in \mathbb{Z}_+ \times \mathcal{F} \times (F^\times)^\sim$ *such that*

$$\delta = \delta(k,f,\chi).$$

Proof. (i) Suppose that $\delta(k,f,\chi) = \delta(k',f',\chi')$. Then $\varphi(k,f,\chi)_\psi$ and $\varphi(k',f',\chi')_\psi$ must be in the same orbit of the Weyl group. This immediately gives $k = k'$, $f = f'$ and furthermore $\chi' \in \chi X_f$.

(ii) Let δ be an essentially square integrable subquotient of $\chi_1 \times \ldots \times \chi_n \rtimes \chi$ where $\varphi = \chi_1 \otimes \ldots \otimes \chi_n \otimes \chi$ is regular. We may first assume that $e(\chi_i) \geq 0$, $1 \leq i \leq n$, since each character is associated to a character of such type. We decompose

$$\{\chi_1, \ldots, \chi_n\} = \Delta_1 \cup \ldots \cup \Delta_s$$

as in (ii) of Proposition 8.2. Let us follow the notation introduced in (ii) of Proposition 8.2. We have

$$s(\varphi) = n - s + p + t + \epsilon.$$

Obviously, we can choose $r \geq 0$ such that

$$s = 2p + t + \epsilon + r.$$

Since $\chi_1 \times \ldots \times \chi_n \rtimes \chi$ contains an essentially square integrable subquotient, we have $s(\varphi) = n$ i.e.

$$\begin{aligned} n &= n - s + p + t + \epsilon = n - 2p - t - \epsilon - r + p + t + \epsilon \\ &= n - p - r. \end{aligned}$$

Clearly, $p = r = 0$ and $s = t + \epsilon$. Thus, each segment either starts with ν or with $\{\lambda, \nu\lambda\}$ where λ is a character of order two. This shows that φ is associated with a character of the form $\varphi(k, f, \chi)_\psi$. Thus $\delta = \delta(k, f, \chi)$. $\square$

We shall need the following

Lemma 8.6. *Let $(k, f, \chi) \in \mathbb{Z}_+ \times \mathcal{F} \times (F^\times)^\sim$. Let $n = k + ord(f)$. Then*

$$X_{Sp(n,F)}(\delta(k, f, \chi)) = X_f,$$

where X_f denotes the subgroup generated by the support of f.

Proof. Let $\delta(k, f, \chi)$ be a subquotient of $\chi_1 \times \ldots \times \chi_n \rtimes \chi$. Then

$$X_{Sp(n,F)}(\chi_1 \times \ldots \times \chi_n \rtimes \chi) = X_f$$

by Lemma 7.4. We know

$$X_{Sp(n,F)}(\delta(k, f, \chi)) \subseteq X_{Sp(n,F)}(\chi_1 \times \ldots \times \chi_n \rtimes \chi)$$

from Lemma 7.3. To prove the lemma it is enough to show that the opposite inclusion holds.

Suppose that $\psi \in X_{Sp(n,F)}(\chi_1 \times \ldots \times \chi_n \rtimes \chi)$. Write

$$\chi_1 \times \ldots \times \chi_n \rtimes \chi = \delta(k, f, \chi) + \sigma_1 + \cdots + \sigma_r.$$

in $R[G]$ as a sum of irreducible representations of $GSp(n, F)$. Now

$$\psi(\chi_1 \times \ldots \times \chi_n \rtimes \chi) = \chi_1 \times \ldots \times \chi_n \rtimes \chi$$

implies

$$\psi\delta(k, f, \chi) + \psi\sigma_1 + \cdots + \psi\sigma_r = \delta(k, f, \chi) + \sigma_1 + \cdots + \sigma_r.$$

Note that $\psi\delta(k, f, \chi)$ is an essentially square integrable representation. Thus

$$\psi\delta(k, f, \chi) = \delta(k, f, \chi),$$

since $\chi_1 \times \ldots \times \chi_n \rtimes \chi$ contains exactly one essentially square integrable subquotient. Therefore $\psi \in X_{Sp(n,F)}(\delta(k, f, \chi))$. $\square$

Remarks 8.7.

(i) *Note that*

$$\chi'\delta(k,f,\chi)=\delta(k,f,\chi\chi').$$

(ii) *We have that $\delta(k,f,\chi)$ is an unramified representation if and only if either $f=0$ or the support of f consists of the unique unramified character of order two of $F^\times$.*

(iii) *For $f\in\mathcal{F}$,*

$$card\ X_f=2^{\text{card }(\text{supp}\,f)}.$$

Notation 8.8. *If $f\in\mathcal{F}$, then we shall sometimes write formally f as*

$$\Sigma f(\chi)\chi$$

where the sum runs over the characters of order two.

9. On square integrable representations of $Sp(n)$

In this section we shall fix for each n a non-degenerate character θ_n of the unipotent subgroup N_α of the standard minimal parabolic subgroup of $Sp(n,F)$ (and also of $GSp(n,F)$). Then $\alpha=(\underbrace{1,\dots,1}_{n\text{ times}})$. It is not important for our purposes to write θ_n more explicitly, but one can fix a non-trivial character of the additive group of F and then write θ_n explicitly in terms of that character.

Let $(k,f,\chi)\in\mathbb{Z}_+\times\mathcal{F}\times(F^\times)^\sim$, where $k+$ ord $(f)=n$. Then the representation $\delta(k,f,\chi)$ has a Whittaker model by Propositions 4. and 5. of [R1]. Therefore

$$\delta(k,f,\chi)|Sp(n,F)$$

is a multiplicity one representation (see for example Proposition 2.8 of [T3]). Thus the length of $\delta(k,f,\chi)|Sp(n,F)$ is

$$2^{\text{card }(\text{supp})f}$$

by Lemma 8.6 and Remarks 8.7, (iii).

Remark 9.1. *The Jacquet module of $\delta(k,f,\chi)$ for a minimal parabolic subgroup has length at least $2^{\text{card }(\text{supp}\,f)}$.*

Now we shall describe a parametrization of irreducible constituents of

$$\delta(k,f,\chi)|Sp(n,F).$$

Part (iv) of Proposition 4.3 implies

$$\delta(k,f,\chi)|Sp(n,F) = \delta(k,f,1_{F^\times})|Sp(n,F).$$

If $k \neq k'$ or $f \neq f'$ then $\delta(k,f,1_{F^\times})|Sp(n,F)$ and $\delta(k',f',1_{F^\times})|Sp(n,F)$ have no composition factor in common (Remark 8.7 (i), Theorem 8.5 (i) and [T3], Corollary 2.5).

For $a \in F^\times$ set

$$\lambda(a) = \begin{bmatrix} I_n & 0 \\ 0 & aI_n \end{bmatrix} \in GSp(n,F).$$

We define a new non degenerate character $(\theta_n)_a$ by

$$(\theta_n)_a(u) = \theta_n(\lambda(a)u\lambda(a)^{-1}).$$

We use observations of Remark 2.9 of [T3] at this point. For each $a \in F^\times$ there exists exactly one irreducible constituent σ of $\delta(k,f,1_{F^\times})|Sp(n,F)$ which has a Whittaker model with respect to $(\theta_n)_a$.

Set

$$X_f^\perp = \left\{x \in F^\times, \chi(x) = 1 \quad \text{for all} \quad \chi \in X_f\right\}.$$

If σ has a Whittaker model with respect to $(\theta_n)_a$ and $a' \in X_f^\perp$, then σ has a Whittaker model with respect to $(\theta_n)_{aa'}$. In this way we obtain a parametrization of irreducible constituents of $\delta(k,f,1_{F^\times})|Sp(n,F)$ by $F^\times/X_f^\perp$ as follows. For $aX_f^\perp \in F^\times/X_f^\perp$, there is a unique irreducible subrepresentation σ of $\delta(k,f,1_{F^\times})|Sp(n,F)$ which has a Whittaker model with respect to $(\theta_n)_a$. We denote

$$\sigma = \delta(k,f,aX_f^\perp).$$

Because of the canonical isomorphism

$$X_f \cong \left(F^\times/X_f^\perp\right)^\wedge,$$

the Pontryagin duality gives a canonical identification

$$\hat{X}_f \cong F^\times/X_f^\perp.$$

Therefore, we have a parametrization of irreducible subrepresentations of

$$\delta(k,f,1_{F^\times})|Sp(n,F)$$

by $\hat{X}_f$. They are denoted by

$$\delta(k,f,\kappa), \kappa \in \hat{X}_f.$$

Now $\delta(k,f,\kappa)$ are irreducible square integrable representations of $Sp(n,F)$.

A character φ of the maximal standard torus in $Sp(n,F)$ will be called weakly regular if it is a restriction of a regular character of the maximal split torus in $Sp(n,F)$. Note that $\varphi = \chi_1 \otimes \ldots \otimes \chi_n \otimes 1$ is weakly regular if and only if conditions (i), (ii) and (iii) of (b), Proposition 8.1, are satisfied, for $\chi_1 \otimes \ldots \otimes \chi_n \otimes 1_{F^\times}$. This shows that if a weakly regular character φ is a restriction of any other character φ', then φ' is regular. Clearly, if φ is regular, then it is weakly regular.

For an enumeration ψ of $\{\chi \in (F^\times)^\wedge;\ \chi^2 = 1_{F^\times},\ \chi \neq 1_{F^\times}\}$ by an initial segment $\{1, 2, \ldots, i\}$ of positive integers, and for $(k, f) \in \mathbb{Z}_+ \times \mathcal{F}$, set

$$\varphi(k,f)_\psi = \nu^k \otimes \nu^{k-1} \otimes \ldots \otimes \nu^1 \otimes \nu^{f(\psi(1))-1}\psi(1) \otimes \ldots \otimes \psi(1) \otimes \ldots$$

$$\ldots \otimes \nu^{f(\psi(r))-1}\psi(r) \otimes \ldots \otimes \psi(r) \otimes 1,$$

and

$$\sigma(k,f) = \nu^k \times \ldots \times \nu^1 \times \nu^{f(\psi(1))-1}\psi(1) \times \ldots \times \psi(r) \rtimes 1.$$

As before, $\sigma(k,f) \in \mathcal{R}[S]$ is well defined and

$$\sigma(k, f, 1_{F^\times})|Sp(n,F) = \sigma(k,f).$$

Remark 9.2. *Characters $\varphi(k,f)_\psi$ are weakly regular. Note that $\varphi(k,f)_\psi$ is regular if and only if $f = 0$ (Proposition 8.1., (a)).*

Theorem 9.3.

(i) *Representations $\delta(k,f,\kappa)$, $k \in \mathbb{Z}_+, f \in \mathcal{F}$, $\kappa \in \hat{X}_f$, are irreducible square integrable representations of $Sp(n,F)$ $(n = k + ord(f))$. We have*

$$\delta(k,f,\kappa) = \delta(k',f',\kappa')$$

if and only if $k = k', f = f'$ and $\kappa = \kappa'$.

(ii) *Representations $\delta(k,f,\kappa)$, $n = k + ord(f)$, appear as subquotients of non-unitary principal series representations of $Sp(n,F)$. If $\delta(k,f,\kappa)$ appears as a subquotient of $\chi_1 \times \ldots \times \chi_n \rtimes 1$, then $\chi_1 \otimes \ldots \otimes \chi_n \otimes 1$ is weakly regular and it is in the orbit of the action of the Weyl group determined by $\varphi(k,f)_\psi$.*

(iii) *If δ is an irreducible square integrable subquotient of some non-unitary principal series representation $\chi_1 \times \ldots \times \chi_n \rtimes 1$ where $\chi_1 \otimes \ldots \otimes \chi_n \otimes 1$ is weakly regular, then there exist $k \in \mathbb{Z}_+$, $f \in \mathcal{F}$ and $\kappa \in \hat{X}_f$, $k + ord(f) = n$, such that*

$$\delta = \delta(k,f,\kappa).$$

Proof. Only (iii) is not proved, but it follows from Theorem 8.5 and Proposition 2.7., (iii) of [T3]. $\square$

10. An example

Recall that the Steinberg representation of $Sp(n,F)$ is a subquotient of

$$\nu^n \times \nu^{n-1} \times \ldots \times \nu^2 \times \nu \rtimes 1.$$

W. Casselman proved that the above non-unitary principal series representation is multiplicity one, and that its length is 2^n (see [BlWh]). It has only one square integrable subquotient. All subquotients different from $1_{Sp(n,F)}$ and $St_{Sp(n,F)}$ are not unitarizable.

We shall give now an example of the non-unitary principal series representation of $Sp(n,F)$, in which square integrable subquotients appear. This representation is in many aspects opposite to the above one. Take a positive integer n and a field F such that the index of the squares $(F^\times)^2$ in $F^\times$ is at least 2^n (if $n \geq 3$ then the residual characteristic must be two). Take different $(\mathbb{Z}/2\mathbb{Z})$-linearly independent characters $\psi_1, \ldots, \psi_n$ of order two. Then the representation

$$\pi = \nu\psi_1 \times \psi_1 \times \nu\psi_2 \times \psi_2 \times \ldots \times \nu\psi_n \times \psi_n \rtimes 1$$

of $Sp(2n,F)$ has exactly 2^n irreducible square integrable subquotients. Their multiplicities are one. The representation

$$L((\nu^{1/2}\psi_1 St_{GL(2)}, \nu^{1/2}\psi_2 St_{GL(2)}, \ldots, \nu^{1/2}\psi_n St_{GL(2)}, 1))$$

has multiplicity 2^n in π. If $n=1$, then we know from [SaT] that all irreducible subquotients of π are unitarizable.

We are going to give a detailed analysis of the case $n=2$. One can apply the same ideas to the above representations when $n \geq 3$.

Take any two different characters ψ_1, ψ_2 of order two. They exist for any field F. Then the representation

$$\pi_1 = \nu\psi_1 \times \psi_1 \times \nu\psi_2 \times \psi_2 \rtimes \nu^{-1}$$

of $GSp(4,F)$ is multiplicity free and of length 16. Exactly one factor is square integrable. We shall write the evident factors. First note that $\psi_1 \rtimes \delta(0, 2\psi_2, \nu^{-1})$ and $\psi_2 \rtimes \delta(0, 2\psi_1, \nu^{-1})$ are irreducible. One gets it easily from the fact

$$s(\psi_1 \otimes \nu\psi_2 \otimes \psi_2 \otimes \nu^{-1}) = s(\nu\psi_2 \otimes \psi_2 \otimes \nu^{-1}),$$

which implies that $\psi_1 \rtimes (\nu\psi_2 \times \psi_2 \rtimes \nu^{-1})$ and $\nu\psi_2 \times \psi_2 \rtimes \nu^{-1}$ are of the same length.

We have the following sixteen factors which are divided into nine groups:

1.	(i)	$\delta(0, 2\psi_1 + 2\psi_2, \nu^{-1})$
2.	(ii)	$L((\nu\psi_1, \psi_1 \rtimes \delta(0, 2\psi_2, \nu^{-1})))$
3.	(iii)	$L((\nu\psi_2, \psi_2 \rtimes \delta(0, 2\psi_1, \nu^{-1})))$
4.	(iv)	$L((\nu^{1/2}\psi_1 St_{GL(2)}, \delta(0, 2\psi_2, \nu^{-1})))$
	(v)	$L((\nu^{1/2}\psi_1 St_{GL(2)}, \delta(0, 2\psi_2, \nu^{-1}\psi_1)))$
5.	(vi)	$L((\nu^{1/2}\psi_2 St_{GL(2)}, \delta(0, 2\psi_1, \nu^{-1})))$
	(vii)	$L((\nu^{1/2}\psi_2 St_{GL(2)}, \delta(0, 2\psi_1, \nu^{-1}\psi_2)))$
6.	$(viii)$	$L((\nu\psi_1, \nu\psi_2, \psi_1 \times \psi_2 \rtimes \nu^{-1}))$
7.	(ix)	$L((\nu\psi_1, \nu^{1/2}\psi_2 St_{GL(2)}, \psi_1 \rtimes \nu^{-1}))$
	(x)	$L((\nu\psi_1, \nu^{1/2}\psi_2 St_{GL(2)}, \psi_1 \rtimes \nu^{-1}\psi_2))$
8.	(xi)	$L((\nu\psi_2, \nu^{1/2}\psi_1 St_{GL(2)}, \psi_2 \rtimes \nu^{-1}))$
	(xii)	$L((\nu\psi_2, \nu^{1/2}\psi_1 St_{GL(2)}, \psi_2 \rtimes \nu^{-1}\psi_1))$
9.	$(xiii)$	$L((\nu^{1/2}\psi_1 St_{GL(2)}, \nu^{1/2}\psi_2 St_{GL(2)}, \nu^{-1}))$
	(xiv)	$L((\nu^{1/2}\psi_1 St_{GL(2)}, \nu^{1/2}\psi_2 St_{GL(2)}, \nu^{-1}\psi_1))$
	(xv)	$L((\nu^{1/2}\psi_1 St_{GL(2)}, \nu^{1/2}\psi_2 St_{GL(2)}, \nu^{-1}\psi_2))$
	(xvi)	$L((\nu^{1/2}\psi_1 St_{GL(2)}, \nu^{1/2}\psi_2 St_{GL(2)}, \nu^{-1}\psi_1\psi_2))$

All of the above representations are different. Therefore, the above representations exhaust all irreducible subquotients of π_1.

We are now going to describe the representation

$$\pi = \nu\psi_1 \times \psi_1 \times \nu\psi_2 \times \psi_2 \rtimes 1$$

of $Sp(4, F)$. First, $\delta(0, 2\psi_1, +2\psi_2, \nu^{-1})|Sp(4, F)$ is a multiplicity one representation. It is of length four. Denote the irreducible constituents by $\delta(0, 2\psi_1 + 2\psi_2)_i$, $i = 1, 2, 3, 4$. Furthermore, $X_{Sp(n,F)}(\psi_1 \rtimes \delta(0, 2\psi_2, \nu^{-1}))$ has four elements and $\psi_1 \rtimes \delta(0, 2\psi_2, \nu^{-1})|Sp(3, F)$ is a multiplicity one representation, since $\psi_1 \rtimes \delta(0, 2\psi_2, \nu^{-1})$ has a Whittaker model by Propositions 4 and 6 of [R1]. Thus $\psi_1 \rtimes \delta(0, 2\psi_2, \nu^{-1})|Sp(3, F)$ is a multiplicity one representation of length four. Denote its irreducible constituents by $T_i^1, i = 1, 2, 3, 4$. Analogously, we introduce $T_i^2, i = 1, 2, 3, 4$, for $\psi_2 \rtimes \delta(0, 2\psi_1, \nu^{-1})$. Irreducible constituents of $\delta(0, 2\psi_i, \sigma)|Sp(2, F)$ were denoted by $\delta(0, 2\psi_i, \pm 1)$.

The representation $\psi_1 \times \psi_2 \rtimes 1$ of $Sp(2, F)$ has length four. It is a multiplicity one representation. We denote irreducible factors by T_i^3, $i = 1, 2, 3, 4$.

The irreducible factors of $\psi_i \rtimes 1$ will be denoted by $T_j^{\psi_i}, j = 1, 2$. Using the sixth section, we get the following list of all irreducible subquotients of π with multiplicities:

	Irreducible Subquotients	Multiplicities
1.	$\delta(0, 2\psi_1 + 2\psi_2)_i, \quad i = 1, 2, 3, 4$	1
2.	$L((\nu\psi_1, T_i^1)), \quad i = 1, 2, 3, 4$	1
3.	$L((\nu\psi_2, T_i^2)), \quad i = 1, 2, 3, 4$	1
4.	$L((\nu^{1/2}\psi_1 St_{GL(2)}, \delta(0, 2\psi_2, \epsilon))), \quad \epsilon \in \{\pm 1\}$	2
5.	$L((\nu^{1/2}\psi_2 St_{GL(2)}, \delta(0, 2\psi_1, \epsilon))), \quad \epsilon \in \{\pm 1\}$	2
6.	$L((\nu\psi_1, \nu\psi_2, T_i^3)), \quad i = 1, 2, 3, 4$	1
7.	$L((\nu\psi_1, \nu^{1/2}\psi_2 St_{GL(2)}, T_i^{\psi_1})), \quad i = 1, 2$	2
8.	$L((\nu\psi_2, \nu^{1/2}\psi_1 St_{GL(2)}, T_i^{\psi_2})), \quad i = 1, 2$	2
9.	$L((\nu^{1/2}\psi_1 St_{GL(2)}, \nu^{1/2}\psi_2 St_{GL(2)}, 1))$	4

Thus the representation

$$\pi = \nu\psi_1 \times \psi_1 \times \nu\psi_2 \times \psi_2 \rtimes 1$$

is of length 36. It has 25 different irreducible subquotients. It has four irreducible square integrable subquotients. They are of multiplicity one. The above example is very different from the $GL(n)$ (or $SL(n)$) case.

References

[Ba] Barbasch, D., *The unitary dual for complex classical groups*, Invent. Math. **96** (1989), 103-176.

[BnDKa] Bernstein, J., Deligne, P. and Kazhdan, D., *Trace Paley-Wiener theorem for reductive p-adic groups*, J. Analyse Math **42** (1986), 180-192.

[BnZ1] Bernstein, I. N. and Zelevinsky, A.V., *Representations of the group $GL(n,F)$, where F is a local non-Archimedean field*, Uspekhi Mat. Nauk. **31** (1976), 5-70.

[BnZ2] Bernstein, J. and Zelevinsky, A.V., *Induced representations of reductive p-adic groups I*, Ann. Sci. École Norm Sup. **10** (1977), 441-472.

[BlWh] Borel, A. and Wallach, N., *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Princeton University Press, Princeton, 1980.

[Bu] Bourbaki, N., *Algébre*, Hermann, Paris, 1947-59.

[Cr] Cartier, P., *Representations of p-adic groups; a survey*, Symp. Pure Math. 33, part 1 ,pp. 111-155, Amer. Math. Soc., Providence, Rhone Island, 1979.

[Cs] Casselman, W., *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint.

[F] Faddeev, D.K, *On multiplication of representations of classical groups over finite field with representations of the full linear group (in Russian)*, Vestnik Leningradskogo Universiteta **13** (1976), 35-40.

[GbKn] Gelbart, S.S. and Knapp, A. W., *L-indistinguishability and R groups for the special linear group*, Advan. in Math. **43** (1982), 101-121.

[GfKa] Gelfand, I.M. and Kazhdan D.A., *Representations of $GL(n,k)$*, Lie Groups and their Representations, Halstead Press, Budapest, 1974, pp. 95-118.

[Go] Goldberg, D., *Reducibility of induced representations for $Sp(2n)$ and $SO(n)$*, preprint.

[Gu] Gustafson, R., *The degenerate principal series for $Sp(2n)$*, Mem. of the Amer. Math. Society **248** (1981), 1-81.

[Jc] Jacquet, H, *Generic representations*, Non-Comutative Harmonic Analysis, Lecture Notes in Math. 587, Spring Verlag, Berlin, 1977, pp. 91-101.

[Jn] Jantzen, C., *Degenerate principal series for symplectic groups*, to appear in Mem. of the Amer. Math. Society.

[KaL] Kazhdan, D. and Lusztig, G., *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), 153-215.

[Ke] Keys, D., *On the decomposition of reducible principal series representations of p-adic Chevalley groups*, Pacific J. Math **101** (1982), 351-388.

[M] Moy, A., *Representations of GSp(4) over a p-adic field: parts 1 and 2*, Compositio Math. **66** (1988), 237-328.

[R1] Rodier, F., *Décomposition de la série principale des groupes réductifs p-adiques*, tative Harmonic Analysis, Lecture Notes in Math. 880, Springer-Verlag, Berlin, 1981.

[R2] Rodier, F., *Sur les représentations non ramifiées des groupes réductifs p-adiques; l'example de GSp(4)*, Bull. Soc. Math. France **116** (1988), 15-42.

[SaT] Sally, P.J. and Tadić, M., *Induced representations and classifications for* $GSp(2,F)$ *and* $Sp(2,F)$, to appear in Mémoires Soc. Math. France.

[Sh1] Shahidi, F., *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. **132** (1990), 273-330.

[Si1] Silberger, A., *Isogeny restrictions of irreducible representations are finite direct sums of irreducible admissible representations*, Proceedings Amer. Math. Soc. **73** (1979), 263-264.

[Si2] Silberger, A., *Discrete series and classifications for p-adic groups I*, Amer. J. Math. **103** (1981), 1231 - 1321.

[T1] Tadić, M., *Classification of unitary representations in irreducible representations of general linear group (non-archimedean case)*, Ann. Sci. École Norm. Sup **19** (1986), 335-382.

[T2] Tadić, M., *Induced representations of* $GL(n,A)$ *for p-adic division algebras* A, J. reine angew. Math. **405** (1990), 48-77.

[T3] Tadić, M., *Notes on representations of non-archimedian* $SL(n)$, Pacific J. Math. **152** (1992), 375-396.

[T4] Tadić, M., *On Jacquet modules of induced representations of p-adic symplectic groups*, Harmonic Analysis on Reductive Groups, Proceedings, Bowdoin College 1989, Progress in Mathematics 101, Birkhäuser, Boston, 1991, pp. 305-314.

[T5] Tadić, M., *A structure arising from induction and restriction of representations of classical p-adic groups*, preprint.

[Wd] Waldspurger, J.-L., *Un exercice sur* $GSp(4,F)$ *et les représentations de Weil*, Bull. Soc. Math. France **115** (1987), 35-69.

[Wi] Winarsky, N., *Reducibility of principal series representations of p-adic Chevalley groups*, Amer. J. Math. **100** (1978), 941-956.

[Z1] Zelevinsky, A.V., *Induced representations of reductive p-adic groups II, On irreducible representations of* $GL(n)$, Ann. Sci École Norm Sup. **13** (1980), 165-210.

[Z2] Zelevinsky A. V., *Representations of Finite Classical Groups, A Hopf Algebra Approach*, Lecture Notes in Math 869, Springer-Verlag, Berlin, 1981.

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