Mathematischen Annalen vol. 305, 1996, pages 419-438

# CORRESPONDENCE ON CHARACTERS OF IRREDUCIBLE UNITARY REPRESENTATIONS OF $GL(n,\mathbb{C})$

Marko Tadić

**Running head: Correspondence on characters**
**Mathematics Subject Classification (MOS) number: 22E46**

## Introduction

Shimura's work on modular forms motivates consideration of a correspondence between irreducible representations of global $GL(r)$ and its $n$-fold cover if the ground field contains the $n$-th roots of unity (see [KP1], [KP2]). This correspondence should satisfy certain character identity. The main tool for getting such correspondences is the trace formula which compares traces of two unitary representations. This global formula enables one to prove a local correspondence of analogous type in a number of cases and suggests that there should be in general a correspondence of such a type on all local levels.

We now restrict our attention to the local case of $GL(n,\mathbb{C})$ (recall that if $n > 2$, then all local archimedean factors are complex). Denote by $D$ (resp. $Z$) the subgroup of all diagonal matrices in $GL(n,\mathbb{C})$ (resp. the center of $GL(n,\mathbb{C})$). If a character $\psi$ of $\mathbb{C}^\times$ is $n$-th power of some character $\phi$, then such $\phi$ is unique and we shall denote it by $\psi^{1/n}$. We can now describe the condition which the correspondence should satisfy in the complex case. Let $\Delta$ be the Weyl factor (see the second section). For an irreducible representation $\pi$ with the character $\Theta_\pi$ whose central character $\omega_\pi$ is $n$-th power of a character, D.A. Kazhdan and S.J. Patterson defined a function $\widetilde{\Theta}_\pi$ on regular semi-simple elements of $GL(r,\mathbb{C})$, constant on conjugacy classes, defined by

$$(\Delta\widetilde{\Theta}_\pi)(h) = n^{-(r-1)} \sum_{\substack{\eta Z \in D/Z;\\ \eta\in D \text{ and } z\in Z\\ \text{such that } \eta^n z = h}} (\Delta\Theta_\pi)(\eta)\omega_\pi^{1/n}(z)$$

on regular elements of $D$ (we could assume only that $\pi$ is of finite length). The function $\widetilde{\Theta}_\pi$ is a virtual character. We shall denote by $\pi^\#$ the virtual representation whose (formal) character is $\widetilde{\Theta}_\pi$ (by a virtual representation we shall mean an integral linear combination of irreducible representations). We extend $^\#$ to a mapping $\#$ on representations whose central characters can be arbitrary: $\#(\pi) = 0$ if the central character of $\pi$ is not $n$-th power of some character (see the second section). S.J. Patterson obtained a formula for the action of $^\#$ on the non-unitary principal series representations. He proved in a number of cases that for irreducible $\pi$, $\widetilde{\Theta}_\pi$ is either 0, or it is a character of some irreducible representation, up to a sign. He conjectured that this holds in general.

In this paper we consider irreducible unitary representations (this class of representations is most closely related with the global case). We show in this paper that irreducible unitary representations of $GL(r,\mathbb{C})$ behaves well with respect to this correspondence. More precisely, we prove the following

**Theorem A.** *If $\pi$ is an irreducible unitary representation of $GL(r,\mathbb{C})$, then $\#(\pi)$ is either 0 or an irreducible unitary representation, up to a sign. If additionally $\pi$ is spherical, then $\pi^{\#}$ is always a spherical irreducible unitary representation.*

Moreover, we obtain an explicit formula for the action of $^{\#}$ on the unitary dual $\widehat{GL(n,\mathbb{C})}$ of $GL(r,\mathbb{C})$ (Theorem 4.5). We now describe that formula. If $\pi_1$ and $\pi_2$ are representations of $GL(r_1,\mathbb{C})$ and $GL(r_2,\mathbb{C})$, then $\pi_1\times\pi_2$ shall denote parabolically induced representation of $GL(r_1+r_2,\mathbb{C})$ by $\pi_1\otimes\pi_2$ (see the first section). For an irreducible representation $\pi$ of $GL(r,\mathbb{C})$ and a non-negative integer $k$ define

$$\mathrm{string}(k,n,\pi)=(|\mathrm{det}|^{\frac{-(k-1)}{n}}\pi)\times(|\mathrm{det}|^{\frac{-(k-1)+2}{n}}\pi)\times(|\mathrm{det}|^{\frac{-(k-1)+4}{n}}\pi)\times\cdots\times(|\mathrm{det}|^{\frac{(k-1)}{n}}\pi)$$

(we take $\mathrm{string}(0,n,\pi)$ to be the trivial representation of $GL(0,\mathbb{C})$). For $d\in\mathbb{Z}$ we define the unitary character $t_d$ of $\mathbb{C}^\times$ by $t_d(z)=z/|z|$. The trivial (one dimensional representation) of a group $G$ will be denoted by $1_G$.

**Theorem B.**
*(i) If $\pi_1,\dots,\pi_k$ are irreducible representations of $GL(r_1,\mathbb{C}),\dots,GL(r_k,\mathbb{C})$, then*

$$\#(\pi_1\times\cdots\times\pi_k)=\#(\pi_1)\times\cdots\times\#(\pi_k).$$

*(ii) Let $\phi$ be a character of $\mathbb{C}^\times$ trivial on $\{z\in\mathbb{C};|z|=1\}$. Then $\#([(\phi\ t_d)\circ det]\ 1_{GL(r)})\neq 0$ if and only if $n|d$ or $n|r$. Write $r=pn+q$ and $d=vn+d'$ with $p,q,v,d'\in\mathbb{Z}$ such that $0\le q\le n-1$, $0\le d'\le n-1$. Then*

$$\begin{aligned}&([(\phi\ t_d)\circ \mathit{det}]\ 1_{GL(r)}))^{\#}\\&\qquad=[(\phi^{1/n}\ t_{d/n})\circ \mathit{det}]\ \big(\mathit{string}(q,n,1_{GL(p+1)})\times \mathit{string}(n-q,n,1_{GL(p)})\big)\ \text{ if } n|d;\end{aligned}$$

$$\begin{aligned}&([(\phi\ t_d)\circ \mathit{det}]\ 1_{GL(r)}))^{\#}=(-1)^{\frac{r}{n}(n-d')d'}[\phi^{1/n}\circ \mathit{det}]\\&\qquad\big(\mathit{string}(d',n,(t_{v+1}\circ \mathit{det})\ 1_{GL(r/n)})\times \mathit{string}(n-d',n,(t_v\circ \mathit{det})\ 1_{GL(r/n)})\big)\ \text{ if } n|r.\end{aligned}$$

Since each irreducible unitary representation is parabolically induced from a one-dimensional representation, the above theorem provides an explicit formula for the action of $^{\#}$ on the irreducible unitary representations.

Now we shall give few examples of the action of $^{\#}$, from which one can get better idea about the correspondence. We shall consider the simplest case, the case of $n=2$ in the following examples:

$$\begin{aligned}(1_{GL(2k,\mathbb{C})})^{\#}&=(|\mathrm{det}|^{-1/2}\,1_{GL(k,\mathbb{C})})\times(|\mathrm{det}|^{1/2}\,1_{GL(k,\mathbb{C})}),\\(1_{GL(2k+1,\mathbb{C})})^{\#}&=1_{GL(k+1,\mathbb{C})}\times 1_{GL(k,\mathbb{C})},\\((t_1\circ\mathrm{det})\,1_{GL(2k,\mathbb{C})})^{\#}&=(-1)^k\ 1_{GL(k,\mathbb{C})}\times((t_1\circ\mathrm{det})\,1_{GL(k,\mathbb{C})}).\end{aligned}$$

The correspondence $^\#$ can be introduced in a very simple way, which shows the interest of this correspondence for representation theory. Let $R$ be the sum of groups of all virtual representations of all $GL(n, \mathbb{C})$. Then $R$ is in a natural way a ring (see the first section; multiplication in $R$ is defined using parabolic induction). The ring $R$ is a polynomial ring over all characters $\chi$ of $\mathbb{C}^\times$. Now the correspondence $^\#$ can be characterized as a restriction of the ring endomorphism of $R$ which sends a character $\chi$ of $\mathbb{C}^\times$ to $\chi^{1/n}$ if $\chi$ is an $n$-the power of a character, and to 0 otherwise (essentially, this is just the Patterson's description of the action of $^\#$ on the non-unitary principal series). Clearly, it is interesting to ask what happens with irreducible (unitary) representation under such purely algebraically defined map.

Note that a number of papers consider local metaplectic correspondences (see references for some of these papers).

The paper is organized into four sections. In the first section we introduce basic notation and recall basic facts regarding the non-unitary and the unitary dual of $GL(r, \mathbb{C})$. We introduce algebra $R$ and recall its principal properties. The second section introduces the correspondence $^\#$ and considers its basic properties with respect to the algebraic structure of $R$. The third section is the computation of $(1_{GL(r,\mathbb{C})})^\#$. In the first part of the fourth section we get the explicit formula for $\pi^\#$ when $\pi$ is an irreducible unitary representation of $GL(r, \mathbb{C})$. The main tool in our computation of the explicit formula is the algebra $R$ and its algebraic structure, and the Zuckerman's formula for the character of the trivial representation, in the case of $GL(r, \mathbb{C})$. We are thankful to D. Vogan for explaining us the Zuckerman's formula in this setting. In the second part of the section we show from the explicit formula for $^\#$, that $^\#$ carries always an irreducible unitary representation to 0 or to an irreducible unitary representation, up to a sign.

This work came as a consequence of conversations with S.J. Patterson. We are thankful to him for turning our attention to this problem. We are also thankful to him for a number of interesting discussions regarding metaplectic correspondences, and for comments regarding this paper. He noticed an interesting similarity of the correspondence with the Adams operations on Grothendieck groups (see Corollary 7.6 and Proposition 7.4 of [Br tD]).

This work was done during our stay in Göttingen as a guest of SFB 170. We want to thank SFB 170 for their kind hospitality and support.

## 1. Preliminaries

In this section we shall introduce notation, and briefly recall of the facts about $GL(r, \mathbb{C})$ which we shall use in the paper. One can find in [T3] more explanation about notation, and also references for the facts that we quote here. Since in this paper we shall consider general linear groups over the complex field only, we shall denote $GL(r, \mathbb{C})$ simply by $GL(r)$.

The square of the usual absolute value $|\ \ |$ on $\mathbb{C}$ will be denoted by $|\ \ |_{\mathbb{C}}$. We identify characters of $\mathbb{C}^\times$ with characters of $GL(r)$, using the determinant homomorphism. The character of $GL(r)$ corresponding to $|\ \ |_{\mathbb{C}}$ will be denoted by $\nu$.

Let $\mathfrak{g}_r = \mathfrak{gl}(r, \mathbb{C})$. Denote by $K_r$ the maximal compact subgroup $U(r)$ of all unitary matrices in $GL(r)$. Denote by $\widetilde{GL(r)}$ the set of all equivalence classes of irreducible (admissible) $(\mathfrak{g}_r, K_r)$-modules. Let $R_r$ be the free abelian group over the basis $\widetilde{GL(r)}$ (it is the Grothendieck group of the category of all $(\mathfrak{g}_r, K_r)$-modules of finite length). For a $(\mathfrak{g}_r, K_r)$-module $\pi$ of finite length, we denote the semi simplification of $\pi$ by s.s.$(\pi)$. We shall consider s.s.$(\pi)$ as an element of $R_r$.

For admissible finite length $(\mathfrak{g}_{r_i}, K_{r_i})$-modules $\pi_i$ $(i = 1, 2)$, we consider $\pi_1 \otimes \pi_2$ as a $(\mathfrak{g}_{r_1} \times \mathfrak{g}_{r_i}, K_{r_1} \times K_{r_2})$-module.We identify $G_{r_1} \times G_{r_2}$ with a Levi factor of the following parabolic subgroup

$$\left\{ \begin{bmatrix} g_1 & * \\ 0 & g_2 \end{bmatrix} ; g_i \in G_{r_i} \right\}$$

in an obvious way. The $(\mathfrak{g}_{r_1+r_2}, K_{r_1+r_2})$-module parabolically induced by $\pi_1 \otimes \pi_2$ from the above parabolic subgroup is denoted by $\pi_1 \times \pi_2$. Let $R = \oplus_{r \geq 0} R_r$. Lift $\times$ to a $\mathbb{Z}$-bilinear mapping $\times : R \times R \to R$. With this multiplication $R$ is an associative commutative graded ring (this is the same ring considered in [T2]). The ring $R$ is a polynomial ring over $\widetilde{\mathbb{C}^\times}$.

For a fixed character $\chi$ of $\mathbb{C}^\times$, lift $\pi \mapsto \chi\pi$, $\pi \in \widetilde{GL(r)}, r \geq 0$, to a $\mathbb{Z}$-linear map on $R$ which we shall denote again by

$$\chi : R \to R. \tag{1-1}$$

Then the map $\chi$ on $R$ is an automorphism of the graded ring.

The set of all unitarizable classes in $\widetilde{GL(r)}$ will be denoted by $\widehat{GL(r)}$. This set is in a natural bijection with the set of all equivalence classes of irreducible unitary representations of $GL(r)$, i.e. with the unitary dual of $GL(r)$. Therefore, we shall consider $\widehat{GL(r)}$ also as the set of all equivalence classes of irreducible unitary representations of $GL(r)$.

Now we shall write the unitary dual of $GL(r)$ (see [T1] or [V2]; in [T1] proof uses a result announced by A.A. Kirillov for which there is no published proof, our notation here follows [T1] ). The trivial representation of a group $G$ on one dimensional complex vector space will be denoted by $1_G$.

**1.1. Theorem.** *Set*

$$B = \{\chi \, 1_{GL(r)}, \;\; [(\nu^\alpha \chi) \, 1_{GL(r)}] \times [(\nu^{-\alpha} \chi) \, 1_{GL(r)}], \;\; \chi \in \widehat{\mathbb{C}^\times}, \;\; r \geq 1, \;\; 0 < \alpha < 1/2\}.$$

*Then*

(i) *If* $\sigma_1, \dots, \sigma_k \in B$, *then* $\sigma_1 \times \ldots \times \sigma_k \in \widehat{GL(p)}$ *for some* $p$.

(ii) *If* $\pi \in \widehat{GL(p)}$, *then there exist* $\sigma_1, \dots, \sigma_m \in B$, *unique up to a permutation, such that*

$$\pi \cong \sigma_1 \times \ldots \times \sigma_m. \quad \square$$

Note that $\widehat{\mathbb{C}^\times}$ is the group of all unitary characters of $\mathbb{C}^\times$. Denote the determinant homomorphism of $GL(r)$ by $\det_r : GL(r) \to \mathbb{C}^\times$. Then $\chi \, 1_{GL(r)} = \chi \circ \det_r$ and $[(\nu^\alpha \chi) \, 1_{GL(r)}] \times [(\nu^{-\alpha} \chi) \, 1_{GL(r)}] = [(\nu^\alpha \chi) \circ \det_r] \times [(\nu^{-\alpha} \chi) \circ \det_r]$.

We shall now fix two parameterizations of characters of $\mathbb{C}^\times$. First note that $\widetilde{\mathbb{C}^\times}$ is the group of all (not necessarily unitary) characters of $\mathbb{C}^\times$. Denote $\mathbb{T} = \{z \in \mathbb{C}; |z|_{\mathbb{C}} = 1\}$.

Then $\mathbb{C} \cong \mathbb{T} \times \mathbb{R}_+^\times$, and further $\mathbb{C}^\times \cong \mathbb{T} \times \mathbb{R}$. One can write a character $\chi$ of $\mathbb{C}^\times$ uniquely as $\chi(z) = (z/|z|)^d \phi(z)$ with $d \in \mathbb{Z}$ and $\phi \in \widetilde{\mathbb{C}/\mathbb{T}}$. This character will be denoted by $\chi(d)^{(\phi)}$. Thus

$$\chi(d)^{(\phi)}(z) = (z/|z|)^d \phi(z), \ d \in \mathbb{Z}, \ \phi \in \widetilde{\mathbb{C}/\mathbb{T}}. \tag{1-2}$$

We have

$$\chi(d)^{(\phi)} \chi(d')^{(\phi')} = \chi(d+d')^{(\phi\phi')}.$$

There exists a unique $\beta \in \mathbb{C}$ so that $\phi = \nu^\beta$.

If $x, y \in \mathbb{C}$ and $x - y \in \mathbb{Z}$, then the following formula defines a character $\gamma(x,y)$ of $\mathbb{C}^\times$:

$$\gamma(x,y)(z) = (z/|z|)^{x-y} |z|^{x+y} = \chi(x-y)^{(\nu^{(x+y)/2})}. \tag{1-3}$$

Each character can be written as some $\gamma(x,y)$, and $\gamma(x,y) = \gamma(x',y')$ implies $x = x'$ and $y = y'$ We collect now same evident properties of this parameterization of $\widetilde{\mathbb{C}^\times}$:

$$\chi(d)^{(\nu^\beta)} = \gamma(\beta + d/2, \beta - d/2), \qquad d \in \mathbb{Z}, \beta \in \mathbb{C}, \tag{1-4}$$

$$\gamma(x,y)\gamma(x',y') = \gamma(x+x', y+y'), \ x, y, x', y' \in \mathbb{C}, x-y, x'-y' \in \mathbb{Z}, \tag{1-5}$$

$$\gamma(d/2, -d/2)(z) = (z/|z|)^d = \chi(d)^{(\nu^0)}(z), \quad d \in \mathbb{Z}, z \in \mathbb{C}, \tag{1-6}$$

$$\gamma(\beta, \beta) = \nu^\beta, \qquad \beta \in \mathbb{C}, \tag{1-7}$$

$$\gamma(p,q)(z) = z^p \bar{z}^q, \qquad p, q \in \mathbb{Z}, z \in \mathbb{C}. \tag{1-8}$$

Now we shall say a few words about the non-unitary dual. Let $M(\mathbb{C}^\times)$ be the set of all finite multisets in $\widetilde{C^\times}$. Take $d = (\chi_1, \chi_2, \ldots, \chi_k) \in M(D)$. Write $\chi_i = \chi(d_i)^{(\nu^{\beta_i})}$. Chose a permutation $p$ of $\{1, 2, \ldots, k\}$ such that $\mathrm{Re}(\beta_{p(1)}) \geq \mathrm{Re}(\beta_{p(2)}) \geq \cdots \geq \mathrm{Re}(\beta_{p(k)})$. The $(\mathfrak{g}_k, K_k)$-module $\chi_{p(1)} \times \chi_{p(2)} \times \cdots \times \chi_{p(k)}$ has a unique irreducible quotient whose class depends only on $d$, not on $p$. Denote this irreducible quotient by $L(d)$. This is one way to express the Langlands classification for the group $GL(k, \mathbb{C})$ ([BoW], [J]). One gets directly that

$$\chi L(\chi_1, \chi_2, \ldots, \chi_k) \cong L(\chi\chi_1, \chi\chi_2, \ldots, \chi\chi_k) \tag{1-9}$$

for a character $\chi$ of $F^\times$. We shall use the following property often

(1-10) $L(\chi_1, \chi_2, \ldots, \chi_k, \chi_1', \chi_2', \ldots, \chi_{k'}')$ is a subquotient of $L(\chi_1, \chi_2, \ldots, \chi_k) \times L(\chi_1', \chi_2', \ldots, \chi_{k'}')$.

Denote by $W_r$ the group of all permutations of $\{1, 2, \ldots, r\}$. The parity of a permutation $w \in W_r$ will be denoted by $\mathrm{sgn}(w)$.

We write now a Zuckerman's formula in the case of $GL(r)$ (Proposition 9.4.16 of [V1], see also [T3]). We have in $R$

$$1_{\mathrm{GL}(r)} = \sum_{w \in W_r} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{r} \gamma(i - (r+1)/2, w(i) - (r+1)/2). \tag{1-11}$$

From this one obtains that for $x, y \in \mathbb{C}$ such that $x - y \in \mathbb{Z}$, we have

$$\text{(1-12)} \qquad \gamma(x,y)\gamma((1+r)/2,(1+r)/2)\, 1_{GL(r)} = \sum_{w \in W_r} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{r} \gamma(x+i, y+w(i)).$$

For us it will be useful the following slight modification of the above formulas. Denote by $W_r'$ the group of all permutations of the set $\{0, 1, 2, \dots, r-1\}$. Then

$$\text{(1-13)} \quad \gamma(x,y)\gamma((r-1)/2,(r-1)/2)\, 1_{GL(r)} = \sum_{w \in W_r'} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \gamma(x+i, y+w(i)),$$

$$\text{(1-14)} \qquad 1_{GL(r)} = \gamma(-(r-1)/2, -(r-1)/2) \sum_{w \in W_r'} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \gamma(i, w(i)).$$

## 2. Correspondence

Let $n$ be an integer $\geq 2$. If $\psi$ is a character of $\mathbb{C}^\times$ such that there exists a character $\phi$ of $\mathbb{C}^\times$ so that $\phi^n = \psi$, then $\phi$ is uniquely determined by $\psi$. We shall say that $\psi$ has $n$-th root character and denote $\phi$ by $\psi^{1/n}$. The map $\phi \mapsto \phi^n$ is a monomorphism on $\widetilde{\mathbb{C}^\times}$. The image of this monomorphism will be denoted by $\widetilde{\mathbb{C}^\times}^{\,n}$. In our parameterizations of characters introduced in the last section, $\chi(d)^{(\nu^\beta)}$ has an $n$-th root character if and only if $n|d$. If this is the case,

$$(\chi(d)^{(\nu^\beta)})^{1/n} = \chi(d/n)^{(\nu^{\beta/n})}.$$

Further, $\gamma(x,y)$ has an $n$-th root character if and only if $n|(x-y)$. If this is the case, then

$$\gamma(x,y)^{1/n} = \gamma(x/n, y/n).$$

We shall identify $\mathbb{C}^\times$ with the center of $GL(r)$ in the standard way. For a character $\psi$ of $\mathbb{C}^\times$, denote by $\widetilde{GL(r)}_\psi$ the set of all equivalence classes of representations in $\widetilde{GL(r)}$ which have the central character equal to $\psi$. Denote by $R_r(\psi)$ the subgroup of $R_r$ generated by $\widetilde{GL(r)}_\psi$. If $\pi_1$ has the central character $\psi_1$ and $\pi_2$ has $\psi_2$, then $\pi_1 \times \pi_2$ has the central character $\psi_1\psi_2$. Therefore

$$\text{(2-1)} \qquad R_{r_1}(\psi_1) \times R_{r_2}(\psi_2) \subseteq R_{r_1+r_2}(\psi_1\psi_2).$$

The set of all monomials $\chi_1 \times \chi_2 \times \cdots \times \chi_r$, $\chi_i \in \widetilde{\mathbb{C}^\times}$ form a basis of the free abelian group $R_r$. The central character of $\chi_1 \times \chi_2 \times \cdots \times \chi_r$ is $\chi_1\chi_2 \dots \chi_r$. Now we have one obvious

**2.1. Lemma.**
*(i)*

$$R_r = \bigoplus_{\psi \in (\mathbb{C}^\times)^\sim} R_r(\psi).$$

*(ii) The group* $R_r(\psi)$ *is a free abelian group over all monomials* $\chi_1 \times \chi_2 \times \cdots \times \chi_r$, $\chi_i \in \widetilde{\mathbb{C}^\times}$, *such that* $\chi_1\chi_2 \dots \chi_r = \psi$. ☐

Denote by $\widetilde{GL(r)}^{(n)}$ the set of all $\pi \in \widetilde{GL(r)}$ such that the central character of $\pi$ is in $\widetilde{\mathbb{C}^\times}^n$. Denote by $R_r(n)$ the subgroup of $R_r$ generated $\widetilde{GL(r)}^{(n)}$. Obviously,

$$R_r(n) = \underset{\psi \in (\mathbb{C}^\times)^{\sim}}{\oplus} R_r(\psi^n).$$

From the above relation and (2-1) we conclude that $R(n) = \underset{r \geq 0}{\oplus} R_r(n)$ is a subring of $R$.

There exists a unique ring homomorphism $\# : R \to R$ which satisfies

$$\#(\chi) = \begin{cases} \chi^{1/n}, & \text{for} \quad \chi \in \widetilde{\mathbb{C}^\times}^n; \\ 0, & \text{for} \quad \chi \in \widetilde{\mathbb{C}^\times} \backslash \widetilde{\mathbb{C}^\times}^n. \end{cases} \tag{2-2}$$

Note that then for $\chi_i \in \widetilde{\mathbb{C}^\times}$ we have

$$\#(\chi_1 \times \chi_2 \times \cdots \times \chi_r) = \begin{cases} \chi_1^{1/n} \times \chi_2^{1/n} \times \cdots \times \chi_r^{1/n}, & \text{if all} \quad \chi_i \in \widetilde{\mathbb{C}^\times}^n; \\ 0, & \text{if some} \quad \chi_i \in \widetilde{\mathbb{C}^\times} \backslash \widetilde{\mathbb{C}^\times}^n. \end{cases} \tag{2-3}$$

One gets directly looking at the central character:

**2.2. Lemma.** *We have* $\#(\pi) = 0$ *for* $\pi \in \widetilde{GL(r)} \backslash \widetilde{GL(r)}^{(n)}$. ☐

Let $\chi \in \widetilde{\mathbb{C}^\times}$. Consider $\chi$ as a ring automorphism of $R$ (see the first section). Note that

$$\chi(R(n)) \subseteq R(n) \quad \text{if} \quad \chi \in \widetilde{\mathbb{C}^\times}^n. \tag{2-4}$$

One checks directly that

$$\# \circ \chi = \chi^{1/n} \circ \# \quad \text{if} \quad \chi \in \widetilde{\mathbb{C}^\times}^n. \tag{2-5}$$

We denote the restriction $\#|R(n)$ by ${}^{\#}$. Thus

$${}^{\#} : R(n) \to R.$$

Now (2-4) implies that (2-5) holds also for ${}^{\#}$:

**2.3. Lemma.**

$${}^{\#} \circ \chi = \chi^{1/n} \circ {}^{\#} \quad \textit{if} \quad \chi \in \widetilde{\mathbb{C}^\times}^n. \tag{2-6}$$

Denote by ${}^{\sim} : R \to R$ the contragredient automorphism (this is a ring automorphism). Then obviously this automorphism preserves $R(n)$ and commutes with ${}^{\#}$ (see the action on the polynomial generators).

The group of all diagonal matrices in $GL(r)$ will be denoted by $D$. The center of $GL(r)$ will be denoted by $Z$. Denote by $\text{diag}(\eta_1,\eta_2,\dots,\eta_r)$ the diagonal matrix which has elements $\eta_1,\eta_2,\dots,\eta_r$ on the diagonal. Let $\Delta$ be the Weyl factor. This is a function on regular semi simple elements of $GL(r)$, constant on conjugacy classes, defined by

$$\Delta(\text{diag}(\eta_1,\eta_2,\dots,\eta_r)) = \prod_{1\le i<j\le r} (|\eta_i-\eta_j|_{\mathbb{C}}/|\eta_i\eta_j|_{\mathbb{C}}^{1/2}) = \prod_{1\le i<j\le r} (|\eta_i-\eta_j|^2/|\eta_i\eta_j|)$$

on $D$ (more precisely, on regular elements of $D$). Note that $\Delta$ is constant on classes mod $Z$, on which it is defined.

Let $\pi$ be a $(\mathfrak{g}_r,K_r)$-module of finite length. Suppose that $\pi$ has a central character. Denote by $\Theta_\pi$ (resp. $\omega_\pi$) the character (resp. the central character) of $\pi$. Suppose that $\omega_\pi\in\widetilde{\mathbb{C}^\times}^n$. D.A. Kazhdan and S.J. Patterson introduced the function $\widetilde{\Theta}_\pi$ on regular semi simple elements of $GL(r)$, constant on conjugacy classes, defined by

$$(\Delta\widetilde{\Theta}_\pi)(h) = n^{-(r-1)} \sum_{\substack{\eta Z\in D/Z;\\ \eta\in D \text{ and } z\in Z\\ \text{such that } \eta^n z=h}} (\Delta\Theta_\pi)(\eta)\omega_\pi^{1/n}(z)$$

on $D$. Then $\widetilde{\Theta}_\pi$ is a virtual character. S.J. Patterson proved in a number of cases that if $\pi$ is irreducible, then $\widetilde{\Theta}_\pi$ is either 0, or it is a character of some irreducible $(\mathfrak{g}_r,K_r)$-module, up to a sign. He conjectured that this holds in general for irreducible $\pi$. He showed that for characters $\chi_1,\chi_2,\dots,\chi_r$ of $\mathbb{C}^\times$ for which $\chi_1\chi_2\dots\chi_r\in\widetilde{\mathbb{C}^\times}^n$,

$$\widetilde{\Theta}\chi_1\times\chi_2\times\dots\times\chi_r = \Theta\chi_1^{1/n}\times\chi_2^{1/n}\times\dots\times\chi_r^{1/n}$$

if all $\chi_i\in\widetilde{\mathbb{C}^\times}^n$, and 0 otherwise. This follows directly from the character formula

$$\begin{aligned}&\Theta\chi_1\times\chi_2\times\dots\times\chi_r(\text{diag}(\eta_1,\eta_2,\dots,\eta_r))\\ &\qquad = \Delta(\text{diag}(\eta_1,\eta_2,\dots,\eta_r))^{-1}\sum_{w\in W_r}\chi_1(\eta_{w(1)})\chi_2(\eta_{w(2)})\dots\chi_r(\eta_{w(r)}).\end{aligned}$$

Since $R$ is a polynomial algebra over $\widetilde{\mathbb{C}^\times}$, we have

$$\widetilde{\Theta}_\pi = \Theta_{\pi^\#}$$

for $\pi\in\widetilde{GL(r)}^{(n)}$. Here we extend $\Theta$ to a function on whole $R$ in an obvious way: $\Theta_{\sum k_i\pi_i} = \sum k_i\Theta_{\pi_i}$ for $\pi_i$ irreducible $(\mathfrak{g}_{r_i},K_{r_i})$-modules and $k_i\in\mathbb{Z}$.

Let us note that this is only the complex case of the the general philosophy.

## 3. Trivial representation

Let $\pi$ be an irreducible $(\mathfrak{g}_r,K_r)$-module and $k$ a non-negative integer. If $k>0$, we define $\text{string}(k,n,\pi)\in R$ by the formula

$$\text{string}(k,n,\pi) = \nu^{\frac{-(k-1)/2}{n}}\pi\times\nu^{\frac{-(k-1)/2+1}{n}}\pi\times\dots\times\nu^{\frac{(k-1)/2}{n}}\pi.$$

We take $\text{string}(0,n,\pi)$ to be the trivial representation of $GL(0)$, i.e. identity of $R$. Note that for a character $\chi$ of $\mathbb{C}^\times$ we have $\chi(\text{string}(k,n,\pi)) = \text{string}(k,n,\chi\pi)$

One proves directly the following simple technical lemma which will enable us to decompose a sum in the proof of the next proposition. We assume $n,r\ge 2$.

**3.1. Lemma.** *Write $r = pn + q$, with $p, q \in \mathbb{Z}$ such that $0 \leq q \leq n - 1$.*
*(i) Let*

$$W'_r(n) = \{w \in W'_r; n|(w(i) - i) \text{ for all } 0 \leq i \leq r - 1\}.$$

*For $0 \leq \ell \leq min(n, r) - 1$ denote by*

$$W'_r(n; \ell) = \{w \in W'_r; w(i) = i \text{ if } n \nmid (i - \ell)\}.$$

*Then $W'_r(n)$ is a subgroup of $W'_r$, $W'_r(n; \ell)$ are subgroups of $W'_r(n)$ and $W'_r(n)$ is a direct product of $W'_r(n; \ell)$, $\ell = 0, 1, 2, \ldots, min(n, r) - 1$.*
*(ii) Let $0 \leq \ell \leq q - 1$ (resp. $q \leq \ell \leq min(n, r) - 1$). For $w \in W'_{p+1}$ (resp. $w \in W'_p$) define $w^* \in W'_r$ by*

$$w^*(j) = \begin{cases} j, & \text{if } \ n \nmid (i - \ell); \\ \ell + nw(i), & \text{if } \ j = \ell + ni. \end{cases}$$

*Then $w \mapsto w^*$ is an isomorphism of $W'_{p+1}$ (resp. $W'_p$) onto $W'_r(n; \ell)$. Further, $sgn(w) = sgn(w^*)$.* $\square$

**3.2. Proposition.** *Write $r = pn + q$, with $p, q \in \mathbb{Z}$ such that $0 \leq q < n$. Then we have*

$$(1_{GL(r)})^{\#} = string(q, n, 1_{GL(p+1)}) \times string(n - q, n, 1_{GL(p)}). \quad \square$$

*Proof.* From (1-14) we get

$$(1_{GL(r)})^{\#} = \gamma(-(r-1)/2, -(r-1)/2)^{1/n} \sum_{w \in W'_r} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \#(\gamma(i, w(i))).$$

We shall use now Lemma 3.1, to decompose the sum over $W'_r(n)$ in our calculation of $(1_{GL(r)})^{\#}$. From the above formula we have

$$\gamma((r-1)/2n, (r-1)/2n) \ \ (1_{GL(r)})^{\#} = \sum_{w \in W'_r} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \#(\gamma(i, w(i)))$$

$$= \sum_{w \in W'_r(n)} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \gamma(i, w(i))^{\#} = \sum_{w_0, w_1, \ldots, w_{q-1} \in W'_{p+1}} \quad \sum_{w_q, w_{q+1} \ldots, w_{\min(n,r)-1} \in W'_p}$$

$$(\prod_{\ell=0}^{q-1} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{p} \gamma(\ell + n\, i, \ell + n\, w_\ell(i))^{\#}) \times (\prod_{\ell=q}^{\min(n,r)-1} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{p-1} \gamma(\ell + n\, i, \ell + n\, w(i))^{\#})$$

$$= (\prod_{\ell=0}^{q-1} \sum_{w_\ell \in W'_{p+1}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{p} \gamma(\ell + n\, i, \ell + n\, w_\ell(i))^{\#})$$

$$\times (\prod_{\ell=q}^{\min(n,r)-1} \sum_{w_\ell \in W'_p} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{p-1} \gamma(\ell + n\, i, \ell + n\, w(i))^{\#})$$

$$= \Big(\prod_{\ell=0}^{q-1} \sum_{w_\ell\in W'_{p+1}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{p} \gamma(\ell/n+i, \ell/n+w_\ell(i))\Big)$$

$$\times \Big( \prod_{\ell=q}^{\min(n,r)-1} \sum_{w_\ell\in W'_{p}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{p-1} \gamma(\ell/n+i, \ell/n+w(i))\Big)$$

$$= \Big(\prod_{\ell=0}^{q-1} \gamma(\ell/n,\ell/n) \sum_{w_\ell\in W'_{p+1}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{p} \gamma(i, w_\ell(i))\Big)$$

$$\times \Big( \prod_{\ell=q}^{\min(n,r)-1} \gamma(\ell/n,\ell/n) \sum_{w_\ell\in W'_{p}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{p-1} \gamma(i, w(i))\Big)$$

$$= \Big(\prod_{\ell=0}^{q-1} \gamma(\ell/n,\ell/n)\gamma(p/2,p/2)\ 1_{GL(p+1)}\Big)$$

$$\times \Big( \prod_{\ell=q}^{\min(n,r)-1} \gamma(\ell/n,\ell/n)\gamma((p-1)/2,(p-1)/2)\ 1_{GL(p)}\Big).$$

From here

$$(1_{GL(r)})^{\#} = \Big(\prod_{\ell=0}^{q-1} \gamma\big(\frac{(\ell+pn/2-(r-1)/2)}{n}, \frac{(\ell+pn/2-(r-1)/2)}{n}\big)\ 1_{GL(p+1)}\Big)$$

$$\times \Big( \prod_{\ell=q}^{\min(n,r)-1} \gamma\big(\frac{\ell+(p-1)n/2-(r-1)/2}{n}, \frac{\ell+(p-1)n/2-(r-1)/2}{n}\big)\ 1_{GL(p)}\Big)$$

$$= \Big(\prod_{\ell=0}^{q-1} \gamma\big(\frac{(\ell-(q-1)/2)}{n}, \frac{(\ell-(q-1)/2)}{n}\big)\ 1_{GL(p+1)}\Big)$$

$$\times \Big( \prod_{\ell=q}^{\min(n,r)-1} \gamma\big(\frac{\ell-(q+n-1)/2}{n}, \frac{\ell-(q+n-1)/2}{n}\big)\ 1_{GL(p)}\Big)$$

$$= \mathrm{string}(q,n,1_{GL(p+1)}) \times \Big( \prod_{\ell'=0}^{\min(n,r)-q-1} \gamma\big(\frac{\ell'-(n-q-1)/2}{n}, \frac{\ell'-(n-q-1)/2}{n}\big)\ 1_{GL(p)}\Big)$$

$$= \mathrm{string}(q,n,1_{GL(p+1)}) \times \mathrm{string}(\min(n,r)-q,n,1_{GL(p)})$$

$$= \mathrm{string}(q,n,1_{GL(p+1)}) \times \mathrm{string}(n-q,n,1_{GL(p)}). \quad \square$$

*3.3. Remark.* Let $n > r$. Then $p = 0$, $q = r$. Therefore $\mathrm{string}(n-q,n,1_{GL(p)})$=$\mathrm{string}(n-r,n,1_{GL(0)})$ is identity of $R$. Clearly, $\mathrm{string}(\min(n,r)-q,n,1_{GL(p)})$=$\mathrm{string}(0,n,1_{GL(0)})$ is also identity.

## 4. General irreducible unitary representation

We assume $r, n \geq 2$.

**4.1. Lemma.** *Take* $d \in \{1, 2, \dots, n-1\}$. *If the set*

$$X_r(n,d) = \{w \in W_r';\ n|(d + w(i) - i) \text{ for all } 0 \leq i \leq r-1\}$$

*is non-empty, then* $n|r$.

*Proof.* Write $r = pn + q$ where $p, q \in \mathbb{Z}$ such that $0 \leq q \leq n-1$. Suppose that $X_r(n,d)$ is non-empty. Since elements in $X_r(n,d) \subseteq W_r'$ are bijections, then for each $i \in \{0, 1, \dots, n-1\}$ it must hold

$$\begin{aligned}&\operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, r-1]) \\ &\qquad = \operatorname{card}(\{i - d + kn; k \in \mathbb{Z}\} \cap [0, r-1]) = \operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d, d+r-1]).\end{aligned}$$

From the above relation we get

$$\begin{aligned}&\operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d-1]) + \operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d, r-1]) \\ &\qquad = \operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d, r-1]) + \operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [r, d+r-1]).\end{aligned}$$

Thus for each $i \in \{0, 1, \dots, n-1\}$ we have

$$\operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d-1]) = \operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [r, d+r-1]).$$

The last relation can be written as

$$\operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d-1]) = \operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [pn+q, pn+q+d-1]).$$

Therefore, for each $i \in \{0, 1, \dots, n-1\}$

$$\operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d-1]) = \operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [q, q+d-1]),$$

i.e.
$$\operatorname{card}(\{i\} \cap [0, d-1]) = \operatorname{card}(\{i + kn; k \in \mathbb{Z}\} \cap [q, q+d-1]).$$

Suppose that $q \geq 1$. The case of $i = d-1$ implies $q \leq d-1$. The case $i = d$ implies $d < q$. Therefore, we got a contradiction. This contradiction finishes the proof of the lemma. □

Note that the set defined in the last lemma can be described also in the following way

$$X_r(n,d) = \{w \in W_r'; n|(d + i - w^{-1}(i)) \text{ for all } 0 \leq i \leq r-1\}.$$

**4.2. Lemma.** *Suppose* $r = pn$ *(*$p \in \mathbb{Z}$*) and* $d \in \{1, 2, \dots, n-1\}$*. Then:*
*(i)* $W'_r(n)X_r(n,d)W'_r(n) = X_r(n,d)$.
*(ii)* $X_r(n,d)$ *normalizes* $W'_r(n)$.
*(iii) For any* $w \in X_r(n,d)$ *we have* $X_r(n,d) = wW'_r(n) = W'_r(n)\,w$.
*(iv) Each* $i \in \{0,1,2,\dots,r-1\}$ *write* $i = s(i)n + t(i)$ *where* $s(i), t(i) \in \mathbb{Z}$ *and* $0 \le t(i) \le n-1$*. Define* $w_{(n,d)} \in W'_r$ *by*

$$w_{(n,d)}(j) = \begin{cases} j + (n-d), & \text{if} \quad t(i) \le d-1; \\ j-d, & \text{if} \quad t(i) \ge d. \end{cases}$$

*Then* $w_{(n,d)} \in X_r(n,d)$ *and* $\operatorname{sgn}(w_{(n,d)}) = (-1)^{\frac{r}{n}(n-d)d}$.

*Proof.* Take $w \in W'_r(n)$ and $w_x \in X_r(n,d)$. Then

$$\begin{aligned} d + ww_x(i) - i &= (d + w_x(i) - i) + (w(w_x(i)) - w_x(i)), \\ d + w_xw(i) - i &= (d + w_x(w(i)) - w(i)) + (w(i) - i). \end{aligned}$$

This implies $ww_x, w_xw \in X_r(n,d)$. Further

$$\begin{aligned} w_xww_x^{-1}(i) - i &= (d + w_x(ww_x^{-1}(i)) - ww_x^{-1}(i)) + (-d + ww_x^{-1}(i) - i) \\ =&(d + w_x(ww_x^{-1}(i)) - ww_x^{-1}(i)) + (w(w_x^{-1}(i)) - w_x^{-1}(i)) - (d + i - w_x^{-1}(i)). \end{aligned}$$

Thus $w_xww_x^{-1} \in W'_r(n)$. So, we have proved (i) and (ii).

Let $w_1, w_2 \in X_r(n,d)$. Then

$$w_1w_2^{-1}(i) - i = (d + w_1(w_2^{-1}(i)) - w_2^{-1}(i)) - (d + i - w_2^{-1}(i)).$$

Thus $w_1w_2^{-1} \in W'_r(n)$. This proves (iii).

Now we shall prove (iv). Evidently, $w_{(n,d)}$ is injective. Also, it is easy to see that

$$w_{(n,d)}(\{0,1,2,\dots,r-1\}) \subseteq \{0,1,2,\dots,r-1\}.$$

Thus, $w_{(n,d)} \in W'_r$. From the formula for $w_{(n,d)}$ one sees directly $w_{(n,d)} \in X_r(n,d)$. It remains only to calculate the signature of $w_{(n,d)}$. Note that

$$\begin{aligned} &w_{(n,d)}(\{0,1,2,\dots,n-1\}) \subseteq \{0,1,2,\dots,n-1\}, \\ &w_{(n,d)}(i+n) = w_{(n,d)}(i) + n \text{ for all } 0 \le i \le r-1-n. \end{aligned}$$

It is easy to see that $\operatorname{card}\{(i,j); 0 \le i < j \le n-1 \text{ and } w(i) > w(j)\} = d(n-d)$. Therefore $\operatorname{sgn}(w_{(n,d)}) = (-1)^{p(n-d)d}$. □

The last lemma, together with Lemma 3.1, gives a complete description of sets $X_r(n,d)$. The following lemma is obvious.

**4.3. Lemma.** *Let* $\pi = L(\gamma(x_1,y_1),\gamma(x_2,y_2),\dots,\gamma(x_r,y_r))$, $x_i, y_i \in \mathbb{C}$, *be such that* $y_1 - x_1 = y_2 - x_2 = \cdots = y_r - x_r = d$ $(d \in \mathbb{Z})$. *Suppose that* $\pi \in \widetilde{GL(r)}^{(n)}$. *Then* $n|rd$. *Assume that* $1 \le d \le n-1$. *Write* $r = pn + q$ *where* $p, q \in \mathbb{Z}$ *such that* $0 \le q \le n-1$. *Then* $n|qd$. $\square$

**4.4. Proposition.** *Let* $d \in \{1,2,\dots,n-1\}$.
*(i) Suppose that* $\#(\chi(d)^{(\nu^0)}1_{GL(r)}) \neq 0$. *Then* $n|r$.
*(ii) Suppose that* $r = pn$, *with* $p \in \mathbb{Z}$. *Then we have*

$$(\chi(d)^{(\nu^0)}1_{GL(r)})^\# = (-1)^{\frac{r}{n}(n-d)d} string(d,n,\chi(1)^{(\nu^0)}1_{GL(p)}) \times string(n-d,n,1_{GL(p)}).$$

*Proof.* We have $\quad \chi(-d)^{(\nu^0)}\ 1_{GL(r)} = \gamma(-d/2,d/2)\ 1_{GL(r)}$

$$= \gamma(-(r-1)/2,-(r-1)/2)\gamma(-d/2,d/2) \sum_{w \in W'_r} (-1)^{\operatorname{sgn}(w)} \prod_{i=0}^{r-1} \gamma(i,w(i))$$

$$= \gamma(-(r+d-1)/2,-(r+d-1)/2)\gamma(0,d) \sum_{w \in W'_r} (-1)^{\operatorname{sgn}(w)} \prod_{i=0}^{r-1} \gamma(i,w(i))$$

$$= \gamma(-(r+d-1)/2,-(r+d-1)/2) \sum_{w \in W'_r} (-1)^{\operatorname{sgn}(w)} \prod_{i=0}^{r-1} \gamma(i,d+w(i)).$$

Thus $\quad \gamma((r+d-1)/2n,(r+d-1)/2n)\ \ \#(\chi(-d)^{(\nu^0)}1_{GL(r)})$

$$= \sum_{w \in W'_r} (-1)^{\operatorname{sgn}(w)} \prod_{i=0}^{r-1} \#(\gamma(i,d+w(i))) = \sum_{w \in X_r(n,d)} (-1)^{\operatorname{sgn}(w)} \prod_{i=0}^{r-1} \#(\gamma(i,d+w(i))).$$

Now from Lemma 4.1 follows (i) in the lemma. In further, we assume that $r = pn$ and continue our calculation

$$\gamma((r+d-1)/2n,(r+d-1)/2n)\ \ (\chi(-d)^{(\nu^0)}1_{GL(r)})^\#$$

$$= \sum_{w \in X_r(n,d)} (-1)^{\operatorname{sgn}(w)} \prod_{i=0}^{r-1} \gamma(i,d+w(i))^\# = \sum_{w \in w_{(n,d)}W'_r(n)} (-1)^{\operatorname{sgn}(w)} \prod_{i=0}^{r-1} \gamma(i,d+w(i))^\#$$

$$= (-1)^{\operatorname{sgn}(w_{(n,d)})} \sum_{w \in W'_r(n)} (-1)^{\operatorname{sgn}(w)} \prod_{i=0}^{r-1} \gamma(i,d+w_{(n,d)}w(i))^\#$$

$$= (-1)^{p(n-d)d} \sum_{w \in W'_r(n)} (-1)^{\operatorname{sgn}(w)} \prod_{\ell=0}^{n-1} \Big(\prod_{j=0}^{p-1} \gamma(\ell+nj,d+w_{(n,d)}w(\ell+nj))^\#\Big)$$

$$= (-1)^{p(n-d)d} \sum_{w_0,w_1,\dots,w_{n-1} \in W'_{p-1}} \prod_{\ell=0}^{n-1} \Big((-1)^{\operatorname{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell+nj,d+w_{(n,d)}(\ell+nw_\ell(j)))^\#\Big)$$

$$
\begin{aligned}
&= (-1)^{p(n-d)d} \sum_{w_0,w_1,\dots,w_{n-1}\in W'_p} [\prod_{\ell=0}^{d-1}((-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell+nj, d+w_{(n,d)}(\ell+nw_\ell(j)))^{\#})] \\
&\qquad \times [\prod_{\ell=d}^{n-1}((-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell+nj, d+w_{(n,d)}(\ell+nw_\ell(j)))^{\#})]
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{p(n-d)d} \sum_{w_0,w_1,\dots,w_{n-1}\in W'_p} [\prod_{\ell=0}^{d-1}((-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell+nj, \ell+n+nw_\ell(j))^{\#})] \\
&\qquad \times [\prod_{\ell=d}^{n-1}((-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell+nj, \ell+nw_\ell(j))^{\#})]
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{p(n-d)d}[\prod_{\ell=0}^{d-1}( \sum_{w_\ell\in W'_p} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell+nj, \ell+n+nw_\ell(j))^{\#})] \\
&\qquad \times [\prod_{\ell=d}^{n-1}( \sum_{w_\ell\in W'_p} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell+nj, \ell+nw_\ell(j))^{\#})]
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{p(n-d)d}[\prod_{\ell=0}^{d-1}( \sum_{w_\ell\in W'_p} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell/n+j, \ell/n+1+w_\ell(j)))] \\
&\qquad \times [\prod_{\ell=d}^{n-1}( \sum_{w_\ell\in W'_p} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(\ell/n+j, \ell/n+w_\ell(j)))]
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{p(n-d)d}[\prod_{\ell=0}^{d-1}(\gamma(\ell/n, \ell/n+1) \sum_{w_\ell\in W'_p} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(j, w_\ell(j)))] \\
&\qquad \times [\prod_{\ell=d}^{n-1}(\gamma(\ell/n, \ell/n) \sum_{w_\ell\in W'_p} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{j=0}^{p-1} \gamma(j, w_\ell(j)))]
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{p(n-d)d}[\prod_{\ell=0}^{d-1} \gamma(\ell/n, \ell/n+1)\gamma((p-1)/2, (p-1)/2)\ 1_{GL(p)}] \\
&\qquad \times [\prod_{\ell=d}^{n-1} \gamma(\ell/n, \ell/n)\gamma((p-1)/2, (p-1)/2)\ 1_{GL(p)}]
\end{aligned}
$$

$$= (-1)^{p(n-d)d}[\prod_{\ell=0}^{d-1}\gamma(-1/2,1/2)\gamma(\ell/n+1/2,\ell/n+1/2)\gamma((p-1)/2,(p-1)/2)\ 1_{GL(p)}]$$
$$\times [\prod_{\ell'=0}^{n-d-1}\gamma((\ell'+d)/n,(\ell'+d)/n)\gamma((p-1)/2,(p-1)/2)\ 1_{GL(p)}]$$

$$= (-1)^{p(n-d)d}[\prod_{\ell=0}^{d-1}\gamma(-1/2,1/2)\gamma(\ell/n+p/2,\ell/n+p/2)\ 1_{GL(p)}]$$
$$\times [\prod_{\ell'=0}^{n-d-1}\gamma((\ell'+d)/n,(\ell'+d)/n)\gamma((p-1)/2,(p-1)/2)\ 1_{GL(p)}].$$

Thus $\quad (\chi(-d)^{(\nu^0)}1_{GL(r)})^{\#} = (-1)^{p(n-d)d}$

$$[\prod_{\ell=0}^{d-1}\gamma(-1/2,1/2)\gamma(\frac{-(r+d-1)/2+\ell+pn/2}{n},\frac{-(r+d-1)/2+\ell+pn/2}{n})\ 1_{GL(p)}]$$
$$\times[\prod_{\ell'=0}^{n-d-1}\gamma(\frac{-(r-d-1)/2+\ell'+(p-1)n/2}{n},\frac{-(r-d-1)/2+\ell'+(p-1)n/2}{n})1_{GL(p)}]$$

$$= (-1)^{p(n-d)d}[\prod_{\ell=0}^{d-1}\gamma(-1/2,1/2)\gamma(\frac{-(d-1)/2+\ell}{n},\frac{-(d-1)/2+\ell}{n})\ 1_{GL(p)}]\times$$
$$[\prod_{\ell=0}^{n-d-1}\gamma(\frac{-(n-d-1)/2+\ell}{n},\frac{-(n-d-1)/2+\ell}{n})\ 1_{GL(p)}]$$

$$= (-1)^{\frac{r}{n}(n-d)d}\ \text{string}(d,n,\chi(-1)^{(\nu^0)}1_{GL(p)})\times\text{string}(n-d,n,1_{GL(p)}).$$

Passing to the contragredients, we get the formula in (ii) of the lemma. □

The following theorem completely describes the action of $^{\#}$ on the irreducible unitary representations of $GL(r)$.

**4.5. Theorem.**
*(i) If* $\pi_1,\dots,\pi_k \in R$, *then* $\#(\pi_1\times\cdots\times\pi_k) = \#(\pi_1)\times\cdots\times\#(\pi_k)$.
*(ii) Each* $\pi\in\widehat{GL(r)}$ *can be written as*

$$\pi = \prod_{i=1}^{k}(\chi(d_i)^{(\phi_i)}1_{GL(r_i)})\times\prod_{j=k+1}^{\ell}\Big((\nu^{\alpha_j}\chi(d_j)^{(\psi_j)}1_{GL(r_j)})\times(\nu^{-\alpha_j}\chi(d_j)^{(\psi_j)}1_{GL(r_j)})\Big),$$

*where* $\phi_i,\psi_j\in\widehat{\mathbb{C}^\times/\mathbb{T}}$, $0<\alpha_j<1/2$ *and* $\sum_{i=1}^{k} r_i + 2\sum_{j=k+1}^{\ell} r_j = r$. *Now* $\pi\in\widehat{GL(r)}^{(n)}$ *if and only if* $n|(\sum_{i=1}^{k} r_i d_i + 2\sum_{j=k+1}^{\ell} r_j d_j)$. *Further* $\#(\pi)\neq 0$ *if and only if for each* $t\in\{1,2,\dots,\ell\}$, $n|d_t$ *or* $n|r_t$.

*(iii) Let* $\phi \in \widetilde{\mathbb{C}^\times/\mathbb{T}}$ *and* $\alpha \in \mathbb{C}$. *Then* $\#(\nu^\alpha \chi(d)^{(\phi)} 1_{GL(r)}) \neq 0$ *if and only if* $n|d$ *or* $n|r$. *Write* $r = pn+q$ *and* $d = vn+d'$ *with* $p, q, v, d' \in \mathbb{Z}$ *such that* $0 \leq q \leq n-1$, $0 \leq d' \leq n-1$. *Then*

$$\begin{aligned}(4\text{-}1)\quad & (\nu^\alpha \chi(d)^{(\phi)} 1_{GL(r)}))^\# \\ & = \nu^{\alpha/n} \chi(d/n)^{(\phi^{1/n})} \left(\textit{string}(q, n, 1_{GL(p+1)}) \times \textit{string}(n-q, n, 1_{GL(p)})\right) \ \textit{if } n|d;\end{aligned}$$

$$\begin{aligned}(4\text{-}2)\quad & (\nu^\alpha \chi(d)^{(\phi)} 1_{GL(r)}))^\# = (-1)^{\frac{r}{n}(n-d')d'} \nu^{\alpha/n} \chi(v)^{(\phi^{1/n})} \\ & \left(\textit{string}(d', n, \chi(1)^{(\nu^0)} 1_{GL(r/n)}) \times \textit{string}(n-d', n, 1_{GL(r/n)})\right) \ \textit{if } n|r.\end{aligned}$$

*(iv) For an irreducible unitary representation* $\pi \in \widehat{GL(r)}^{(n)}$, $\pi^\#$ *is either* 0, *or* $\pi^\# \in \widehat{GL(r)}$, *or* $-\pi^\# \in \widehat{GL(r)}$.

*(v) If* $\pi$ *is a spherical irreducible unitary representation in* $\widehat{GL(r)}$, *then* $\pi^\#$ *is again a spherical irreducible unitary representation.*

*Proof.* Only the statement (iv) remains to prove. It is enough to prove the statement only for elements in $B$.

First look at the case of $\chi(d)^{(\phi)} 1_{GL(r)}$. Suppose that $d|n$. Then string$(q, n, 1_{GL(p+1)}) =$

$$(\nu^{\frac{-(q-1)/2}{n}} 1_{GL(p+1)} \times \nu^{\frac{(q-1)/2}{n}} 1_{GL(p+1)}) \times (\nu^{\frac{-(q-1)/2+1}{n}} 1_{GL(p+1)} \times \nu^{\frac{(q-1)/2-1}{n}} 1_{GL(p+1)}) \times \cdots .$$

Obviously $(q-1)/(2n) < 1/2$. This implies that string$(q, n, 1_{GL(p+1)})$ is irreducible unitary (see Theorem 1.1). Similarly, string$(n-q, n, 1_{GL(p)})$ is irreducible unitary since $(n-q-1)/(2n) < 1/2$. This implies that $(\chi(d)^{(\phi)} 1_{GL(r)}))^\#$ is irreducible unitary up to a sign, in this case.

Suppose that $n|r$. Now string$(d', n, \chi(1)^{(\nu^0)} 1_{GL(r/n)})$ is irreducible unitary since $(d'-1)/(2n) < 1/2$ and string$(n-d', n, 1_{GL(r/n)})$ is irreducible unitary since $(n-d'-1)/(2n) < 1/2$. From this follows again that $(\chi(d)^{(\phi)} 1_{GL(r)}))^\#$ is irreducible unitary up to a sign.

In the other cases, $(\chi(d)^{(\phi)} 1_{GL(r)}))^\# = 0$.

It remains to prove (iv) for the remaining elements of $B$, i.e. for $(\nu^\alpha \chi(d)^{(\psi)} 1_{GL(r)}) \times (\nu^{-\alpha} \chi(d)^{(\psi)} 1_{GL(r)}) \in B$.

Consider first the case of $n|r$. Note that $((\nu^\alpha \chi(d)^{(\phi)} 1_{GL(r)}) \times (\nu^{-\alpha} \chi(d)^{(\phi)} 1_{GL(r)}))^\# =$

$$\begin{aligned}& \nu^{\alpha/n} \chi(v)^{(\phi^{1/n})} \left(\text{string}(d', n, \chi(1)^{(\nu^0)} 1_{GL(r/n)}) \times \text{string}(n-d', n, 1_{GL(r/n)})\right) \\ & \qquad \times \nu^{-\alpha/n} \chi(v)^{(\phi^{1/n})} \left(\text{string}(d', n, \chi(1)^{(\nu^0)} 1_{GL(r/n)}) \times \text{string}(n-d', n, 1_{GL(r/n)})\right).\end{aligned}$$

From this we see that it is enough to prove that

$$\begin{aligned}\pi_1 = {} & \nu^{\alpha/n} \chi(v)^{(\phi^{1/n})} \left(\text{string}(d', n, \chi(1)^{(\nu^0)} 1_{GL(r/n)})\right) \\ & \qquad \times \nu^{-\alpha/n} \chi(v)^{(\phi^{1/n})} \left(\text{string}(d', n, \chi(1)^{(\nu^0)} 1_{GL(r/n)})\right),\end{aligned}$$

$$\pi_2 = \nu^{\alpha/n}\chi(v)^{(\phi^{1/n})}\big(\mathrm{string}(n-d', n, 1_{GL(r/n)})\big)$$
$$\times\, \nu^{-\alpha/n}\chi(v)^{(\phi^{1/n})}\big(\mathrm{string}(n-d', n, 1_{GL(r/n)})\big)$$

are irreducible unitary. Consider $\pi_1 =$

$$\nu^{\alpha/n}\chi(v{+}1)^{(\phi^{1/n})}\Big(\prod_{\ell=0}^{d'-1}\nu^{\frac{\ell-(d'-1)/2}{n}}1_{GL(r/n)}\Big){\times}\nu^{-\alpha/n}\chi(v{+}1)^{(\phi^{1/n})}\Big(\prod_{\ell=0}^{d'-1}\nu^{\frac{\ell-(d'-1)/2}{n}}1_{GL(r/n)}\Big)$$
$$= \chi(v+1)^{(\phi^{1/n})}\Big[\Big(\prod_{\ell=0}^{d'-1}\nu^{\frac{\alpha+\ell-(d'-1)/2}{n}}1_{GL(r/n)}\Big)\times \nu^{-\alpha/n}\Big(\prod_{\ell=0}^{d'-1}\nu^{\frac{(d'-1)-\ell-(d'-1)/2}{n}}1_{GL(r/n)}\Big)\Big]$$
$$= \chi(v+1)^{(\phi^{1/n})}\Big[\Big(\prod_{\ell=0}^{d'-1}\nu^{\frac{\alpha+\ell-(d'-1)/2}{n}}1_{GL(r/n)}\Big)\times \Big(\prod_{\ell=0}^{d'-1}\nu^{\frac{-\alpha+(d'-1)/2-\ell}{n}}1_{GL(r/n)}\Big)\Big]$$
$$= \chi(v+1)^{(\phi^{1/n})}\Big[\prod_{\ell=0}^{d'-1}\big(\nu^{\frac{\alpha+\ell-(d'-1)/2}{n}}1_{GL(r/n)}\big)\times \big(\nu^{\frac{-\alpha+(d'-1)/2-\ell}{n}}1_{GL(r/n)}\big)\Big].$$

Note that $-1/2 < \frac{\alpha+\ell-(d'-1)/2}{n} < 1/2$, $-1/2 < \frac{-\alpha+(d'-1)/2-\ell}{n} < 1/2$ for $0 \le d' \le n-1$, $0 \le \ell \le d'-1$ and $0 < \alpha < 1/2$. This implies that $\pi_1$ is irreducible unitary.

In a similar way we get

$$\pi_2 = \chi(v)^{(\phi^{1/n})}\Big(\prod_{\ell=0}^{n-d'-1}\big(\nu^{\frac{\alpha+\ell-(n-d'-1)/2}{n}}1_{GL(r/n)}\big)\times \big(\nu^{\frac{-\alpha+(n-d'-1)/2-\ell}{n}}1_{GL(r/n)}\big)\Big).$$

Since $-1/2 < \frac{\alpha+\ell-(n-d'-1)/2}{n} < 1/2$, $-1/2 < \frac{-\alpha+(n-d'-1)/2-\ell}{n} < 1/2$ for $0 \le d' \le n-1$, $0 \le \ell \le n-d'-1$ and $0 < \alpha < 1/2$, we get that $\pi_2$ is irreducible unitary.

Consider now the case of $n|d$. Then

$$((\nu^{\alpha}\chi(d)^{(\phi)}1_{GL(r)})) \times (\nu^{-\alpha}\chi(d)^{(\phi)}1_{GL(r)}))^{\#}$$
$$= \nu^{\alpha/n}\chi(d/n)^{(\phi^{1/n})}\big(\mathrm{string}(q,n,1_{GL(p+1)}) \times \mathrm{string}(n-q,n,1_{GL(p)})\big)$$
$$\times\, \nu^{-\alpha/n}\chi(d/n)^{(\phi^{1/n})}\big(\mathrm{string}(q,n,1_{GL(p+1)}) \times \mathrm{string}(n-q,n,1_{GL(p)})\big)$$
$$= \chi(d/n)^{(\phi^{1/n})}[(\nu^{\alpha/n}\mathrm{string}(q,n,1_{GL(p+1)}) \times \nu^{-\alpha/n}\mathrm{string}(q,n,1_{GL(p+1)}))$$
$$\times\, (\nu^{\alpha/n}\mathrm{string}(n-q,n,1_{GL(p)}) \times \nu^{-\alpha/n}\mathrm{string}(n-q,n,1_{GL(p)}))]$$

Denote
$$\pi_3 = \nu^{\alpha/n}\mathrm{string}(q,n,1_{GL(p+1)}) \times \nu^{-\alpha/n}\mathrm{string}(q,n,1_{GL(p+1)})$$
$$\pi_4 = \nu^{\alpha/n}\mathrm{string}(n-q,n,1_{GL(p)}) \times \nu^{-\alpha/n}\mathrm{string}(n-q,n,1_{GL(p)})$$

Now
$$\pi_3 = \prod_{\ell=0}^{q-1}\big(\nu^{\frac{\alpha+\ell-(q-1)/2}{n}}1_{GL(p+1)}\big)\times\big(\nu^{\frac{-\alpha+(q-1)/2-\ell}{n}}1_{GL(p+1)}\big),$$
$$\pi_2 = \prod_{\ell=0}^{n-q-1}\big(\nu^{\frac{\alpha+\ell-(n-q-1)/2}{n}}1_{GL(p)}\big)\times\big(\nu^{\frac{-\alpha+(n-q-1)/2-\ell}{n}}1_{GL(p)}\big).$$

Since $-1/2 < \frac{\alpha+\ell-(q-1)/2}{n} < 1/2$, $-1/2 < \frac{-\alpha+(q-1)/2-\ell}{n} < 1/2$ for $0 \leq q \leq n-1$, $0 \leq \ell \leq q-1$ and $0 < \alpha < 1/2$, $\pi_3$ is irreducible unitary. Further we get easily that $-1/2 < \frac{\alpha+\ell-(n-q-1)/2}{n} < 1/2$, $-1/2 < \frac{-\alpha+(n-q-1)/2-\ell}{n} < 1/2$ for $0 \leq q \leq n-1$, $0 \leq \ell \leq n-q-1$ and $0 < \alpha < 1/2$. From this follows that $\pi_4$ is irreducible unitary.

In the other cases we have $((\nu^{\alpha}\chi(d)^{(\psi)}1_{GL(r)}) \times (\nu^{-\alpha}\chi(d)^{(\psi)}1_{GL(r)}))^{\#} = 0$.

The proof of (iv) is now complete. □

## REFERENCES

[Bl] Blondel, C., *Les représentations supercuspidales des groupes métaplectiques sur $GL(2)$ et leurs caractèrs*, Mémoires Société Mathématique de France **18** (1985).

[BoW] Borel, A., Wallach, N., *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Princeton University Press, Princeton, 1980.

[Br tD] Bröcker, T., tom Dieck, T., *Representations of Compact Lie Groups*, Springer-Verlag, New York, 1985.

[F] Flicker, Y., *Automorphic forms on covering groups of $GL(2)$*, Inv. Math. **57** (1980), 119-182.

[FK] Flicker, Y., Kazhdan, D.A., *Metaplectic correspondence*, Publ. Math. IHES **64** (1986), 53-110.

[H] Huang, J.-S., *Metapletic correspondences and unitary representations*, Compositio Math. **80** (1991), 309-322.

[J] Jacquet, H., *Generic representations*, Non-Commutative Harmonic Analysis, Lecture Notes in Math. 587, Spring Verlag, Berlin, 1977, pp. 91-101.

[KP1] Kazhdan, D.A., Patterson, S.J., *Metaplectic forms*, Publ. Math. IHES **59** (1984), 35-142.

[KP2] Kazhdan, D.A., Patterson, S.J., *Towards a generalized Shimura Correspondence*, Advan. in Math. **60** (1986), 161-234.

[S1] Savin, G., *Local Shimura correspondence*, Math. Ann. **280** (1988), 185-190.

[S2] Savin, G., *An analogue of the Weil representation for $G_2$*, J. reine angew. Math. **434** (1993), 115-126.

[T1] Tadić, M., *Unitary representations of general linear group over real and complex field*, preprint MPI/SFB 85-22 Bonn (1985).

[T2] ———, *An external approach to unitary representations,*, Bulletin Amer. Math. Soc. **28** (1993), no. 2, 215-252.

[T3] ———, *On characters of irreducible unitary representations of general linear groups (Math. Goettingensis issue 2(1994))*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **65** (1995) (to appear).

[V1] Vogan, D.A., *Representations of real reductive groups*, Birkhäuser, Boston, 1987.

[V2] ———, *The unitary dual of $GL(n)$ over an archimedean field*, Invent. Math. **82** (1986), 449-505.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA 30, 41000 ZAGREB, CROATIA