Jacquet modules and induced representations*

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1. Introduction

In this paper, we shall review some possible applications of Jacquet modules to the study of parabolically induced representations of reductive p-adic groups.

Let us start with a few brief remarks about the history of Jacquet modules. Jacquet modules do not make their appearance in representation theory until the end of the 1960's. Using them, H. Jacquet was able to obtain some important basic results in the representation theory of p-adic groups (e.g., his subrepresentation theorem and the equivalence of the different characterizations of cuspidal [Jc]). These results are still somewhat qualitative in nature. To use Jacquet modules as a calculational tool, more quantitative results are needed. A major step in this direction was taken by W. Casselman, who calculated the Jordan-Hölder series for the Jacquet module of an induced representation, among other things ([Cs]). A significant number of his results were obtained independently by J. Bernstein and A.V. Zelevinsky ([B-Z]).

Before going on to discuss how one can use Jacquet modules to study reducibility questions for induced representations, let us indicate some limitations to this approach. For example, consider $SL(2)$. If $\chi^2 = 1$ with $\chi \neq 1$, the representation $\mathrm{Ind}_{P_\emptyset}^{SL(2)}(\chi)$ is reducible. However, one cannot obtain this reducibility through a simple use of Jacquet modules. A similar situation occurs with $\mathrm{Ind}_{P_\emptyset}^{SL(2)}(1)$, except that in this case the representation is irre-

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ducible, and it is its irreducibility which cannot be shown through a simple use of Jacquet modules.

Let us make one simple remark about the examples above. In principle, one can use Frobenius reciprocity to determine the reducibility of parabolically induced representations when the inducing representation is unitary. However, in practice, one does not know the exact structure of the Jacquet modules in some of the most interesting cases, only the composition factors that arise. Even for the example $\mathrm{Ind}_P^{SL(2)}(\chi)$, $\chi^2 = 1$, $\chi \neq 1$, one needs other tools to determine reducibility. On the other hand, in the non-unitary case, even knowing the structure of the Jacquet modules is not necessarily enough to determine reducibility. For example, one knows the Jacquet module structure for $\mathrm{Ind}_P^{SL(2)}(|\cdot|_F^\alpha)$, $\alpha \in \mathbb{R} \setminus \{0\}$, but this is not enough to give irreducibility for $\alpha \neq \pm 1$.

After this, one might wonder what the prospects are for using Jacquet modules in more complicated situations. Surprisingly, Jacquet modules are much more powerful in more complicated situations. The reason for this is simple: there are more standard parabolic subgroups. Therefore, one can compare information on Jacquet modules coming from different parabolic subgroups. For this approach to be most effective, one needs to direct one's attention to Jacquet modules with respect to large parabolic subgroups (which was not usually done in the early applications of this use of Jacquet modules). We note that this approach is particularly convenient for classical groups because the Levi factors of their parabolic subgroups are direct products of general linear groups and smaller classical groups. And, we understand induced representations for general linear groups from the Bernstein-Zelevinsky theory ([B-Z],[Zl]). Further, the representation theory of general linear groups is relatively simple in comparison with that of other classical groups.

Our primary interest in the use of Jacquet modules was in understanding the non-cuspidal square-integrable representations and, more generally, the non-cuspidal tempered representations. The latter problem is clearly related to understanding reducibility of parabolic induction. But, the first problem is also very much related to understanding reducibility of parabolic induction, and the reducibility problems involved are even more complicated than in the tempered case. (The reason for this is that one must look for non-cuspidal square-integrable representations as subquotients of parabolically induced representations where the inducing representation is non-unitary.) In the long run, we are also motivated by the desire to develop tools for addressing the unitarizability problem for classical p-adic groups (other than general linear groups).

Most of the paper is based on the author's original notes of the lecture given at AMS-IMS-SIAM Joint Summer Research Conference "Representation theory of reductive groups" (University of Washington, Seattle, 1997). Time constraints forced a number of items to be omitted during the talk. They have been included here. C. Jantzen wrote up the notes of the lecture (using the author's notes in the process). We are very thankful to him for improving the style of the original notes and for a number of useful additions (in particular, Example 2.2).

We now describe the contents section by section. In the second section, we describe two simple criteria for reducibility and irreducibility of parabolically induced representations. The third section introduces notation for the classical groups. In the fourth section, we mainly consider examples of the reducibility of parabolically induced representations, and discuss the occurrence of square-integrable subquotients (particular attention is paid to the case of non-generic reducibilities). In the last section, we discuss a possible general strategy for getting non-cuspidal square-integrable representations of classical groups.

2. Simple criteria for reducibility and irreducibility

We first need some ideas on how to deal with reducibility questions for parabolically induced representations. As an illustration, we shall give two simple, but useful, recipes for proving reducibility or irreducibility.

Let G be a connected reductive group over a local non-archimedean field F (later, when we start to deal with classical groups, we shall assume $\mathrm{char} F \neq 2$). For a smooth representation π of G and a parabolic subgroup $P = MN$, we denote by $r_M^G(\pi)$ the normalized Jacquet module of π with respect to P (notation of Bernstein-Zelevinsky). We remind the reader that if V is the space of π, then $r_M^G(\pi)$ has space: $V/V(N)$, where $V(N) = span\{\pi(n)v - v \mid n \in N, v \in V\}$

action: $(r_M^G(\pi))(m)[v + V(N)] = \delta_P^{-\frac{1}{2}}(m)\pi(m)v + V(N)$,

where δ_P denotes the modular function for P (since M normalizes N, this defines an action). For much of our discussion, we shall essentially be dealing with semi-simplifications of representations. More precisely, we work in the Grothendieck group $\mathfrak{R}(G)$ of the category of smooth finite-length representations of G. Recall that two representations π_1, π_2 have $\pi_1 = \pi_2$ in $\mathfrak{R}(G)$ if $m(\rho, \pi_1) = m(\rho, \pi_2)$ for all smooth irreducible representations ρ, where $m(\rho, \pi_i)$ denotes the multiplicity of ρ in π_i. Then, r_M^G lifts to a mapping

$$r_M^G : \mathfrak{R}(G) \longrightarrow \mathfrak{R}(M).$$

We note that $\mathfrak{R}(G)$ admits a natural partial order: $\pi_1 \leq \pi_2$ if $m(\rho, \pi_1) \leq m(\rho, \pi_2)$ for all smooth irreducible ρ.

We now give simple criteria for reducibility and irreducibility of induced representations. For more details on these criteria, the reader is referred to section 3 of [Td3].

Reducibility criteria: **(RC)**

Suppose $P_0 = M_0N_0$ and $P = MN$ are standard parabolic subgroups. Further, suppose σ is a smooth irreducible representation of M_0; π, Π smooth finite-length representations of G. Suppose that

1. $Ind_{P_0}^G(\sigma) \leq \Pi$, $\pi \leq \Pi$.
2. $r_M^G(Ind_{P_0}^G(\sigma)) + r_M^G(\pi) \not\leq r_M^G(\Pi)$
3. $r_M^G(Ind_{P_0}^G(\sigma)) \not\leq r_M^G(\pi)$.

Then, $Ind_{P_0}^G(\sigma)$ is reducible.

Remarks:

1. One can almost always choose π and Π to be parabolically induced representations.
2. It is easy to get upper and lower estimates on the Jacquet modules of common irreducible subquotients of $\mathrm{Ind}_{P_0}^G(\sigma)$ and π; often they will give the Jacquet modules of the common irreducible subquotients exactly.

We now take up the question of showing irreducibility. First, fix a minimal parabolic subgroup $P_\emptyset$ of G. Then, we can (and do) choose Levi decompositions $P = MN$ of standard parabolic subgroups so that

$$r^{M_1}_{M_2} \circ r^G_{M_1} = r^G_{M_2}$$

for standard parabolics $P_1 \supset P_2$.

Suppose that σ is an irreducible representation of M_0 and $\mathrm{Ind}^G_{P_0}(\sigma)$ reduces. Write $\mathrm{Ind}^G_{P_0}(\sigma) = \pi_1 + \pi_2$, $\pi_1 > 0$, $\pi_2 > 0$, in the Grothendieck group $\mathfrak{R}(G)$. For any standard parabolic $P = MN$, let

$$T_{i,P} = r^G_M(\pi_i), \quad i = 1, 2$$

(viewed as element of $\mathfrak{R}(M)$). Then, the following must hold:

1. $T_{i,P} \geq 0$ and $T_{1,P} \neq 0$ if and only if $T_{2,P} \neq 0$
2. $T_{1,P} + T_{2,P} = r^G_M(\mathrm{Ind}^G_{P_0}(\sigma))$
3. $r^{M_1}_{M_2}(T_{i,P_1}) = T_{i,P_2}$ when $P_1 \supset P_2$.

Irreducibility criteria: **(IC)**

Let σ be an irreducible representation of M_0 and consider $\mathrm{Ind}^G_{P_0}(\sigma)$. If one can show there is no system of $T_{i,P}$'s as above ($i = 1, 2$; P running through the standard parabolics), then one has shown the irreducibility of $\mathrm{Ind}^G_{P_0}(\sigma)$.

Remark: When using this approach, it is usually possible to produce three proper standard parabolic subgroups P, P_1, P_2 with $P \subset P_1, P_2$, and $P \neq P_1, P_2$ for which one can already show that 1.-3. above cannot be satisfied.

2.1. Example: A simple example of application of this criteria which often works is the following: assume $P = MN$, $P_1 = M_1N_1$, $P_2 = M_2N_2$ are proper standard parabolic subgroups with $P \subset P_1, P_2$ and $P \neq P_1, P_2$. Suppose there exists an irreducible subquotient τ_1 of $r^G_{M_1}(\mathrm{Ind}^G_{P_0}(\sigma))$ such that for any irreducible subquotient τ_2 of $r^G_{M_2}(\mathrm{Ind}^G_{P_0}(\sigma))$, we have

$$r^{M_1}_M(\tau_1) + r^{M_2}_M(\tau_2) \not\leq r^G_M(\mathrm{Ind}^G_{P_0}(\sigma)).$$

Then, $\mathrm{Ind}_{P_0}^G(\sigma)$ is irreducible.

To see that this holds, suppose $\mathrm{Ind}_{P_0}^G(\sigma) = \pi_1 + \pi_2$ as above. Without loss of generality, assume τ_1 is a subquotient for $r_{M_1}^G(\pi_1) = T_{1,P_1}$. Choose any irreducible subquotient τ_2 of $r_{M_2}^G(\pi_2) = T_{2,P_2}$. Then, by 2.,3.

$$r_M^G(\mathrm{Ind}_{P_0}^G(\sigma)) = T_{1,P} + T_{2,P} = r_M^{M_1}(T_{1,P_1}) + r_M^{M_2}(T_{2,P_2}) \geq r_M^{M_1}(\tau_1) + r_M^{M_2}(\tau_2),$$

contradicting the assumption $r_M^{M_1}(\tau_1) + r_M^{M_2}(\tau_2) \not\leq r_M^G(\mathrm{Ind}_{P_0}^G(\sigma))$.

We now give an example, to illustrate these ideas in a concrete situation.

2.2. Example: Let $G = GL(3, F) = GL(3)$. Let $P_\emptyset, P_1, P_2$ denote the three (proper) standard parabolic subgroups ($P_\emptyset$ minimal). These have Levi factors $A = F^\times \times F^\times \times F^\times$, $M_1 = F^\times \times GL(2, F)$, $M_2 = GL(2, F) \times F^\times$, respectively. We shall apply **(IC)** and **(RC)** to the induced representation $\rho = \mathrm{Ind}_{P_2}^G(\chi_1 \circ det_{GL(2)} \otimes \chi_2)$, where χ_1, χ_2 are characters (not necessarily unitary) of $F^\times$.

The semisimplified Jacquet modules for ρ may be calculated using Lemma 2.12 of [B-Z] or the results in chapter 6 of [Cs]. We tabulate them below:

$$\begin{tikzcd}
\underline{r_{M_1}^G(\rho)} & \underline{r_A^G(\rho)} & \underline{r_{M_2}^G(\rho)} \\
|\cdot|^{-\frac{1}{2}}\chi_1 \otimes \sigma_1 \arrow[r, dash] \arrow[dr, dash] & |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \otimes \chi_2 \arrow[r, dash] & \chi_1 \circ det_{GL(2)} \otimes \chi_2 \\
 & |\cdot|^{-\frac{1}{2}}\chi_1 \otimes \chi_2 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \arrow[r, dash] & \sigma_2 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \\
\chi_2 \otimes \chi_1 \circ det_{GL(2)} \arrow[r, dash] & \chi_2 \otimes |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \arrow[ur, dash] &
\end{tikzcd}$$

where $\sigma_1 = \mathrm{Ind}_P^{GL(2)}(|\cdot|^{\frac{1}{2}}\chi_1 \otimes \chi_2)$ and $\sigma_2 = \mathrm{Ind}_P^{GL(2)}(|\cdot|^{-\frac{1}{2}}\chi_1 \otimes \chi_2)$ (here, P is the standard minimal parabolic subgroup of $GL(2)$). For a subquotient τ of $r_{M_1}^G$ (resp. $r_{M_2}^G$), the lines indicate which terms in $r_A^G(\rho)$ come from $r_A^{M_1}(\tau)$ (resp. $r_A^{M_2}(\tau)$).

We now apply **(IC)**, using the form of **(IC)** which appears in the preceding example. Assume $\chi_2 \neq |\cdot|^{\pm\frac{1}{2}}\chi_1, |\cdot|^{\pm\frac{3}{2}}\chi_1$. Take $\tau_1 = |\cdot|^{-\frac{1}{2}}\chi_1 \otimes$

$(\mathrm{Ind}_P^{GL(2)}(|\cdot|^{\frac{1}{2}}\chi_1\otimes\chi_2))$ (a subquotient of $r_{M_1}^G(\rho)$) and $\tau_2'=\chi_1\circ det_{GL(2)}\otimes\chi_2$, $\tau_2''=(\mathrm{Ind}_P^{GL(2)}(|\cdot|^{-\frac{1}{2}}\chi_1\otimes\chi_2))\otimes|\cdot|^{\frac{1}{2}}\chi_1$. For $\chi_2\neq|\cdot|^{\pm\frac{1}{2}}\chi_1,|\cdot|^{\pm\frac{3}{2}}\chi_1$, these are all irreducible (in general, $\mathrm{Ind}_P^{GL(2)}(\chi\otimes\chi')$ is reducible if and only if $\chi'=|\cdot|^{\pm1}\chi$). Then,

$$r_A^{M_1}(\tau_1)+r_A^{M_2}(\tau_2')>2\cdot|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1\otimes\chi_2$$

$$r_A^{M_1}(\tau_1)+r_A^{M_2}(\tau_2'')>2\cdot|\cdot|^{-\frac{1}{2}}\chi_1\otimes\chi_2\otimes|\cdot|^{\frac{1}{2}}\chi_1.$$

Therefore, it is easy to see that $r_A^{M_1}(\tau_1)+r_A^{M_2}(\tau_2)\not\leq r_A^G(\rho)$ for $\tau_2=\tau_2'$ or τ_2''. By the preceding example, **(IC)** now implies irreducibility.

Next, we apply **(RC)** when $\chi_2=|\cdot|^{\pm\frac{3}{2}}\chi_1$. Say $|\cdot|^{-\frac{3}{2}}\chi_1$, for concreteness. Take

$$\Pi=\mathrm{Ind}_{P_\emptyset}^G(|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{3}{2}}\chi_1)$$

$$\pi=\mathrm{Ind}_{P_2}^G((|det_{GL(2)}|^{-1}\chi_1\circ det_{GL(2)})\otimes|\cdot|^{\frac{1}{2}}\chi_1).$$

We have the following semisimplified Jacquet modules:

$$\begin{aligned}
r_A^G(\rho)\ &=|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{3}{2}}\chi_1+|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1\\
&+|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1
\end{aligned}$$

$$\begin{aligned}
r_A^G(\Pi)\ &=|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{3}{2}}\chi_1+|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1\\
&+|\cdot|^{\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{3}{2}}\chi_1+|\cdot|^{\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{-\frac{1}{2}}\chi_1\\
&+|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1+|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{1}{2}}\chi_1
\end{aligned}$$

$$\begin{aligned}
r_A^G(\pi)\ &=|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{-\frac{1}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1+|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{1}{2}}\chi_1\\
&+|\cdot|^{\frac{1}{2}}\chi_1\otimes|\cdot|^{-\frac{3}{2}}\chi_1\otimes|\cdot|^{-\frac{1}{2}}\chi_1.
\end{aligned}$$

Therefore, one can immediately check that applying **(RC)** with $M=A$ gives reducibility. The case $\chi_2=|\cdot|^{\frac{3}{2}}\chi_1$ is similar. (We note that for both $\chi_2=|\cdot|^{\pm\frac{3}{2}}\chi_1$, ρ contains a one-dimensional subquotient).

A final word: the case $\chi_2=|\cdot|^{\pm\frac{1}{2}}\chi_1$ cannot be resolved using **(IC)** or **(RC)** (it is irreducible–see [Zl]). We shall encounter this phenomenon again in section 4.

3. General linear, symplectic and orthogonal groups

To present some applications of these ideas in the case of classical groups, and say a few words about their proofs, we first need to introduce some notation.

First, let us recall some of the notation of Bernstein-Zelevinsky for general linear groups ([B-Z]). For representations π_1, π_2 of $GL(n_1), GL(n_2)$, they let $\pi_1 \times \pi_2$ denote the representation obtained by inducing $\pi_2 \otimes \pi_2$ from the parabolic subgroup

$$P_{(n_1,n_2)} = M_{(n_1,n_2)} N_{(n_1,n_2)} = \left\{ \left(\begin{array}{c|c} GL(n_1) & * \\ \hline 0 & GL(n_2) \end{array} \right) \right\}.$$

Note that $M_{(n_1,n_2)} \cong GL(n_1) \times GL(n_2)$.

They let $R_n = \mathfrak{R}(GL(n))$ (Grothendieck group) and set

$$R = \bigoplus_{n \geq 0} R_n.$$

Then, $\times$ lifts to R in a natural way, giving a multiplication $\times : R \times R \longrightarrow R$, which extends bilinearly to give $m : R \otimes R \longrightarrow R$. They also introduced a comultiplication

$$m^* : R \longrightarrow R \otimes R$$

defined on irreducible representations π by

$$m^*(\pi) = \sum_{k=0}^{n} s.s.(r_{M_{(k,n-k)}}^{GL(n)}(\pi)) \in R \otimes R,$$

then extended it additively (in the above formula, $s.s.(\tau)$ denotes the semi-simplification of τ).

Bernstein and Zelevinsky showed that with this multiplication and comultiplication, R has the structure of a Hopf algebra. But, they did not really use this structure in their treatment of p-adic general linear groups.

Next, let us recall the definitions of some classical groups. Assume char$F \neq 2$. Let I_n denote the $n \times n$ identity matrix; J_n the $n \times n$ matrix with $j_{a,b} = 1$ if $a + b = n + 1$, $j_{a,b} = 0$ if not. If ^{t} denotes transpose and ${}^\tau$ transpose with respect to the other diagonal (antidiagonal), then

$$Sp(n) = Sp(n, F) = \left\{ g \in GL(2n, F) \mid {}^t g \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right\},$$

$$SO(2n+1) = SO(2n+1, F) = \{ g \in SL(2n+1, F) \mid {}^\tau g g = I_{2n+1} \}.$$

In what follows, we use S_n to denote $Sp(n)$ or $SO(2n + 1)$. The similarity in structures allows us to treat both families simultaneously. This does not imply that the results will be the same in the concrete situations, since generalized rank one reducibilities can be different (an example of such concrete situations with different (generalized) rank one reducibilities are representations with Iwahori fixed vectors).

We now introduce some notation as in [Td1]. Recall that the maximal standard parabolic subgroups of S_n have the form

$$P_{(k)} = M_{(k)} N_{(k)} = \left\{ \left(\begin{array}{c|c|c} g & * & * \\ \hline 0 & h & * \\ \hline 0 & 0 & {}^\tau g^{-1} \end{array} \right) \in S_n \;\middle|\; g \in GL(k),\ h \in S_{n-k} \right\}.$$

Note that $M_{(k)} \cong GL(k) \times S_{n-k}$ in the obvious way. If π is a representation of $GL(k)$ and σ is a representation of S_{n-k}, we let $\pi \rtimes \sigma$ denote the representation of S_n obtained by inducing $\pi \otimes \sigma$ from $P_{(k)}$. (We use $\rtimes$ to make it clear that we are multiplying a representation of $GL(k)$ and a representation of S_{n-k}; any similarity to the notation for semidirect product is purely coincidental.)

Now, we set

$$R(S) = \bigoplus_{n \geq 0} \mathfrak{R}(S_n),$$

a direct sum of Grothendieck groups. We lift $\rtimes$ to $\rtimes : R \times R(S) \longrightarrow R(S)$, and $R(S)$ is an R-module in this way. By bilinearity, we may extend $\rtimes$ to $R \otimes R(S)$. We now define a comodule structure: if σ is an irreducible representation of S_n, set

$$\mu^*(\sigma) = \sum_{k=0}^{n} s.s.(r^{S_n}_{M_{(k)}}(\sigma)) \in R \otimes R(S).$$

We extend μ^* additively to $\mu^* : R(S) \longrightarrow R \otimes R(S)$.

If π is an admissible representation of $GL(p)$, σ an irreducible cuspidal representation of S_k, and τ a subquotient of $\pi \rtimes \sigma$, then we set

$$s_{GL}(\tau) = r^{S_{k+p}}_{M_{(p)}}(\tau).$$

This is often a very important Jacquet module; very convenient for calculations since it essentially requires only the GL-theory.

For admissible finite-length representations π of $GL(k)$ and σ of S_q, we would like a simple formula for $\mu^*(\pi \rtimes \sigma)$. This would enable us to calculate factors in a Jordan-Hölder series for Jacquet modules of $\pi \rtimes \sigma$. To this end, let $s : R \otimes R \longrightarrow R \otimes R$ denote transposition: $s(\sum x_i \otimes y_i) = \sum y_i \otimes x_i$. Set

$$M^* = (m \otimes 1) \circ (\tilde{} \otimes m^*) \circ s \circ m^* : R \longrightarrow R \otimes R,$$

where $\tilde{}$ denotes contragredient. Note that $R \otimes R(S)$ is an $R \otimes R$-module in the natural way: specifically, $(\pi_1 \otimes \pi_2) \rtimes (\pi_3 \otimes \sigma) = (\pi_1 \times \pi_3) \otimes (\pi_2 \rtimes \sigma)$. We have the following formula ([Td2]):

3.1. Theorem: *For admissible finite-length representations π of $GL(p)$ and σ of S_q, we have*

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma).$$

We shall need some additional notation for general linear groups (cf. [B-Z],[Zl]). Let $\nu = |det|_F$ on $GL(n)$, where $|\cdot|_F$ denotes the modulus character of F. For an irreducible cuspidal representation ρ of $GL(p)$, the set

$$\{\rho, \nu\rho, \ldots, \nu^k\rho\} = [\rho, \nu^k\rho]$$

is called a segment in the set of (equivalence classes of) irreducible cuspidal representations of general linear groups. We shall frequently denote such segments by Δ (and simply call them segments). For such a segment, we have

$$\delta(\Delta) \hookrightarrow \nu^k\rho \times \nu^{k-1}\rho \times \ldots \times \nu\rho \times \rho \longrightarrow \zeta(\Delta),$$

with $\delta(\Delta)$ the unique irreducible subrepresentation and $\zeta(\Delta)$ the unique irreducible quotient. Then, $\delta(\Delta)$ is an essentially square-integrable representation (i.e, after a twist by a character of the group, the matrix coefficients are square-integrable mod the center of the group), and Bernstein showed that every essentially square-integrable representation of a general linear group arises in this way. Moreover, we note that

$$m^*(\delta([\rho, \nu^k\rho])) = \sum_{i=-1}^{k} \delta([\nu^{i+1}\rho, \nu^k\rho]) \otimes \delta([\rho, \nu^i\rho]).$$

We now review the Langlands classification for general linear groups. Any irreducible essentially square-integrable representation δ of a general linear group can be written $\delta = \nu^{e(\delta)}\delta^u$, with $e(\delta) \in \mathbb{R}$ and δ^u unitarizable. For irreducible essentially square-integrable representations $\delta_1, \ldots, \delta_k$ of general linear groups, choose a permutation p of $1, \ldots, k$ such that

$$e(\delta_{p(1)}) \geq \ldots \geq e(\delta_{p(k)}).$$

Then, $\delta_{p(1)} \times \ldots \times \delta_{p(k)}$ has a unique irreducible quotient, which we denote by $L(\delta_1, \ldots, \delta_k)$. This is (part of) the Langlands classification for general linear groups.

4. Reducibility and square-integrability

Now, we start to analyze induced representations of the classical groups S_n. Our goal is to study reducibility questions, with an eye toward finding square-integrable subquotients.

The first case to consider is the following: let ρ be an irreducible unitarizable cuspidal representation of $GL(n)$, σ a similar representation of S_m, and $\alpha \in \mathbb{R}$. If $\nu^\alpha\rho \rtimes \sigma$ reduces, then $\rho \cong \tilde{\rho}$. We cannot say anything about

α using only Jacquet modules (of the groups S_n[1]). This is essentially the same situation as with $SL(2)$ (discussed above), and the same basic problem arises. However, we know from Harish-Chandra's work that for any such ρ, σ with $\rho \cong \tilde{\rho}$, there is a unique $\alpha_0 \geq 0$ such that $\nu^{\alpha_0}\rho \rtimes \sigma$ reduces and $\nu^{\alpha}\rho \rtimes \sigma$ is irreducible for all $\alpha \in \mathbb{R} \setminus \{\pm\alpha_0\}$.

For a given pair (ρ, σ) as above, the key piece of information which we build on is the value of α_0 (rather than the actual representations ρ, σ). However, we should point out that the question of determining α_0 for a given pair (ρ, σ) is a difficult one.

What is known about α_0? If ρ is a character of $F^\times = GL(1)$ and σ is the trivial representation of S_0 (the trivial group), then the values of α_0 have been known for decades. The first case other than $SL(2) = Sp(1)$ or $SO(3)$ was settled by J.-L. Waldspurger in the '80's ([Wl]). F. Shahidi made great progress toward understanding α_0. He showed that if σ is generic and $\operatorname{char} F = 0$, then

$$\alpha_0 \in \left\{0, \frac{1}{2}, 1\right\}$$

([Sh2]). We shall call this type of reducibility "generic reducibility," whether or not σ is actually generic. Shahidi also determined α_0 explicitly for some cases with σ the trivial representation of S_0. Further results in this case were recently obtained by F. Murnaghan and J. Repka ([M-R]).

In the case of non-generic σ, M. Reeder ([Rd2]) and C. Mœglin ([Mg]) independently showed that one could have non-generic reducibility, i.e., $\alpha_0 \notin \{0, \frac{1}{2}, 1\}$. Further, Mœglin formulated a conjecture expressing α_0 in terms of the local Langlands correspondence. However, a discussion of this conjecture would be too great a digression.

It is expected that

$$\alpha_0 \in \frac{1}{2}\mathbb{Z}.$$

This would follow, e.g., from Shahidi's conjecture on the existence of a generic representation in each L^2 L-packet ([Sh1]).

[1] It is still possible to get complete answers in some cases using Jacquet modules. In [Td4] we give a simple method with which we determined reducibility points for $Sp(n)$ in certain cases. The trick is that we also took into consideration the group $GSp(n)$. A particular case of this method gives the (well-known) reducibility points of $\mathrm{Ind}_{P_\emptyset}^{SL(2)}(\nu^\alpha\chi)$, where $\alpha \in \mathbb{R}$ and χ is a character of order two of $F^\times$ ($\alpha = 0$ is the only reducibility point). In this case, one needs to bring $GL(2)$ into consideration.

In the case of general linear groups, we have the following fact from Bernstein-Zelevinsky.

Fact: *Let δ_i, $i = 1, 2$, be an irreducible essentially square-integrable representation of $GL(n_i)$. If $\delta_1 \times \delta_2$ reduces and δ_1 or δ_2 is cuspidal, then $\delta_1 \times \delta_2$ contains an essentially square-integrable subquotient. Starting with cuspidal representations, each essentially square-integrable representation arises this way in a finite number of steps.*

This is one of the reasons we studied reducibility questions for representations of the form $\delta \rtimes \sigma$ with δ essentially square-integrable and σ cuspidal or vice-versa.

Since much less is known about non-generic than generic reducibilities, it will be of more interest to focus mainly on non-generic reducibilities. Let us note that even in the non-generic setting, non-generic reducibilities should not occur too often (this follows from the Mœglin conjecture). The study of what happens for the case of non-generic reducibilities is still in its infancy, so the results we shall discuss are simpler. In fact, the existence of such reducibilities seems to date from 1996; we first learned of them from M. Reeder ([Rd1]).

Let ρ be an irreducible, unitarizable cuspidal representation of $GL(p)$; σ an irreducible cuspidal representation of S_q. Suppose that $\nu^\alpha \rho \rtimes \sigma$ reduces for some $\alpha > 1$, $\alpha \in \frac{1}{2}\mathbb{Z}$ (i.e., non-generic reducibility). Then, $\nu^\alpha \rho \rtimes \sigma$ contains a unique square-integrable subquotient, which we denote by $\delta(\nu^\alpha \rho, \sigma)$.

Let ρ_0 be an irreducible, unitarizable cuspidal representation of $GL(p_0)$. Question: for which $\beta \in \mathbb{R}$ does

$$\nu^\beta \rho_0 \rtimes \delta(\nu^\alpha \rho, \sigma) \tag{4-1}$$

reduce? To answer this, we consider two fundamentally different cases.

Case 1: $\rho \not\cong \rho_0$

This case is easy. Using **(RC)** and **(IC)**, one can show that

$$\text{(4-1) reduces} \iff \nu^\beta \rho_0 \rtimes \sigma \text{ reduces.}$$

Further, if (4-1) reduces and $\beta \neq 0$, then $\nu^{\beta}\rho_0 \rtimes \delta(\nu^{\alpha}\rho, \sigma)$ contains a unique square-integrable subquotient. If (4-1) reduces and $\beta = 0$, there is no square-integrable subquotient.

Case 2: $\rho \cong \rho_0$

First, from **(IC)**, we have that (4-1) is irreducible for $\beta \in \mathbb{R} \setminus \{\pm(\alpha - 1), \pm\alpha, \pm(\alpha+1)\}$. Using **(RC)**, we see that (4-1) is reducible for $\beta = \pm(\alpha - 1), \pm(\alpha+1)$. Later, we shall say more about what happens at the reducibility points.

Neither **(IC)** nor **(RC)** can be applied to decide what happens with

$$\nu^{\alpha}\rho \rtimes \delta(\nu^{\alpha}\rho, \sigma).$$

(This is similar to what happened earlier with our $GL(3)$ example when $\chi_2 = |\cdot|^{\pm\frac{1}{2}}\chi_1$.) It is important to determine what happens here. Failure to solve this will have a ripple effect: as the rank of the classical group increases, there will be an increasing number of induced representations which cannot be fully analyzed by Jacquet module methods.

Fact: *If $\alpha \in \frac{1}{2}\mathbb{Z}$ with $\alpha \geq 1$, then $\nu^{\alpha}\rho \rtimes \delta(\nu^{\alpha}\rho, \sigma)$ is irreducible.*

Sketch of proof: We shall illustrate the proof with the case $\alpha = 1$. Suppose that $\Pi = \nu\rho \rtimes \delta(\nu\rho, \sigma)$ reduces. Since

$$\text{length}(s_{GL}(\Pi)) = 2$$

and

$$\nu\rho \times \nu\rho \otimes \sigma \leq s_{GL}(\Pi),$$

there exists an irreducible subquotient π of Π such that

$$s_{GL}(\pi) = \nu\rho \times \nu\rho \otimes \sigma.$$

This and Theorem 3.1 imply

$$\delta([\nu^{-1}\rho, \nu\rho]) \otimes \sigma \not\leq s_{GL}(\rho \rtimes \pi). \tag{4-2}$$

Further, one easily gets that

$$\rho \rtimes \pi \leq \nu^{-1}\rho \times \rho \times \nu\rho \rtimes \sigma$$

and

$$\delta([\nu^{-1}\rho, \nu\rho]) \rtimes \sigma \leq \nu^{-1}\rho \times \rho \times \nu\rho \rtimes \sigma.$$

The multiplicity of $\nu\rho \times \delta([\rho, \nu\rho]) \otimes \sigma$ in s_{GL} of all three of these representations is 2. Therefore, $\rho \rtimes \pi$ and $\delta([\nu^{-1}\rho, \nu\rho]) \rtimes \sigma$ must have a common irreducible subquotient τ (since the Jacquet functor is exact). Thus, (4-2) implies $\delta([\nu^{-1}\rho, \nu\rho]) \otimes \sigma \not\leq s_{GL}(\tau)$. On the other hand, since $\delta([\nu^{-1}\rho, \nu\rho]) \rtimes \sigma$ is unitary, Frobenius reciprocity tells us that $\delta([\nu^{-1}\rho, \nu\rho]) \otimes \sigma \leq s_{GL}(\tau)$, a contradiction. Therefore, Π is irreducible.

For $\alpha > 1$, assuming reducibility, one introduces π in a similar way as above and uses $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \pi$, $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \sigma$ and $\nu^{\alpha}\rho \times \nu^{-\alpha}\rho \times \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma$ to produce a similar contradiction (cf. [Td3]). $\square$

One might well ask how many problems we shall encounter in applying these methods to more complicated situations (when it is not easy to settle the above relatively simple question). In principle, after settling this delicate case and a similar one, we can settle reducibility for a large family of induced representations, essentially just using **(IC)** and **(RC)**. For example:

4.1. Examples:

1. Let χ be a character of $F^{\times}$ and $St_{Sp(n)}$ the Steinberg representation of Sp_n. Then, for $n \geq 1$, $\chi \rtimes St_{Sp(n)}$ is reducible if and only if $\chi = \nu^{\pm(n+1)}$ or $\chi^2 = 1_{F^{\times}}$.

2. Suppose that σ is cuspidal generic, Δ is a segment and char, $(F) = 0$. Then, $\delta(\Delta) \rtimes \sigma$ is reducible if and only if $\tau \rtimes \sigma$ is reducible for some $\tau \in \Delta$. (Note that the assumption σ generic is just to constrain the kinds of reducibility which can occur.)

One can apply a generalized Zelevinsky involution ([Ab], [S-S]) to the parabolically induced representations considered in the above two examples, to obtain reducibility results for degenerate principal series for the Siegel parabolic and the $F^{\times} \times Sp_{n-1}$ parabolic. Actually, C. Jantzen settled the reducibility of any degenerate principal series of these groups, and in the maximal parabolic case, obtained lengths, Langlands data of the irreducible subquotients, and determined the lattice of subrepresentations ([Jn1], [Jn2]). Moreover, his results are a bit more general: rather than just using one-dimensional representations, segment representations are used.

Let us now return to that representation $\nu^\beta\rho \rtimes \delta(\nu^\alpha\rho, \sigma)$, $\alpha \geq 1$, that we considered earlier. We have two pairs of reducibility points

$$\beta = \pm(\alpha+1), \pm(\alpha-1).$$

At $\beta = \pm(\alpha+1)$, we get a square-integrable subquotient. This square-integrable subquotient belongs to a family of square-integrable representations which are closely related to the Steinberg representations. In particular, for $\ell \geq 1$, the representation $\delta([\nu^\alpha\rho, \nu^{\alpha+\ell}\rho]) \rtimes \sigma$ contains a unique square-integrable subrepresentation, which we denote by $\delta([\nu^\alpha\rho, \nu^{\alpha+\ell}\rho], \sigma)$. If $G = Sp_n$, ρ is the trivial representation of $F^\times = GL(1)$, and σ the trivial representation of $Sp(0)$ (so that $\alpha = 1$), then $\delta([\nu\rho, \nu^n\rho], \sigma)$ is the Steinberg representation of Sp_n. We note that

$$\mu^*(\delta([\nu^\alpha\rho, \nu^{\alpha+\ell}\rho], \sigma)) = \sum_{i=-1}^{\ell} \delta([\nu^{\alpha+i+1}\rho, \nu^{\alpha+\ell}\rho]) \otimes \delta([\nu^\alpha\rho, \nu^{\alpha+i}\rho], \sigma).$$

The reducibility at $\beta = \pm(\alpha-1)$ starts a second family of more unusual square-integrable representations. Let $k \in \mathbb{Z}$ be such that $0 < \alpha - k < \alpha$. Then,

$$\zeta([\nu^{\alpha-k}\rho, \nu^\alpha\rho]) \rtimes \sigma$$

contains a unique irreducible subrepresentation, which we denote by

$$\delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma).$$

We note that

$$\mu^*(\delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma)) = \sum_{i=0}^{k+1} \zeta([\nu^{\alpha-k}\rho, \nu^{\alpha-i}\rho]) \otimes \delta([\nu^{\alpha-i+1}\rho, \nu^\alpha\rho], \sigma).$$

These representation are square-integrable. Notice that the presence of ζ's in the Jacquet modules is in marked contrast to the square-integrable representations we have encountered above.

If we continue this approach, the next place we might look for square-integrable representations is among the subquotients of

$$\nu^{\beta}\rho \rtimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+1}\rho], \sigma)$$

when β is a reducibility point. Now, the reducibility points are $\beta = \pm(\alpha+2)$ and $\beta = \pm(\alpha-1)$. When $\beta = \pm(\alpha+2)$, one of the subquotients is the square-integrable representation $\delta([\nu^{\alpha}\rho, \nu^{\alpha+2}\rho], \sigma)$, which we have already encountered.

When $\beta = \pm(\alpha-1)$, there is also a square-integrable subquotient. This subquotient is the beginning of the following family of square-integrable representations: for $k, \ell \in \mathbb{Z}$ such that $0 < \alpha - k \leq \alpha \leq \alpha + \ell$, the representation

$$\delta([\nu^{\alpha+1}\rho, \nu^{\alpha+\ell}\rho]) \times \zeta([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho]) \rtimes \sigma$$

contains a unique irreducible subrepresentation, which we denote by

$$\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+\ell}\rho], \sigma).$$

This representation is square-integrable, and we have

$$s_{GL}(\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+\ell}\rho, \sigma)) = L(\nu^{\alpha-k}\rho, \nu^{\alpha-k+1}\rho, \ldots, \nu^{\alpha-1}\rho, \delta([\nu^{\alpha}\rho, \nu^{\alpha+\ell}\rho])) \otimes \sigma.$$

Note that for $k=0$ or $\ell=0$, this reduces to one of the representations we have already considered.

If we continue with this strategy, starting with $\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+\ell}\rho], \sigma)$, we usually just get new members of the same family. But, there are two interesting directions one can go. We shall just indicate what happens in these directions with a pair of examples.

4.2. Examples:

1. Suppose that $\nu^{\frac{3}{2}}\rho \rtimes \sigma$ reduces ([Rd2]). Then, $\nu^{\beta}\rho \rtimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho], \sigma)$ reduces for
$$\beta \in \left\{\pm\frac{5}{2}, \pm\frac{1}{2}\right\}.$$
For $\beta = \pm\frac{5}{2}$, the square-integrable representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{5}{2}}\rho], \sigma)$ is a subquotient.

When $\beta = \frac{1}{2}$, the induced representations has length two. One of the irreducible subquotients is the Langlands quotient, hence not square-integrable, or even tempered. The other is tempered but not square-integrable. We may ask how this tempered representation arises. In fact, this tempered representation is a subquotient (moreover, a subrepresentation) of

$$\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \delta(\nu^{\frac{3}{2}}\rho, \sigma).$$

It is not hard to show that the full induced representation $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho \rtimes \sigma$ does not contain any square integrable subquotient (cf. [Td5]). So, the prospects of finding interesting new square-integrable representations in this direction do not seem that good.

2. Suppose that $\nu^3\rho \rtimes \sigma$ reduces ([Mg]). Then,

$$\nu^{\beta}\rho \rtimes \delta([\nu^2\rho, \nu^4\rho], \sigma)$$

is reducible if and only if

$$\beta \in \{\pm 1, \pm 3, \pm 5\}.$$

For $\beta = \pm 1, \pm 5$, we get square-integrable representations from the family we have already considered.

Let us look at $\beta = 3$. Take

$$\pi = \delta([\nu^2\rho, \nu^3\rho]) \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma)$$

$$\Pi = \nu^2\rho \times \nu^3\rho \rtimes \delta([\nu^3\rho, \nu^4\rho], \sigma)$$

and apply **(RC)**. Then, there is a unique common irreducible subquotient τ_d of π and $\nu^3\rho \rtimes \delta([\nu^2\rho, \nu^4\rho], \sigma)$. Moreover,

$$s_{GL}(\tau_d) = L(\delta([\nu^2\rho, \nu^3\rho]), \delta([\nu^3\rho, \nu^4\rho])) \otimes \sigma.$$

The Casselman square-integrability criteria implies that τ_d is square-integrable.

A detailed analysis of reducibility points, square-integrable subquotients, etc., obtained by going in this direction is left for the future.

Let us note that using [Jn3], we can combine the square-integrable representations described earlier to obtain new square-integrable representations.

5. Square integrable representations

Having considered the above sequence of examples of square-integrable representations, it is now a natural point at which to ask what might be a general strategy for getting all of the non-cuspidal irreducible square-integrable representations of the groups S_n. We shall roughly describe one possible strategy.

The first step would be to attach square-integrable representations to all segments Δ such that $\delta(\Delta) \rtimes \sigma$ reduces, and further, if $\Delta \cap \tilde{\Delta} \neq \emptyset$, then $\delta(\Delta \cap \tilde{\Delta}) \rtimes \sigma$ also reduces ($\tilde{\Delta}$ denotes $\{\tilde{\pi} | \pi \in \Delta\}$). If $\Delta \cap \tilde{\Delta} = \emptyset$, then the square-integrable representation $\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+\ell}\rho], \sigma) = \delta(\Delta, \sigma)$ considered in the last section is the attached representation (one can find more details about these representations in [Td4]). If not, we attach to each such segment two square-integrable representations. The following theorem describes them:

5.1. Theorem ([Td5]): *Suppose that ρ and σ are irreducible unitarizable cuspidal representations of $GL(p, F)$ and S_q, respectively, such that there exists $\alpha \in (1/2)\,\mathbb{Z}, \alpha \geq 0$ satisfying the following: $\nu^\alpha \rho \rtimes \sigma$ reduces and $\nu^\beta \rho \rtimes \sigma$ is irreducible for $\beta \in (\alpha + \mathbb{Z}) \backslash \{\pm\alpha\}$. Let Δ be a segment such that $e(\delta(\Delta)) > 0$ and $\nu^\alpha \rho \in \Delta \cap \tilde{\Delta}$. Then $\delta(\Delta \cap \tilde{\Delta}) \rtimes \sigma$ reduces into a sum of two inequivalent irreducible tempered subrepresentations τ_1 and τ_2. Each representation $\delta(\Delta \backslash \tilde{\Delta}) \rtimes \tau_i$ contains a unique irreducible subrepresentation, which we denote by*

$$\delta(\Delta, \sigma)_{\tau_i}.$$

The representations $\delta(\Delta, \sigma)_{\tau_i}$, $i = 1, 2$ are inequivalent and square-integrable. Further, they are subrepresentations of $\delta(\Delta) \rtimes \sigma$, and $\delta(\Delta) \rtimes \sigma$ does not contain any other irreducible subrepresentation.

Note that the above theorem describes the representations $\delta(\Delta, \sigma)_{\tau_i}$ as irreducible subrepresentations of standard modules in two different ways. One of these descriptions makes the above theorem of particular interest from the point of view of Whiattaker models. Suppose that σ is a generic (irreducible cuspidal) representation of an odd-orthogonal group. The theorem shows that the standard module (of an odd-orthogonal group) $\delta(\Delta) \rtimes \sigma$, which is induced from a generic representation, has a non-generic irreducible subrepresentation. Therefore, the standard module cannot have an injective Whittaker model. This cannot happen for general linear groups by the results of Jacquet-Shalika ([J-S]).

The other part of the strategy would be to attach sets (packets) of square-integrable representations to sequences of segments as above (subject to certain additional conditions), using the above square-integrable representations attached to single segments as a starting point. An example of such a construction gives the following theorem, which for simplicity we write in the generic setting only (the theorem holds in much wider generality; see [Td6]). Here the representations in packets are parameterized by tempered representations (again).

5.2. Theorem: *Suppose char$(F) = 0$. Let $\Delta_1, \ldots, \Delta_k$ be segments and σ be a non-degenerate irreducible cuspidal representation of S_q such that:*

1. *$e(\delta(\Delta_i)) > 0$, $\delta(\Delta_i) \rtimes \sigma$ reduces, and if $\Delta_i \cap \tilde{\Delta}_i \neq \emptyset$, then $\delta(\Delta_i \cap \tilde{\Delta}_i) \rtimes \sigma$ reduces;*
2. *if $i \neq j$ and $\Delta_i \cap \Delta_j \neq \emptyset$, then either $\Delta_i \cup \tilde{\Delta}_i \subsetneqq \Delta_j \cap \tilde{\Delta}_j$, or $\Delta_j \cup \tilde{\Delta}_j \subsetneqq \Delta_i \cap \tilde{\Delta}_i$.*

Set $l = \text{card}\{i; 1 \leq i \leq k \text{ and } \Delta_i \cap \tilde{\Delta}_i \neq \emptyset\}$. Then $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ decomposes into the sum $\oplus_{j=1}^{2^l} \tau_j$ of 2^l inequivalent irreducible (tempered) representations. Each representation $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau_j$ has a unique irreducible subrepresentation, which we denote by

$$\delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau_j}.$$

The representations $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau_j}$ are square-integrable, and they are inequivalent for different j's. Further, $\delta(\Delta_1, \ldots, \Delta_k, \sigma)_{\tau_j}$ are subrepresentations of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$, and $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$ does not contain any other irreducible subrepresentation.

Let us recall that 2. in Examples 4.1 gives a simple criterion for reducibility of $\delta(\Delta) \rtimes \sigma$ (this criteria explains of the first condition in the theorem).

G. Muić has shown that each generic irreducible square-integrable representation of the groups S_n is one of the representations listed in the above theorem. One can find the details in [Mi] (generic square-integrable representations are related to generic τ_j's).

Suppose that Shahidi's conjecture on the existence of a generic representation in each L^2 L-packet holds. Then the above theorem produces at least one element of each L^2 L-packet (usually, it produces many). Because of this, the above theorem is also of interest in the construction of non-generic square-integrable representations, even those which involve non-generic reducibilities. For example, if the representation $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho \rtimes \sigma$ which we considered in 1. of Examples 4.2 ($\nu^{\frac{3}{2}}\rho \rtimes \sigma$ reduces in that example) were to have a square-integrable subquotient, it would contradict Shahidi's conjecture (and the expected properties of L-packets). (By considering the possible cuspidal supports, one can easily check that $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho \rtimes \sigma$ and the induced representations appearing in Theorem 5.2 have no subquotients in common.) On the other hand, the existence of the square-integrable subquotient τ_d of the representation $\nu^2\rho \times \nu^3\rho \times \nu^3\rho \times \nu^4\rho \rtimes \sigma$ considered in 2. of Examples 4.2 ($\nu^3\rho \rtimes \sigma$ reduces in that example) fits well with Shahidi's conjecture (one can even describe the generic square-integrable representation which should be in the L-packet of τ_d).

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