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ON REGULAR SQUARE INTEGRABLE REPRESENTATIONS OF p-ADIC GROUPS

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Introduction

Let G be a connected reductive group over a local non-archimedean field F (we assume in this paper that char $F \neq 2$). Denote by $P = MN$ a proper parabolic subgroup in G and denote by ρ an irreducible cuspidal representation of M. We shall consider the problem of classifying irreducible (essentially) square integrable subquotients of the parabolically induced representation $\mathrm{Ind}_P^G(\rho)$. These problems are equivalent to the problem of classification of non-cuspidal irreducible square integrable representations of G. Therefore, its solution is quite important to other classification problems.

We have a relatively simple reduction of the problem in the case when P is a maximal parabolic subgroup (then we shall say that we are in the generalized rank one case). In that case we have an irreducible essentially square integrable subquotient if and only if $\mathrm{Ind}_P^G(\rho)$ reduces and if ρ satisfies certain non-unitarity condition (this condition turns out to be just the non-unitarity of ρ for semi simple G, see [C]). In the case of maximal parabolic subgroups, reducibility is understood in the case when P is a minimal parabolic subgroup (then G has split rank one), or G is $\mathrm{GL}(n)$ ([BeZ]). The examination of the reducibility of $\mathrm{Ind}_P^G(\rho)$ in the generalized rank one case was undertaken in F. Shahidi's papers. He made an enormous progress in his work on that hard problem (his work is in the case of char $F = 0$). Shahidi described reducibility in terms of L-functions in [Sd1]. Before Shahidi, J.-L. Waldspurger settled one case in [W]. Although [Sd2] gives a criterion for rank one reducibility in several cases in terms of twisted endoscopy, one likes to get more explicit information about the inducing data. New results in that direction are obtained by C. Mœglin, by G. Muić ([Mi1]), by F. Murnaghan and J. Repka ([MrRp]), by M. Reeder. C. Mœglin made an interesting conjecture which describes the reducibility in the generalized rank one case (in terms of Langlands correspondences).

The problem of classification of non-cuspidal irreducible square integrable representations is solved for $\mathrm{GL}(n, F)$ (see [Z]; for $\mathrm{GL}(n)$ over division algebra see [DKaV]). A partial information exists about classification of irreducible square integrable subquotients of $\mathrm{Ind}_P^G(\rho)$ when P is a minimal parabolic subgroup and G is a connected split reductive group over F: regular irreducible square integrable subquotients are classified in [Ro2] (for the definition of regular representation see below), while the classification is done in [KaLu] when ρ is an unramified character and G has connected center. A number of general facts about non-cuspidal irreducible square integrable representations is obtained in [Sb]. Besides this and a few particular groups, not much is known about classification of non-cuspidal irreducible square integrable representations.

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In this paper we study square integrable representations of groups $\mathrm{GSp}(n,F)$, $\mathrm{Sp}(n,F)$ and $\mathrm{SO}(2n+1,F)$. This paper has two main aims. The first one is to understand which basic properties must be satisfied by ρ when $\mathrm{Ind}_P^G(\rho)$ contains a square integrable subquotient. The second aim is a classification of regular irreducible square integrable representations of these classical groups (one can also interpret our work as a reduction of the problem of the classification of regular irreducible square integrable representations to the problem of the reducibility in the generalized rank one case). We also discuss some consequence of our results in the study of induced representations from generic representations, i.e. those which have Whittaker models (they are also called non-degenerate representations). We shall now describe the results that we have obtained in these directions.

J.N. Bernstein and A.V. Zelevinsky have defined operation $\times$ between admissible representations of general linear groups (see [BeZ] or [Z], or the first section of this paper). We shall consider two series of groups, $\mathrm{Sp}(n,F)$ and $\mathrm{SO}(2n+1,F)$. We shall fix one of these two series, and denote $\mathrm{Sp}(n,F)$ or $\mathrm{SO}(2n+1,F)$ by S_n (depending which series we have fixed). The maximal parabolic subgroups of S_n have Levi factors isomorphic to $\mathrm{GL}(k,F)\times S_{n-k}$. Therefore, for admissible representations π and σ of $\mathrm{GL}(k,F)$ and S_{n-k} respectively, we can define

$$\pi \rtimes \sigma$$

as a representation of S_n which we get by parabolic induction of $\pi\otimes\sigma$ (see the section 6 for precise definition). When we parabolically induce an irreducible admissible representation, then the induced representation can be written as $\pi_1\times\pi_2\times\dots\pi_k\rtimes\sigma$. If ρ is an irreducible cuspidal representation of $\mathrm{GL}(n,F)$, then there is a unique $e(\rho)\in\mathbb{R}$ such that $\rho=|\det|_F^{e(\rho)}\rho^u$, where ρ^u is a unitarizable representation. Let ρ and σ be irreducible cuspidal representations of $\mathrm{GL}(n,F)$ and S_m respectively. In the study of square integrable representations, it will be important to know if the following assumptions on (ρ,σ) hold:

$(\mathcal{R}_G)$ if $\rho\rtimes\sigma$ reduces, then there exists $\alpha_0\in\{0,1/2,1\}$ such that

$$|\det|_F^{\pm\alpha_0}\rho^u\rtimes\sigma \text{ reduce and } |\det|_F^{\beta}\rho^u\rtimes\sigma \text{ is irreducible for } \beta\in\mathbb{R}, |\beta|\neq\alpha_0;$$

$(\mathcal{R}_{(1/2)\mathbb{Z}})$ if $\rho\rtimes\sigma$ reduces, then there exists $\alpha_0\geq 0$ in $(1/2)\mathbb{Z}$ such that

$$|\det|_F^{\pm\alpha_0}\rho^u\rtimes\sigma \text{ reduce and } |\det|_F^{\beta}\rho^u\rtimes\sigma \text{ is irreducible for } \beta\in\mathbb{R}, |\beta|\neq\alpha_0.$$

The analogous assumptions for GSp-groups we denote by (R_G) and $(R_{(1/2)\mathbb{Z}})$ (see section 4). Shahidi has shown that $(\mathcal{R}_G)$ holds if σ is generic ([Sd2]). If a pair (ρ,σ) satisfies $(\mathcal{R}_G)$ (resp. $(\mathcal{R}_{(1/2)\mathbb{Z}})$), then we shall say that ρ and σ have generic reducibility (resp. reducibility in $(1/2)\mathbb{Z}$, or $(1/2)\mathbb{Z}$-reducibility). C. Mœglin and M. Reeder have obtained recently examples of reducibilities in $(1/2)\,\mathbb{Z}$ which are not generic reducibilities. There are no known examples of reducibility which are not in $(1/2)\,\mathbb{Z}$. F. Shahidi has informed us that his Conjecture 9.4 in [Sd1] would imply that $\mathcal{R}_{(1/2)\mathbb{Z}}$ holds in general (Corollary 8.9 in our paper is related to this implication). Mœglin's conjecture also would imply $(\mathcal{R}_{(1/2)\mathbb{Z}})$.

The following theorem shows that among irreducible cuspidal representations of general linear groups only the selfdual play a role in the construction of irreducible non-cuspidal square integrable representations of symplectic and orthogonal groups.

Theorem A. *Let $\rho_1, \rho_2, \dots, \rho_k$ be irreducible cuspidal representations of general linear groups over F, and let σ be a similar representation of S_m. Suppose that $\rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \sigma$ contains a square integrable subquotient. Then:*
(i) $\rho_i^u \cong (\rho_i^u)\tilde{}$ for $i = 1, 2, \dots, k$ ($(\rho_i^u)\tilde{}$ denotes the contragredient representation of ρ_i^u).
(ii) If $(\mathcal{R}_{(1/2)\mathbb{Z}})$ holds, then $e(\rho_i) \in (1/2)\mathbb{Z}$ for $i = 1, 2, \dots, k$.

An additional information about ρ_i's in the above theorem is given in (iii) of Theorem 6.2. The proof of Theorem A is based on simple properties of the operation $\rtimes$. Since such (or a very similar) properties hold also for other classical groups, Theorem A, in a same or a slightly modified version, will hold also for other classical groups. A modification may be necessary in the non-split case. For example, for a unitary group defined by a separable quadratic extension $F \subset E$, the condition $\rho_i^u \cong (\rho_i^u)\tilde{}$ in Theorem A need to be replaced with the condition $\rho_i^u \cong (\mathrm{conj}(\rho_i^u))\tilde{}$ where $\mathrm{conj}(\rho_i^u)$ denotes the representation of $\mathrm{GL}(n, E)$ composed with the non-trivial F-automorphism of E.

An irreducible admissible representation σ of a connected reductive group G will be called regular if there exists a parabolic subgroup $P = MN$ of G and an irreducible cuspidal representation ρ of M such that σ is a subquotient of $\mathrm{Ind}_P^G(\rho)$ and such that all Jacquet modules of $\mathrm{Ind}_P^G(\rho)$ are multiplicity one representations. Recall that each irreducible square integrable representation of $\mathrm{GL}(n)$ is regular (see [Z], and also [DKaV]).

One can attach in a natural way to each non-unitary reducibility in the generalized rank one case of symplectic and of odd-orthogonal group, a sequence of irreducible square integrable representations, which resemble to Steinberg representations (we shall call them square integrable representations of Steinberg type). We shall discuss first the regular square integrable representations which are attached to generic reducibilities. Regular irreducible square integrable representations of symplectic and of odd-orthogonal groups can be considered as certain "combinations" (in the sense of [Jn]) of above square integrable representations of Steinberg type (see Theorem 6.3 for precise statement). In the case of $\mathrm{GSp}(n, F)$, in addition to those of Steinberg type (Proposition 3.1), there are irreducible square integrable representations attached to an easy irreducibility criterion in GSp-setting (Lemma 2.1, the situation $\rho \cong \tilde{\rho}$ and $\sigma \not\cong \omega_\rho \sigma$). We call these square integrable representations of Rodier type (Proposition 3.2). In this way we get in a relatively elementary manner a considerable number of irreducible square integrable representations which are not supported in the minimal parabolic subgroup (see Remark 3.4). Representations of Rodier type are additional "building blocks" in construction of regular irreducible square integrable representations of GSp-groups (see Theorem 3.3). Assuming (R_G) (resp. $(\mathcal{R}_G)$), we show that regular square integrable representations constructed in Theorem 3.3 (resp. 6.3) are all the possible such representations (Theorems 5.3 and 6.4).

To the case of a non-generic non-unitary reducibility, one can attach additional (a little bit unusual) regular irreducible square integrable representations (see Proposition 7.2). Assuming $(\mathcal{R}_{(1/2)\mathbb{Z}})$, each regular irreducible square integrable representation of symplectic or of odd-orthogonal group is a "combination" of such square integrable representations and square integrable representations of Steinberg type (Theorem 7.4).

In contrast to the case of general linear groups, non-regular irreducible square integrable representations of symplectic and orthogonal groups do exist. We construct a wide family of non-regular irreducible square integrable representations in [T7] (see [T7] for more details).

Understanding of regular seems to be an important step in understanding non-regular.

The (essentially) square integrable representations of Theorem 3.3 can be used to see that some basic properties of Whittaker models related to the Langlands classification are quite different in the case of $\mathrm{GL}(n, F)$ and the other classical groups. Let us recall that Langlands classification gives parameterization of all irreducible representations of a reductive group G over a local field as unique irreducible quotients of representations parabolically induced by tempered ones twisted by positive valued characters which satisfy a positivity condition with respect to the roots. These induced representations are called standard modules (this seems to be the most often name in the literature; see the beginning of the seventh and the eighth section for precise definitions in the case of $\mathrm{GL}(n, F)$, $\mathrm{GSp}(n, F)$, $\mathrm{Sp}(n, F)$ and $\mathrm{SO}(2n+1, F)$). H. Jacquet and J.A. Shalika have proved that each standard module of $\mathrm{GL}(n, F)$ has an injective Whittaker module ([JcSk]). One may ask if such strong and useful theorem holds for standard modules of other reductive groups, induced by non-degenerate essentially tempered representations (standard modules induced by non-degenerate essentially tempered representations will be called non-degenerate standard modules; they have Whittaker models and the models are injective for $\mathrm{GL}(n, F)$). Theorem 3.3 provides in a simple way plenty of counter examples for a such statement in the case of $\mathrm{Sp}(n, F)$ (Proposition 8.1).

If a standard module has an injective Whittaker model with respect to a non-degenerate character θ, then each irreducible subrepresentation has a Whittaker model with respect to θ. This implies that the standard module has a unique irreducible subrepresentation. Theorem 3.3 provides us with examples of non-degenerate standard modules with more than one irreducible subrepresentation (and therefore, these standard modules do not have injective Whittaker models; see Proposition 8.1). Still, each of these irreducible subrepresentations is non-degenerate for some non-degenerate character, when $\mathrm{char}\ F = 0$ (this follows from [Mi3]). Thus, a natural question is: are there non-degenerate standard modules with degenerate irreducible subrepresentations (by degenerate irreducible subrepresentation, we shall mean degenerate with respect to any non-degenerate character). This question is interesting in automorphic forms (for example, for L-packets). In the eighth section we shall give positive answer to this question (Corollaries 8.3 and 8.6). In fact, this phenomenon (non-degenerate standard modules with degenerate irreducible subrepresentations) seems to be quite often, when the standard module is reducible.

It is interesting to understand why this difference between general linear groups and other classical groups regarding Whittaker models shows up (i.e. which properties of representation theory of general linear groups and other classical groups enables this difference). It seems that this phenomenon is related to the simple fact that for other classical groups, representations parabolically induced from non-degenerate tempered representations can have degenerate irreducible subrepresentations (this is not the case for general linear groups, and special linear groups, which is related to the irreducibility of tempered induction in the case of general linear groups). Let us explain connection of this fact with construction of non-degenerate standard modules with degenerate irreducible subrepresentations. The rough idea behind the construction is the following. One considers a non-degenerate tempered representation τ of a Levi subgroup $P = MN$ of a classical group G_1, such that $\mathrm{Ind}_P^{G_1}(\tau)$ contains a degenerate irreducible subrepresentations τ_0. If we take any irreducible essentially square integrable representation δ (of a product of general linear groups) such that $\delta \rtimes \tau_0$ is a standard module (of a classical group G), then each irreducible subrepresentation of $\delta \rtimes \tau_0$ is degenerate. If one can now find a non-degenerate standard module π of G, and a representation Π of G such that $\delta \rtimes \tau_0$ and π embed into Π, and that their images in Π intersects non-trivially, then obviously the non-degenerate standard module π has a degenerate irreducible subrepresentation (this is an example that we wanted). To see that intersection is non-trivial (after one has selected $\tau, \tau_0, \delta, \pi$ and Π), it is enough to find a parabolic subgroup $P' = M'N'$ of G such that $r^G_{M'}(\delta \rtimes \tau_0) + r^G_{M'}(\pi) \not\leq r^G_{M'}(\Pi)$ (in the Grothendieck group), where $r^G_{M'}$ denotes the Jacquet functor with respect to the parabolic subgroup $P' = M'N'$. There are very natural candidates for π, τ, δ, Π and τ_0. Let us give one simple example (which have Iwahori fixed vectors) of such candidates in the case of odd-orthogonal groups (this is the simplest case considered in Corollary 8.6; for more examples see the eighth section). The Steinberg (resp. trivial) representation of a connected reductive group G over F will be denoted by St_G (resp 1_G). Take $\tau = \mathrm{St}_{\mathrm{GL}(2,F)} \otimes 1$ (for more explanation regarding notation, see the first section). Considering the representation $|\ |_F^{1/2} \rtimes \mathrm{St}_{\mathrm{SO}(3,F)}$ of $\mathrm{SO}(5,F)$, and comparing Jacquet modules of it and of $\mathrm{St}_{\mathrm{GL}(2,F)} \rtimes 1$, it is easy to see that $\mathrm{Ind}_P^{\mathrm{SO}(5,F)}(\tau) = \mathrm{St}_{\mathrm{GL}(2,F)} \rtimes 1$ reduces. Frobenius reciprocity implies that it reduces into a sum o two (inequivalent) irreducible representations. Since in the case of $\mathrm{SO}(5,F)$ there is only one orbit of non-degenerate characters, one of these irreducible representations is degenerate (with respect to any non-degenerate character). Denote it by τ_0. Take now $\delta = |\ |_F^{3/2}$, $\pi = |\ |_F^{1/2}\mathrm{St}_{\mathrm{GL}(3,F)} \rtimes 1$ and $\Pi = |\ |_F^{3/2} \rtimes \mathrm{St}_{\mathrm{GL}(2,F)} \rtimes 1$ (for P' we take the Siegel parabolic subgroup). In this way one gets that the representation

$$|\ |_F^{1/2}\mathrm{St}_{\mathrm{GL}(3,F)} \rtimes 1$$

of $\mathrm{SO}(7,F)$ contains an irreducible subrepresentation which is degenerate with respect to any non-degenerate character.

This paper follows the approach to the representation theory of classical groups initiated in [T3]. The construction of regular irreducible square integrable representations is based on the structure obtained in [T4]. A part of the results of this paper was announced in [T2].

At the end, we describe the content of each section. The first section introduces notation for $\mathrm{GSp}(n,F)$ and $\mathrm{GL}(n,F)$. In the second section we collect some well-known facts about square integrable representations which we need in the rest. The regular irreducible square integrable representations of $\mathrm{GSp}(n,F)$ related to generic reducibilities are constructed in the third section. The main aim of the fourth section is a proof of Theorem A. We show in the fifth section that we have constructed in the third section all regular irreducible square integrable representations of $\mathrm{GSp}(n,F)$ related to the generic reducibilities. The sixth section deals with regular irreducible square integrable representations of $\mathrm{Sp}(n,F)$ and $\mathrm{SO}(2n+1,F)$ in the case of generic reducibilities, while the seven section deals with regular irreducible square integrable representations of these groups in the case of reducibilities in $(1/2)\mathbb{Z}$. Here we omit proofs, since they are very similar to the proofs in the case of $\mathrm{GSp}(n,F)$ (moreover, they are often more simple than in the case of $\mathrm{GSp}(n,F)$). In the last section we present some families of non-degenerate standard modules with degenerate irreducible subrepresentations.

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1. Preliminaries

We fix a local non-archimedean field F of characteristic different from two. To fix terminology, we first recall some well-known general notions. Let G be the group of F-rational points of a reductive group defined over F. Representations of groups G that we consider in this paper will always be admissible (i.e. each vector in the representation space is fixed by some open subgroup of G, and invariants of open subgroups in the representation space are finite dimensional; see [C]). A representation of G is called cuspidal (resp. square integrable) if it has matrix coefficients compactly supported modulo center (resp. if it has central character which is unitary and if the matrix coefficients are square integrable functions on the quotient of G by the center). A representation π of G is called essentially square integrable if there exists a (not necessarily unitary) character $\chi : G \to \mathbb{C}^\times$ such that $\chi\pi$ is square integrable.

Now we recall some notation for general linear groups that we shall use in the paper. More details about this notation can be found in [BeZ] and [Z]. Let $\alpha = (n_1, \dots, n_k)$ be a partition of a positive integer n. Consider $n \times n$ matrices with entries in F, as block matrices where blocks are of $n_i \times n_j$ sizes, $1 \leqslant i, j \leqslant k$. Denote by M_α^{GL} block-diagonal matrices and denote by P_α^{GL} block-upper triangular matrices. The unipotent radical of P_α^{GL} is denoted by N_α^{GL}.

Let π_1 be an admissible representation of $\mathrm{GL}(n_1, F)$ and let π_2 be an admissible representation of $\mathrm{GL}(n_2, F)$. Then $\pi_1 \times \pi_2$ denotes the parabolically induced representation of $\mathrm{GL}(n_1 + n_2, F)$ from $P_{(n_1,n_2)}^{\mathrm{GL}}$ by $\pi_1 \otimes \pi_2$. Note that the Levi factor $M_{(n_1,n_2)}^{\mathrm{GL}}$ of $P_{(n_1,n_2)}^{\mathrm{GL}}$ is naturally isomorphic to $\mathrm{GL}(n_1, F) \times \mathrm{GL}(n_2, F)$. In this paper we only consider the normalized parabolic induction. Let R_n be the Grothendieck group of the category of all admissible representations of $\mathrm{GL}(n, F)$ of finite length. For an admissible representation π of $\mathrm{GL}(n, F)$, its image in R_n will be denoted by s.s.(π). Set $R = \underset{n \geqslant 0}{\oplus} R_n$. Then $\times$ lifts to a binary operation on R. This operation will be denoted by $\times$ again. In this way R becomes a graded ring.

Let π be an admissible representation of finite length of $\mathrm{GL}(n, F)$. Suppose that $\alpha = (n_1, \dots, n_k)$ is a partition of n. We denote by $r_{\alpha,(n)}(\pi)$, or simply by $r_\alpha(\pi)$, the normalized Jacquet module of π with respect to N_α^{GL}. We consider it as a representation of M_α^{GL}. We can consider s.s. $\big(r_{\alpha,(n)}(\pi)\big) \in R_{n_1} \otimes R_{n_2} \otimes \cdots \otimes R_{n_k}$. Set

$$m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}\ \big(r_{(k,n-k),(n)}(\pi)\big) \in R \otimes R.$$

Lift m^* to an additive mapping $m^* : R \to R \otimes R$. Now R is a graded Hopf algebra.

We now introduce a similar notation for $\mathrm{GSp}(n)$. More details concerning this notation can be found in [T3] and [T4]. The $n \times n$ matrix which has 1's on the second diagonal and which has all other entries equal to 0 will be denoted by J_n. The group of all $(2n) \times (2n)$ matrices with entries in F which satisfy

$${}^tS \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} S = \psi(S) \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$$

for some $\psi(S) \in F^\times$ will be denoted by $\mathrm{GSp}(n, F)$. In the above relation ${}^tS$ denotes the transposed matrix of S.

Let $\alpha = (n_1, \ldots, n_k)$ be a partition of $0 \leqslant m \leqslant n$. Denote by α' the partition $\alpha' = (n_1, \ldots, n_k, 2(n-m), n_k, \ldots, n_1)$ of $2n$. Set

$$P_\alpha = P_{\alpha'}^{\mathrm{GL}} \cap \mathrm{GSp}(n, F), \quad M_\alpha = M_{\alpha'}^{\mathrm{GL}} \cap \mathrm{GSp}(n, F), \quad N_\alpha = N_{\alpha'}^{\mathrm{GL}} \cap \mathrm{GSp}(n, F).$$

Denote by ${}^\tau g$ the transposed matrix of g with respect to the second diagonal. Using the isomorphism

$$(g_1, \ldots, g_k, h) \longmapsto \text{q-diag}\,(g_1, \cdots, g_k, h,\ \psi(h)\ {}^\tau g_k^{-1}, \cdots,\ \psi(h)\ {}^\tau g_1^{-1}),$$

we shall identify $\mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_k, F) \times \mathrm{GSp}(n-m, F)$ with M_α. In the above formula q-diag $(g_1, \cdots, g_k, h,\ \psi(h)\ {}^\tau g_k^{-1}, \cdots,\ \psi(h)\ {}^\tau g_1^{-1})$ denotes the quasi-diagonal matrix which has on the quasi-diagonal matrices $g_1, \cdots, g_k, h,\ \psi(h)\ {}^\tau g_k^{-1}, \cdots,\ \psi(h)\ {}^\tau g_1^{-1}$. Note that $P_\alpha = M_\alpha N_\alpha$ is a Levi decomposition of P_α. We shall denote by $\tilde{G}$ the set of all equivalence classes of irreducible admissible representations of a reductive group G over F. The above identification implies a natural bijection

$$M_\alpha\tilde{} \quad \leftrightarrow \quad \mathrm{GL}(n_1, F)\tilde{} \times \cdots \times \mathrm{GL}(n_k, F)\tilde{} \times \mathrm{GSp}(n-m, F)\tilde{}.$$

Let π be an admissible representation of $\mathrm{GL}(n, F)$ and let σ be an admissible representation of $\mathrm{GSp}(m, F)$. We denote by $\pi \rtimes \sigma$ a parabolically induced representation of $\mathrm{GSp}(n+m, F)$ from $P_{(n)}$ by $\pi \otimes \sigma$. If additionally π' is an admissible representation of $\mathrm{GL}(n', F)$, then

$$(\pi' \times \pi) \rtimes \sigma \cong \pi' \rtimes (\pi \rtimes \sigma) \tag{1-1}$$

([T3], Proposition 4.3, (i), or [T4], Proposition 3.1, (i)).

Denote by $R_n(G)$ the Grothendieck group of the category of all admissible representations of $\mathrm{GSp}(n, F)$ of finite length. Set $R(G) = \underset{n \geqslant 0}{\oplus} R_n(G)$. One can lift $\rtimes$ to an additive mapping $\rtimes : R \times R(G) \to R(G)$. In this way $R(G)$ becomes a graded module over R.

Let σ be an admissible representation of $\mathrm{GSp}(n,F)$ and let $\alpha = (n_1,\dots,n_k)$ be a partition of $m \leqslant n$. The normalized Jacquet module of σ with respect to P_α will be denoted by $s_\alpha(\pi)$. We may consider $\mathrm{s.s.}(s_\alpha(\pi)) \in R_{n_1} \otimes \cdots \otimes R_{n_k} \otimes R_{n-m}(G)$. Set

$$\mu^*(\pi) = \sum_{k=0}^{n} \mathrm{s.s.}(s_{(k)}(\pi)) \in R \otimes R(G).$$

We lift μ^* to an additive mapping $R(G) \to R \otimes R(G)$. There are natural cones of positive elements in R and $R(G)$. Then tensor products among them also carry natural cones of positive elements. The resulting partial orders will be denoted by $\leqslant$. Operations $\times, \rtimes, m^*$ and μ^* transform positive elements to the positive ones.

In the standard way one identifies $F^\times$ with the center of $\mathrm{GL}(n,F)$. In a similar way $F^\times$ will be identified with the center of $\mathrm{GSp}(n,F)$. If τ is an admissible representation of $\mathrm{GL}(n,F)$, or $\mathrm{GSp}(n,F)$, then the central character, if exists, will be denoted by ω_τ. Using the homomorphism $\det : \mathrm{GL}(n,F) \to F^\times$, one identifies characters of $\mathrm{GL}(n,F)$ with the characters of $F^\times$. Let $|\ \ |_F$ be the modulus of F. Then the character $g \mapsto |\det g|_F$ will be denoted by ν. Using the homomorphism $\psi : \mathrm{GSp}(n,F) \to F^\times$ we shall identify characters of $\mathrm{GSp}(n,F)$ with characters of $F^\times$.

Let π be an admissible representation of $\mathrm{GL}(n,F)$ and let σ be an admissible representation of $\mathrm{GSp}(m,F)$. For a character χ of $F^\times$ we have

$$\chi(\pi \rtimes \sigma) \cong \pi \rtimes (\chi\sigma) \tag{1-2}$$

([T3], Proposition 4.3,(ii), or [T4], Proposition 3.1, (ii)). Suppose moreover that π and σ are irreducible (it is enough to suppose that π has a central character). Then we have

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes \omega_\pi \sigma \tag{1-3}$$

in $R(G)$ ([T3], Proposition 4.3, (iii), or [T4], Proposition 3.2, (ii)). Here $\tilde{\pi}$ denotes the contragredient representation of π.

Let π_i be an irreducible admissible representation of $\mathrm{GL}(n_i,F)$, for $i = 1,2,3,4$. Let σ be an irreducible admissible representation of $\mathrm{GSp}(m,F)$. Denote

$$(\pi_1 \otimes \pi_2 \otimes \pi_3) \tilde{\rtimes} (\pi_4 \otimes \sigma) = \tilde{\pi}_1 \times \pi_2 \times \pi_4 \otimes \pi_3 \rtimes \omega_{\pi_1}\sigma.$$

Extend $\tilde{\rtimes}$ to a $\mathbb{Z}$-bilinear mapping $\tilde{\rtimes} : (R \otimes R \otimes R) \times (R \otimes R(G)) \to R \otimes R(G)$. Denote by $s : R \otimes R \to R \otimes R$ the mapping $s(\sum p_i \otimes q_i) = \sum q_i \otimes p_i$. Set $\mathfrak{M}^* = (\mathrm{Id}_R \otimes m^*) \circ s \circ m^*$, where Id_R denotes the identity mapping on R. Now for an admissible representation π of $\mathrm{GL}(n,F)$ of finite length and for an admissible representation σ of finite length Theorem 5.2 of [T4] gives

$$\mu^*(\mathrm{s.s.}(\pi \rtimes \sigma)) = \mathfrak{M}^*(\mathrm{s.s.}(\pi)) \tilde{\rtimes} \mu^*(\mathrm{s.s.}(\sigma)). \tag{1-4}$$

In the remainder, the trivial representation of a group G (on $\mathbb{C}$) will be denoted by 1_G. The trivial representation of the trivial group will be denoted simply by 1.

2. Some basic facts about square integrable representations

At the beginning of this section we shall recall some facts about square integrable representations of general linear groups over F which we shall often use. For more details one can consult [Z].

Let ρ be an irreducible cuspidal representation of some $\mathrm{GL}(m, F)$ and let n be a non-negative integer. The set $\{\rho, \nu\rho, \nu^2\rho, \ldots, \nu^n\rho\}$ is called a segment in cuspidal representations of general linear groups. Such segment will be denoted by $[\rho, \nu^n\rho]$. The number $n+1$ is called the length of the segment. The representation $\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \times \rho$ has a unique irreducible subrepresentation, and a unique irreducible quotient. This irreducible subrepresentation (resp. irreducible quotient) will be denoted by $\delta([\rho, \nu^n\rho])$ (resp. $\mathfrak{s}([\rho, \nu^n\rho])$; note that in the Langlands classification of the seventh section we have $\mathfrak{s}([\rho, \nu^n\rho]) = L((\rho, \nu\rho, \ldots, \nu^{n-1}\rho, \nu^n\rho)))$. Then $\delta([\rho, \nu^n\rho])$ is an essentially square integrable representation (i.e., this representation becomes square integrable modulo center after a twist with a suitable character). It is well-known that $\Delta \mapsto \delta(\Delta)$ is a one-to-one mapping from the set of all segments in cuspidal representations of general linear groups onto the set of all equivalence classes of essentially square integrable representations of general linear groups ([Z]). If $n > n'$, then we take $[\nu^n\rho, \nu^{n'}\rho]$ to be the empty set $\emptyset$. We define $\delta(\emptyset)$ to be identity of R (this is the unique irreducible representation of $\mathrm{GL}(0, F)$). Then we have

$$m^* \left(\delta([\rho, \nu^n\rho])\right) = \sum_{k=0}^{n+1} \delta([\nu^{n+1-k}\rho, \nu^n\rho]) \otimes \delta([\rho, \nu^{n-k}\rho]) \tag{2-1}$$

([Z]). This implies

$$s(m^* \left(\delta([\rho, \nu^n\rho])\right)) = \sum_{k=0}^{n+1} \delta([\rho, \nu^{n-k}\rho]) \otimes \delta([\nu^{n+1-k}\rho, \nu^n\rho]). \tag{2-2}$$

Also

$$r_{(m)^{n+1}} \left(\delta([\rho, \nu^n\rho])\right) = \nu^n\rho \otimes \nu^{n-1}\rho \otimes \cdots \otimes \rho, \tag{2-3}$$

where $(m)^{n+1}$ denotes $(m, m, \ldots, m) \in \mathbb{Z}^{n+1}$. The representation $\delta([\rho, \nu^n\rho])$ is the only irreducible subquotient of $\rho \times \nu\rho \times \cdots \times \nu^n\rho$ such that $\nu^n\rho \otimes \nu^{n-1}\rho \otimes \cdots \otimes \rho$ is a subquotient of the Jacquet module with respect to $P^{\mathrm{GL}}_{(m)^{n+1}}$.

For an irreducible essentially square integrable representation δ of $\mathrm{GL}(n, F)$ there exists a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Denote $\delta^u = \nu^{-e(\delta)}\delta$. Then

$$\delta = \nu^{e(\delta)}\delta^u$$

where $e(\delta) \in \mathbb{R}$ and δ^u is unitarizable.

We now consider GSp-groups. First we have a simple and well known lemma. For the sake of completeness we shall present a proof. The lemma can be proved easily by standard methods of Harish-Chandra, as the referee observed.

2.1. Lemma. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(n,F)$ and let σ be an irreducible cuspidal representation of $GSp(m,F)$. Take $\alpha \in \mathbb{R}$. Suppose that $(\nu^\alpha\rho) \rtimes \sigma$ reduces. Then $\rho \cong \tilde{\rho}$ and $\sigma \cong \omega_\rho\sigma$. In particular, $\omega_\rho^2 = 1_{F^\times}$.*

Proof. There exists a positive valued character χ of $F^\times$ such that $\sigma_0 = \chi\sigma$ is unitarizable. Note that $\nu^\alpha\rho \rtimes \sigma$ is irreducible if and only if $\nu^\alpha\rho \rtimes \sigma_0 = \nu^\alpha\rho \rtimes \chi\sigma \cong \chi(\nu^\alpha\rho \rtimes \sigma)$ is irreducible (the last isomorphism follows from (1-2)). Also $\sigma \cong \omega_\rho\sigma$ if and only if $\chi\sigma \cong \omega_\rho\chi\sigma$, i.e., if and only if $\sigma_0 \cong \omega_\rho\sigma_0$. Therefore, it is enough to prove the lemma when σ is unitarizable. We shall assume that in the rest of the proof. First $\mathfrak{M}^*(\nu^\alpha\rho) = 1\otimes 1\otimes \nu^\alpha\rho + 1\otimes\nu^\alpha\rho\otimes 1 + \nu^\alpha\rho\otimes 1\otimes 1$ and $\mu^*(\sigma) = 1\otimes\sigma$. Now the formula (1-4) implies $\mu^*(\nu^\alpha\rho\rtimes\sigma) = \mathfrak{M}^*(\nu^\alpha\rho)\tilde{\rtimes}\mu^*(\sigma) = 1\otimes\nu^\alpha\rho\rtimes\sigma + \nu^\alpha\rho\otimes\sigma + \nu^{-\alpha}\tilde{\rho}\otimes(\omega_{\nu^\alpha\rho}\sigma)$. Suppose that $\nu^\alpha\rho\rtimes\sigma$ reduces. We consider two cases.

Let $\alpha = 0$. Then $\dim_{\mathbb{C}} \mathrm{End}_{\mathrm{GSp}(n+m,F)}(\nu^\alpha\rho\rtimes\sigma) > 1$. The Frobenius reciprocity implies $\rho\otimes\sigma \cong \tilde{\rho}\otimes(\omega_\rho\sigma)$. Thus $\rho\cong\tilde{\rho}$ and $\sigma\cong\omega_\rho\sigma$.

Now suppose that $\alpha \neq 0$. First we can choose a positive valued character χ of $F^\times$ such that the central character of $\chi((\nu^\alpha\rho)\rtimes\sigma) \cong (\nu^\alpha\rho)\rtimes(\chi\sigma)$ is unitary (the isomorphism follows from (1-2)). Then Proposition 8.1.3 of [C] implies that $\nu^\alpha\rho\rtimes\chi\sigma$ has an essentially square integrable subquotient. Therefore, $\nu^\alpha\rho\rtimes\chi\sigma$ and $\nu^{-\alpha}\rho\rtimes\chi^{-1}\sigma$ have non-disjoint Jordan-Hölder series. Thus $\nu^{-\alpha}\rho\otimes\chi^{-1}\sigma\cong\nu^\alpha\rho\otimes\chi\sigma$ or $\nu^{-\alpha}\rho\otimes\chi^{-1}\sigma\cong\nu^{-\alpha}\tilde{\rho}\otimes\omega_{\nu^\alpha\rho}\chi\sigma$. Since $\nu^{-\alpha}\rho\cong\nu^\alpha\rho$ implies $\alpha = 0$, we have that $\nu^{-\alpha}\rho\otimes\chi^{-1}\sigma\cong\nu^{-\alpha}\tilde{\rho}\otimes\omega_{\nu^\alpha\rho}\chi\sigma$. Thus $\rho\cong\tilde{\rho}$ and $\chi^{-1}\sigma\cong\omega_{\nu^\alpha\rho}\chi\sigma$. The first relation implies $\omega_\rho\cong\overline{\omega}_\rho$. Note that $\omega_{\nu^\alpha\rho} = |\ |_F^{m\alpha}\omega_\rho$. Therefore $\chi^{-1}\sigma\cong|\ |_F^{m\alpha}\omega_\rho\chi\sigma$, and further $\chi^{-2}|\ |_F^{-m\alpha}\sigma\cong\omega_\rho\sigma$. Since the central characters of σ and $\omega_\rho\sigma$ are unitary, and $\chi^{-2}|\ |_F^{-m\alpha}$ is a positive valued character, we have $\chi^{-2}|\ |_F^{-m\alpha} = 1_{F^\times}$. Thus $\sigma\cong\omega_\rho\sigma$. $\square$

2.2. Remark. *Suppose that ρ is an irreducible cuspidal representation of $GL(n,F)$ and suppose that σ is a similar representation of $GSp(m,F)$. Suppose that $\rho\cong\tilde{\rho}$. Let $\alpha\in\mathbb{R}$. Then $\nu^\alpha\rho\rtimes\sigma$ and $\nu^{-\alpha}\rho\rtimes\omega_{\nu^\alpha\rho}\sigma$ have the same Jordan-Hölder series. Note that $\nu^{-\alpha}\rho\rtimes\omega_{\nu^\alpha\rho}\sigma\cong\omega_{\nu^\alpha\rho}(\nu^{-\alpha}\rho\rtimes\sigma)$. Therefore $\nu^\alpha\rho\rtimes\sigma$ reduces if and only if $\nu^{-\alpha}\rho\rtimes\sigma$ reduces.*

We shall now recall the Casselman square integrability criterion in the case of $\mathrm{GSp}(n,F)$ (see [C] and [T3]). Consider the standard inner product on $\mathbb{R}^n$: $((x_i)_{1\le i\le n},(y_i)_{1\le i\le n}) = \sum_{i=1}^n x_iy_i$. For $1\le i\le n$ set

$$\beta_i = (\underbrace{1,1,\dots,1}_{i\ \text{times}},0,0,\dots,0)\in\mathbb{R}^n.$$

Let π be an irreducible smooth representation of $\mathrm{GSp}(n,F)$. We shall suppose that π is not cuspidal. Take a standard proper parabolic subgroup P_α such that $s_\alpha(\pi)\neq 0$, which is minimal among all standard parabolic subgroups which satisfy this property. Let $\alpha = (n_1,\dots,n_\ell)$ and let $n_1+\dots+n_\ell = m$. Suppose that σ is an irreducible subquotient of s_α. Then we can decompose $\sigma = \rho_1\otimes\rho_2\otimes\dots\otimes\rho_\ell\otimes\rho$ where $\rho_i\in\mathrm{GL}(n_i,F)\tilde{}$, and $\rho\in\mathrm{GSp}(n-m,F)\tilde{}$. Set

$$e_*(\sigma) = (\underbrace{e(\rho_1),\dots,e(\rho_1)}_{n_1\ \text{times}},\dots,\underbrace{e(\rho_\ell),\dots,e(\rho_\ell)}_{n_\ell\ \text{times}},\ \underbrace{0,\dots,0}_{n-m\ \text{times}}\ )$$

If π is essentially square integrable, then

$$(2\text{-}4)\qquad (e_*(\sigma),\beta_{n_1})>0,\ (e_*(\sigma),\beta_{n_1+n_2})>0,\ \cdots,\ (e_*(\sigma),\beta_{m-n_\ell})>0,\ (e_*(\sigma),\beta_m)>0.$$

Conversely, if all above inequalities hold for any α and σ as above, then π is essentially square integrable. Further, if instead of >0 the weaker condition ≥ 0 in all relations in (2-4) holds, then π is essentially tempered.

We shall need several times the following well-known fact from the representation theory of general linear groups, which is a (simple) part of the Bernstein-Zelevinsky theory (we present the proof here since we do not know convenient reference for it).

2.3. Lemma. *Let $\rho_1,\rho_2,\dots,\rho_k$ be irreducible cuspidal representations of general linear groups $GL(n_1,F)$, $GL(n_2,F)$, … ,$GL(n_k,F)$ respectively ($n_i\geq 1$). Denote $n=\sum_{i=1}^k n_i$ and let $\alpha=(m_1,m_2,\dots,m_{k'})$ be a partition of n into positive integers.*
(i) If $k'=k$, then

$$(2\text{-}5)\qquad \text{s.s.}(r_{\alpha,(n)}(\rho_1\times\rho_2\times\cdots\times\rho_k))=\sum_{p\in X_\alpha^k}\rho_{p(1)}\otimes\rho_{p(2)}\otimes\cdots\otimes\rho_{p(k)}$$

where X_α^k denotes the set of all permutations p of $\{1,2,\dots,k\}$ such that $m_{p(i)}=m_i$ for all $i\in\{1,2,\dots,k\}$.
(i) If $k'>k$, then $r_{\alpha,(n)}(\rho_1\times\rho_2\times\cdots\times\rho_k)=0$.

Proof. We shall prove the lemma by induction with respect to k. For $k=1$ the lemma obviously holds. Suppose that $k>1$ and that the lemma hold for all $k''<k$. Note that $m^*(\rho_1\times\rho_2\times\cdots\times\rho_k)=\prod_{i=1}^k(\rho_i\otimes 1+1\otimes\rho_i)$. This implies

$$(2\text{-}6)\qquad \text{s.s.}(r_{(m_1,n-m_1),(n)}(\rho_1\times\rho_2\times\cdots\times\rho_k))$$
$$=\sum_{\substack{1\leq j_1<j_2<\cdots<j_l\leq k\\ n_{j_1}+n_{j_2}+\cdots+n_{j_l}=m_1}}\rho_{j_1}\times\rho_{j_2}\times\cdots\times\rho_{j_l}\otimes\prod_{i\in\{1,2,\dots,k\}\setminus\{j_1,j_2,\dots,j_l\}}\rho_i.$$

Now the above formula, the transitivity of Jacquet modules and the inductive assumption (ii), imply that (ii) holds also for k (i.e. $r_{\alpha,(n)}(\rho_1\times\rho_2\times\cdots\times\rho_k)=0$ if $k'>k$).

Suppose that $k'=k$. Now the formula (2-6) after application of the inductive assumption becomes

$$\text{s.s.}(r_{(m_1,n-m_1),(n)}(\rho_1\times\rho_2\times\cdots\times\rho_k))=\sum_{\substack{1\leq j\leq k\\ n_j=m_1}}\rho_j\otimes\prod_{i\in\{1,2,\dots,k\}\setminus\{j\}}\rho_i.$$

The inductive assumption (i) now implies the formula (2-5). $\square$

3. Construction of regular square integrable representations

First we shall give definition of a regular parabolically induced representation (induced from irreducible cuspidal representation). Suppose that G is a connected reductive group over F. Let $P = MN$ be a parabolic subgroup of G. Take an irreducible cuspidal representation ρ of M. We shall say that $\mathrm{Ind}_P^G(\rho)$ is a regular (parabolically induced) representation, if the Jacquet module of $\mathrm{Ind}_P^G(\rho)$ with respect to P is a multiplicity one representation. One can easily see from [BeZ] or [C] that all Jacquet modules of such a regular parabolically induced representation are multiplicity one representations. This condition on $\mathrm{Ind}_P^G(\rho)$ is equivalent to the Casselman's condition in 6.4 of [C] on ρ (ρ is then called regular in [C]).

Let π be an irreducible admissible representation of G. We shall say that π is regular if there exist a parabolic subgroup $P = MN$ in G and an irreducible cuspidal representation ρ of M such that $\mathrm{Ind}_P^G(\rho)$ is regular and π is a subquotient of $\mathrm{Ind}_P^G(\rho)$.

Recall that any irreducible essentially square integrable representation of $\mathrm{GL}(n, F)$ is regular ([Z]).

If π is a representation, then the complex conjugate representation of π will be denoted by $\overline{\pi}$.

3.1. Proposition. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(\ell, F)$ and let σ be an irreducible cuspidal representation of $GSp(m, F)$. Let $\alpha > 0$. Suppose that $\nu^\alpha\rho \rtimes \sigma$ reduces. Let n be a non-negative integer. Then:*
(i) $\rho \cong \tilde{\rho}$ *and* $\sigma \cong \omega_\rho\sigma$.
(ii) The representation $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma$ contains a unique irreducible subrepresentation. This subrepresentation will be denoted by $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$.
(iii) We have $s_{(\ell)^{n+1}}(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)) = \nu^{\alpha+n}\rho \otimes \nu^{\alpha+n-1}\rho \otimes \cdots \otimes \nu^{\alpha+1}\rho \otimes \nu^\alpha\rho \otimes \sigma$.
(iv) If τ is an irreducible representation of $GSp(\ell(n+1)+m, F)$ such that $\nu^{\alpha+n}\rho \otimes \nu^{\alpha+n-1}\rho \otimes \cdots \otimes \nu^{\alpha+1}\rho \otimes \nu^\alpha\rho \otimes \sigma$ is a subquotient of $s_{(\ell)^{n+1}}(\tau)$, then $\tau \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$.
(v) We have $\mu^*(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^\alpha\rho, \nu^{\alpha+k}\rho], \sigma)$, where we assume $\delta(\emptyset, \sigma) = \sigma$ in the above formula.
(vi) The representation $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ is essentially square integrable.
(vii) For a character χ of $F^\times$ we have $\chi\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma) \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \chi\sigma)$.
(viii) $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)^- \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \overline{\sigma})$.
(ix) $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)\tilde{} \cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \omega_{\nu^{-\alpha}\rho}\omega_{\nu^{-(\alpha+1)}\rho} \cdots \omega_{\nu^{-(\alpha+n)}\rho}\tilde{\sigma})$.
(x) Suppose that also ρ', σ', α' and n' satisfy the assumptions from the beginning of the proposition. Then $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma) \cong \delta([\nu^{\alpha'}\rho', \nu^{\alpha'+n'}\rho'], \sigma')$ if and only if $\rho \cong \rho', \sigma \cong \sigma', \alpha = \alpha'$ and $n = n'$.
(xi) The representations $\nu^\alpha\rho \times \nu^{\alpha+1}\rho \times \cdots \times \nu^{\alpha+n}\rho \rtimes \sigma$ and $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma$ are regular (note that they have the same Jordan-Hölder series).

Proof. Lemma 2.1 implies (i). Observe that $\mathfrak{M}^*(\nu^{\alpha+n+1}\rho) = (\nu^{\alpha+n+1}\rho \otimes 1 \otimes 1 + 1 \otimes \nu^{\alpha+n+1}\rho \otimes 1) + 1 \otimes 1 \otimes \nu^{\alpha+n+1}\rho$. The formula (1-4) implies

$$\mathrm{s.s.}(s_{(\ell(n+1))}(\nu^{\alpha+n}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma)) = \sum_{(\varepsilon_i)\in\{\pm 1\}^{n+1}} \nu^{\varepsilon_n(\alpha+n)}\rho \times \cdots \times \nu^{\varepsilon_0\alpha}\rho \otimes \chi_{(\varepsilon_i)}\sigma$$

where $\chi_{(\varepsilon_i)}$ is a suitable character of $F^\times$ depending on $(\varepsilon_i) \in \{\pm 1\}^{n+1}$.

The above formula and [Z] imply that $s_{(\ell)^{n+1}}(\nu^{\alpha+n}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma)$ is of length $2^{n+1}(n+1)!$. Irreducible subquotients are $\nu^{\varepsilon_n(\alpha+p(n))}\rho \otimes \cdots \otimes \nu^{\varepsilon_0(\alpha+p(0))}\rho \otimes \chi_{(\varepsilon_i)}\sigma$ where $(\varepsilon_i) \in \{\pm 1\}^{n+1}$ and p runs over all permutations of the set $\{0, 1, 2, \ldots, n\}$. In particular, we have a multiplicity one representation. Therefore $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$ is a regular representation. This proves (xi). Our definition of regularity, exactness of Jacquet modules and Frobenius reciprocity, imply directly that $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$ contains a unique irreducible subrepresentation (it also follows from Proposition 6.4.1 of [C]; see also [Ro2] for some of such properties of regular representations). This proves (ii). Frobenius reciprocity implies that $\nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$ is a quotient of $s_{(\ell)^{n+1}}(\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma))$. The regularity implies that $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$ is a multiplicity one representation and that $\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)$ is the only irreducible subquotient of $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$ which has $\nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$ for a subquotient of the corresponding Jacquet module.

Suppose that an irreducible representation τ of $\mathrm{GSp}(\ell(n+1)+m, F)$ has $\nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$ for a subquotient of $s_{(\ell)^{n+1}}(\tau)$. We know that τ must be a subrepresentation of some $\mathrm{Ind}_{P_\alpha}^{\mathrm{GSp}(\ell(n+1)+m,F)}(\sigma')$. Since σ' and $\nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$ must be associate ([C]), $\mathrm{Ind}_{P_\alpha}^{\mathrm{GSp}(\ell(n+1)+m,F)}(\sigma')$ and $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$ have the same Jordan-Hölder series. Now the above considerations imply that $\tau \cong \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)$. This proves (iv).

We prove (iii) and (v) by induction on n. For $n = 0$ we know that the statements hold. Let $n \geq 0$ and assume that (iii) and (v) holds for $k \leq n$. First observe that $\pi_1 := \nu^{\alpha+n+1}\rho \rtimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)$ and $\pi_2 := \delta([\nu^{\alpha+n}\rho, \nu^{\alpha+n+1}\rho]) \rtimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+n-1}\rho], \sigma)$ are subrepresentations of $\nu^{\alpha+n+1}\rho \times \nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$. By (1-4) and the inductive assumption we have

$$\text{s.s.}(s_{(\ell(n+2))}(\pi_1)) = \nu^{\alpha+n+1}\rho \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]) \otimes \sigma + \nu^{-\alpha-n-1}\rho \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]) \otimes \omega_{\nu^{\alpha+n+1}\rho}\sigma.$$

Since $\mathfrak{M}^*(\delta([\nu^{\alpha+n}\rho, \nu^{\alpha+n+1}\rho]))$ equals to

$$\begin{aligned}&[1 \otimes \delta([\nu^{\alpha+n}\rho, \nu^{\alpha+n+1}\rho]) \otimes 1 + \nu^{\alpha+n}\rho \otimes \nu^{\alpha+n+1}\rho \otimes 1 + \delta([\nu^{\alpha+n}\rho, \nu^{\alpha+n+1}\rho]) \otimes 1 \otimes 1] \\ &+ [1 \otimes \nu^{\alpha+n+1}\rho \otimes \nu^{\alpha+n}\rho + 1 \otimes 1 \otimes \delta([\nu^{\alpha+n}\rho, \nu^{\alpha+n+1}\rho]) + \nu^{\alpha+n}\rho \otimes 1 \otimes \nu^{\alpha+n+1}\rho],\end{aligned}$$

we have

$$\begin{aligned}\text{s.s.}(s_{(\ell(n+2))}(\pi_2)) = {} & \delta([\nu^{\alpha+n}\rho, \nu^{\alpha+n+1}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+n-1}\rho]) \otimes \sigma \\ & + \nu^{-\alpha-n}\rho \times \nu^{\alpha+n+1}\rho \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+n-1}\rho]) \otimes \omega_{\nu^{\alpha+n}\rho}\sigma \\ & + \delta([\nu^{-\alpha-n-1}\rho, \nu^{-\alpha-n}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+n-1}\rho]) \otimes \omega_{\delta([\nu^{\alpha+n}\rho, \nu^{\alpha+n+1}\rho])}\sigma.\end{aligned}$$

A simple analysis of Jacquet modules for general linear groups imply that $s_{(\ell)^{n+1}}(\pi_1)$ and $s_{(\ell)^{n+1}}(\pi_2)$ have $\nu^{\alpha+n+1}\rho \otimes \nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$ for subquotients. This is also the only irreducible subquotient which appears in both Jacquet modules. Therefore, the intersection of π_1 and π_2 is non-zero. Denote it by π. We have $s_{(\ell)^{n+1}}(\pi) = \nu^{\alpha+n+1}\rho \otimes \nu^{\alpha+n}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$. This implies that π is irreducible. By (iv) we have $\pi = \delta([\nu^{\alpha}\rho, \nu^{\alpha+n+1}\rho], \sigma)$. This proves (iii). Furthermore (iv), together with a similar characterization of the representations $\delta(\Delta)$ of general linear groups (see (2-3) and the comment after the formula), imply the formula in (v). Note that we have also proved that $\delta([\nu^{\alpha}\rho, \nu^{\alpha+n+1}\rho], \sigma)$ is a unique common irreducible subquotient of π_1 and π_2.

From the square integrability criterion (2-4) it is clear that $\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)$ is essentially square integrable. This proves (vi).

The isomorphism $\chi(\nu^{\alpha+n}\rho \times \nu^{\alpha+n+1}\rho \times \cdots \times \nu^{\alpha+1}\rho \times \nu^{\alpha}\rho \rtimes \sigma) \cong \nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha+1}\rho \times \nu^{\alpha}\rho \rtimes \chi\sigma$ implies (vii). Also $(\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma)^{-} \cong \nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \overline{\sigma}$ implies (viii).

We prove (ix) by induction on n. Note that $\delta(\nu^{\alpha}\rho,\sigma)$ can be characterized as a unique essentially square integrable subquotient of $\nu^{\alpha}\rho \rtimes \sigma$. Furthermore, $\delta(\nu^{\alpha}\rho,\sigma)\tilde{}$ is a quotient of $\nu^{-\alpha}\rho \rtimes \tilde{\sigma}$. The representation $\delta(\nu^{\alpha}\rho,\sigma)\tilde{}$ is essentially square integrable and $\nu^{-\alpha}\rho \rtimes \tilde{\sigma}$ and $\nu^{\alpha}\rho \rtimes \omega_{\nu^{-\alpha}\rho}\tilde{\sigma}$ have the same Jordan- Hölder series (see (1-3)). Thus $\delta(\nu^{\alpha}\rho,\sigma)\tilde{} \cong \delta(\nu^{\alpha}\rho,\omega_{\nu^{-\alpha}\rho}\tilde{\sigma})$. This proves (ix) for $n=0$. Fix $n \geq 0$ and suppose that we have proved (ix) for $k \leq n$. Recall that we have proved that $\delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\rho],\sigma)$ can be characterized as a unique common irreducible subquotient of $\nu^{\alpha+n+1}\rho \rtimes \delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\sigma)$ and $\delta([\nu^{\alpha+n}\rho,\nu^{\alpha+n+1}\rho]) \rtimes \delta([\nu^{\alpha}\rho,\nu^{\alpha+n-1}\rho],\sigma)$. From this and the inductive assumption we get that $\delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\rho],\sigma)\tilde{}$ is a common irreducible subquotient of

$$\begin{aligned}&\nu^{-(\alpha+n+1)}\rho \rtimes \delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\omega_{\nu^{-\alpha}\rho}\omega_{\nu^{-\alpha+1}\rho}\cdots\omega_{\nu^{-(\alpha+n)}\rho}\tilde{\sigma}) \qquad \text{and}\\ &\qquad \delta([\nu^{-(\alpha+n+1)}\rho,\nu^{-(\alpha+n)}\rho]) \rtimes \delta([\nu^{\alpha}\rho,\nu^{\alpha+n-1}\rho],\omega_{\nu^{-\alpha}\rho}\omega_{\nu^{-\alpha+1}\rho}\cdots\omega_{\nu^{-(\alpha+n-1)}\rho}\tilde{\sigma}).\end{aligned}$$

Using (1-3) we get the following equalities in $R(G)$

$$\begin{aligned}&\nu^{-(\alpha+n+1)}\rho \rtimes \delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\omega_{\nu^{-\alpha}\rho}\omega_{\nu^{-\alpha+1}\rho}\ldots\omega_{\nu^{-(\alpha+n)}\rho}\tilde{\sigma})\\ &\qquad = \nu^{\alpha+n+1}\rho \rtimes \delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\omega_{\nu^{-\alpha}\rho}\omega_{\nu^{-\alpha+1}\rho}\ldots\omega_{\nu^{-\alpha+n}\rho}\omega_{\nu^{-(\alpha+n+1)}\rho}\tilde{\sigma});\end{aligned}$$

$$\begin{aligned}&\delta([\nu^{-(\alpha+n+1)}\rho,\nu^{-(\alpha+n)}\rho]) \rtimes \delta([\nu^{\alpha}\rho,\nu^{\alpha+n-1}\rho],\omega_{\nu^{-\alpha}\rho}\omega_{\nu^{-\alpha+1}\rho}\ldots\omega_{\nu^{-(\alpha+n-1)}\rho}\tilde{\sigma})\\ &= \delta([\nu^{\alpha+n}\rho,\nu^{\alpha+n+1}\rho]) \rtimes \delta([\nu^{\alpha}\rho,\nu^{\alpha+n-1}\rho],\omega_{\nu^{-\alpha}\rho}\omega_{\nu^{-(\alpha+1)}\rho}\cdots\omega_{\nu^{-(\alpha+n)}\rho}\omega_{\nu^{-(\alpha+n+1)}\rho}\tilde{\sigma})\end{aligned}$$

(since $\omega_{\delta([\nu^{-(\alpha+n+1)}\rho,\nu^{-(\alpha+n)}\rho])} = \omega_{\nu^{-(\alpha+n+1)}\rho\times\nu^{-(\alpha+n)}\rho} = \omega_{\nu^{-(\alpha+n+1)}\rho}\ \omega_{\nu^{-(\alpha+n)}\rho}$). Therefore $\delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\rho],\sigma)\tilde{} \cong \delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\rho],\omega_{\nu^{-\alpha}\rho}\ldots\omega_{\nu^{-(\alpha+n+1)}\rho}\tilde{\sigma})$. This completes the inductive proof of (ix). The statement (x) follows from (iii). ☐

The representations $\delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\sigma)$ will be called essentially square integrable representations of the **Steinberg type**. The following proposition is similar to the previous one. It introduces essentially square integrable representations of a new type. These new representations will be called essentially square integrable representations of the **Rodier type**.

3.2. Proposition. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(\ell,F)$ and let σ be an irreducible cuspidal representation of $GSp(m,F)$. Let n be a non-negative integer. Suppose that $\rho \cong \tilde{\rho}$ and $\omega_{\rho}\sigma \not\cong \sigma$. Then:*
(i) ω_{ρ} is a character of order two
(ii) The representation $\nu^{n}\rho \times \nu^{n-1} \times \cdots \times \nu\rho \times \rho \rtimes \sigma$ has a unique irreducible subrepresentation. We denote this subrepresentation by $\delta([\rho,\nu^{n}\rho],\sigma)$.
(iii) $s.s.(s_{(\ell)^{n+1}}(\delta([\rho,\nu^{n}\rho],\sigma))) = \nu^{n}\rho \otimes \nu^{n-1}\rho \otimes \cdots \otimes \nu\rho \otimes \rho \otimes (\sigma\ +\ \omega_{\rho}\sigma)$.
(iv) Suppose that τ is an irreducible representation of $GSp(\ell(n+1)+m,F)$ such that $\nu^{n}\rho \otimes \cdots \otimes \nu\rho \otimes \rho \otimes (\omega_{\rho})^{p}\sigma$ is a subquotient of $s_{(\ell)^{n+1}}(\tau)$ for $p=1$ or $p=2$. Then $\tau \cong \delta([\rho,\nu^{n}\rho],\sigma)$.

(v) $\mu^*(\delta([\rho,\nu^n\rho],\sigma))=\delta([\rho,\nu^n\rho])\otimes(\sigma+\omega_\rho\sigma)+\sum_{k=0}^n\delta([\nu^{k+1}\rho,\nu^n\rho])\otimes\delta([\rho,\nu^k\rho],\sigma)$.
(vi) The representation $\delta([\rho,\nu^n\rho],\sigma)$ is essentially square integrable for $n\geq 1$. The representation $\delta(\rho,\sigma)$ is by Lemma 2.1 equal to $\rho\rtimes\sigma$, and it is not essentially square integrable.
(vii) $\chi\delta([\rho,\nu^n\rho],\sigma)\cong\delta([\rho,\nu^n\rho],\chi\sigma),\ \chi\in(F^\times)\tilde{}$.
(viii) $\delta([\rho,\nu^n\rho],\sigma)^-\cong\delta([\rho,\nu^n\rho],\overline{\sigma})$.
(ix) $\delta([\rho,\nu^n\rho],\sigma)\tilde{}\cong\delta([\rho,\nu^n\rho],\omega_\rho\omega_{\nu^{-1}\rho}\omega_{\nu^{-2}\rho}\dots\omega_{\nu^{-n}\rho}\tilde{\sigma})$.
(x) Suppose that ρ',σ' and n' satisfy the assumption at the beginning of the proposition. Then $\delta([\rho,\nu^n\rho],\sigma)\cong\delta([\rho',\nu^{n'}\rho'],\sigma')$ if and only if $\rho\cong\rho',n=n',\sigma\cong\sigma'$ or $\rho\cong\rho',n=n',\sigma\cong\omega_\rho\sigma'$.
(xi) The representation $\nu^n\rho\times\nu^{n-1}\times\cdots\times\nu\rho\times\rho\rtimes\sigma$ is regular.

Proof. Excluding one crucial difference, the proof is very similar to the proof of Proposition 3.1. First, $\omega_{\tilde{\rho}}=\omega_\rho^{-1}$ and $\rho\cong\tilde{\rho}$ imply (i). Note that $\rho\rtimes\sigma$ is irreducible by Lemma 2.1. Further s.s.$(s_{(\ell)}(\rho\rtimes\sigma))=\rho\otimes\sigma+\rho\otimes\omega_\rho\sigma$. Using the formula (1-4) one gets by induction

$$\begin{aligned}\text{s.s.}(s_{(\ell(n+1))}&(\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu\rho\times\rho\rtimes\sigma))\\&=\sum_{(\varepsilon_i)\in\{\pm1\}^n}\nu^{\varepsilon_n n}\rho\times\cdots\times\nu^{\varepsilon_2 2}\rho\times\nu^{\varepsilon_1}\rho\rtimes\rho\otimes\chi_{(\varepsilon_i)}\sigma\\&\qquad+\sum_{(\varepsilon_i)\in\{\pm1\}^n}\nu^{\varepsilon_n n}\rho\times\cdots\times\nu^{\varepsilon_2 2}\rho\times\nu^{\varepsilon_1}\rho\times\rho\otimes\chi_{(\varepsilon_i)}\omega_\rho\sigma\end{aligned}$$

where $\chi_{(\varepsilon_i)}$ is a character of $F^\times$ which depends on $(\varepsilon_i)\in\{\pm1\}^n$. From the above formula we see that $\nu^n\rho\otimes\cdots\otimes\nu\rho\otimes\rho\otimes\sigma$ is regular. This proves (xi) and implies (ii). Frobenius reciprocity implies that $\nu^n\rho\otimes\cdots\otimes\nu\rho\otimes\rho\otimes\sigma$ is a quotient of $s_{(\ell)^{n+1}}(\delta([\rho,\nu^n\rho],\sigma))$. The irreducibility of $\rho\rtimes\sigma$ and (1-3) imply $\rho\rtimes\sigma\cong\rho\rtimes\omega_\rho\sigma$. Applying (1-2) we get $\nu^n\rho\times\cdots\times\nu\rho\times\rho\rtimes\sigma\cong\nu^n\rho\times\cdots\times\nu\rho\times\rho\rtimes\omega_\rho\sigma$. Therefore $\nu^n\rho\otimes\cdots\otimes\nu\rho\otimes\rho\otimes\omega_\rho\sigma$ is also a quotient of $s_{(\ell)^{n+1}}(\delta([\rho,\nu^n\rho],\sigma))$. Now (iv) follows in the same way as in the previous proposition. The above isomorphism implies $\delta([\rho,\nu^n\rho],\sigma)\cong\delta([\rho,\nu^n\rho],\omega_\rho\sigma)$.

We shall now prove (iii) and (v). We have seen that (iii) and (v) holds for $n=0$. From (1-4) and (2-1) one gets

$$\text{s.s.}(s_{(2\ell)}(\delta([\rho,\nu\rho])\rtimes\sigma))=\delta([\nu^{-1}\rho,\rho])\otimes\omega_\rho\omega_{\nu\rho}\sigma+\rho\times\nu\rho\otimes\omega_\rho\sigma+\delta([\rho,\nu\rho])\otimes\sigma.$$

This implies

$$\text{s.s.}(s_{(\ell,\ell)}(\delta([\rho,\nu\rho])\rtimes\sigma))=\rho\otimes\nu^{-1}\rho\otimes\omega_\rho\omega_{\nu\rho}\sigma+\rho\otimes\nu\rho\otimes\omega_\rho\sigma+\nu\rho\otimes\rho\otimes\omega_\rho\sigma+\nu\rho\otimes\rho\otimes\sigma.$$

Analogously,

$$\text{s.s.}(s_{(\ell,\ell))}(\delta([\rho,\nu\rho])\rtimes\omega_\rho\sigma))=\rho\otimes\nu^{-1}\rho\otimes\omega_{\nu\rho}\sigma+\rho\otimes\nu\rho\otimes\sigma+\nu\rho\otimes\rho\otimes\sigma+\nu\rho\otimes\rho\otimes\omega_\rho\sigma.$$

From the two last formulas, (iv) and the first part of the proof, one concludes that s.s.$(s_{(\ell,\ell)}(\delta([\rho,\nu\rho],\sigma)))=\nu\rho\otimes\rho\otimes\sigma+\nu\rho\otimes\rho\otimes\omega_\rho\sigma$. Therefore (iii) holds for $n=1$. One gets easily that (v) also holds for $n=1$.

We shall now prove (iii) and (v) by induction on n. Let $n \geq 1$ and suppose that (iii) and (v) hold for $k \leq n$. Then the inductive assumption and (1-4) imply

$$\begin{aligned}
\text{s.s.}(&s_{(\ell(n+2))}(\delta([\nu^n\rho, \nu^{n+1}\rho]) \rtimes \delta([\rho, \nu^{n-1}\rho], \sigma))) \\
&= \delta([\nu^n\rho, \nu^{n+1}\rho]) \times \delta([\rho, \nu^{n-1}\rho]) \otimes (\sigma + \omega_\rho\sigma) \\
&+ \delta([\nu^{-n-1}\rho, \nu^{-n}\rho]) \times \delta([\rho, \nu^{n-1}\rho]) \otimes \omega_{\nu^n\rho}\omega_{\nu^{n+1}\rho}(\sigma + \omega_\rho\sigma) \\
&+ \nu^{-n}\rho \times \nu^{n+1}\rho \times \delta([\rho, \nu^{n-1}\rho]) \otimes \omega_{\nu^n\rho}(\sigma + \omega_\rho\sigma),
\end{aligned}$$

$$\begin{aligned}
\text{s.s.}(&s_{(\ell(n+2))}(\nu^{n+1}\rho \rtimes \delta([\rho, \nu^n\rho], \sigma))) \\
&= \nu^{n+1}\rho \times \delta([\rho, \nu^n\rho]) \otimes (\sigma + \omega_\rho\sigma) + \nu^{-n-1}\rho \times \delta([\rho, \nu^n\rho]) \otimes \omega_{\nu^{n+1}\rho}(\sigma + \omega_\rho\sigma).
\end{aligned}$$

A simple analysis gives now

$$\text{s.s.}(s_{(\ell)^{n+2}}(\delta([\rho, \nu^{n+1}\rho], \sigma))) = \nu^{n+1}\rho \otimes \nu^n\rho \otimes \cdots \otimes \nu\rho \otimes \rho \otimes (\sigma + \omega_\rho\sigma).$$

This proves (iii) for $n+1$. Now (iii) and (iv) imply (v) for $n+1$. This finishes the proof of (iii) and (v). Recall that we have proved that $\delta([\rho, \nu^{n+1}\rho], \sigma)$ is a unique common irreducible subquotient of $\delta([\nu^n\rho, \nu^{n+1}\rho]) \rtimes \delta([\rho, \nu^{n-1}\rho], \sigma)$ and $\nu^{n+1}\rho \rtimes \delta([\rho, \nu^n\rho], \sigma)$ if $n \geq 1$.

The Casselman's square integrability criterion and (iii) give (vi). One gets (vii) and (viii) in the same way as in the Proposition 3.1. One proves (ix) for $\delta([\rho, \nu\rho], \sigma)$ from the fact that $\delta([\rho, \nu\rho], \sigma)$ can be characterized as a unique common irreducible subquotient of $\delta([\rho, \nu\rho]) \rtimes \sigma$ and $\delta([\rho, \nu\rho]) \rtimes \omega_\rho\sigma$ (see the previous part of the proof). One gets (x) by considering the Jacquet modules. The above characterization of $\delta([\rho, \nu^{n+1}\rho], \sigma)$ as a unique common irreducible subquotient of $\delta([\nu^n\rho, \nu^{n+1}\rho]) \rtimes \delta([\rho, \nu^{n-1}\rho], \sigma)$ and $\nu^{n+1}\rho \rtimes \delta([\rho, \nu^n\rho], \sigma)$, $n \geq 1$, enables the inductive prove of (ix). $\square$

3.3. Theorem. *Let ρ_i, $i = 1, 2, \dots, n$, (resp. τ_j, $j = 1, 2, \dots, m$) be unitarizable irreducible cuspidal representations of $GL(a_i, F)$, where $a_i \geq 1$ (resp. $GL(b_j, F)$, where $b_j \geq 1$). The case of $n = 0$ or $m = 0$ is not excluded. Let σ be an irreducible cuspidal representation of $GSp(\ell, F)$. Let X be the group of characters of $F^\times$ generated by the central characters of $\rho_1, \dots, \rho_n$.*

Suppose that $\rho_i \cong \tilde{\rho}_i$ for all $1 \leq i \leq n$ (this implies card $X \leq 2^n$), Card $X = 2^n$ and $\omega\sigma \not\cong \sigma$ for any $\omega \in X \backslash \{1_{F^\times}\}$. Assume that the representations $\tau_1, \dots, \tau_m$ are mutually inequivalent and that for any $j = 1, 2, \dots, m$, there exists $s_j > 0$ such that $\nu^{s_j}\tau_j \rtimes \sigma$ reduces.

Denote $r = a_1 + \cdots + a_n + b_1 + \cdots + b_m$. Let p_i, $i = 1, \dots, n$, and q_j, $j = 1, \dots, m$, be non-negative integers. Set $\Delta_i = [\rho_i, \nu^{p_i}\rho_i]$ and $\Gamma_j = [\nu^{s_j}\tau_j, \nu^{s_j+q_j}\tau_j]$. Then:
(i) The representation

$$(3\text{-}1) \qquad \Big[\prod_{i=1}^{n} \big(\rho_i \times \nu\rho_i \times \cdots \times \nu^{p_i}\rho_i\big)\Big] \times \Big[\prod_{j=1}^{m} \nu^{s_j}\tau_j \times \nu^{s_j+1}\tau_j \times \cdots \times \nu^{s_j+q_j}\tau_j\big)\Big] \rtimes \sigma$$

is regular. Representations $\nu^c\rho_i \rtimes \sigma$ are irreducible for any $c \in \mathbb{R}$.
(ii) The representation $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes \sigma$ contains a unique irreducible subrepresentation. We denote it by $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)$. Moreover, $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)$ can be characterized as a unique irreducible subrepresentation of (3-1).

(iii) $$\text{s.s}(s_{(r)}(\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma))) = \sum_{\omega \in X} \delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \otimes \omega\sigma.$$

(iv) The representation $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)$ is essentially tempered.
(v) The representation $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)$ is essentially square integrable if and only if all p_i are positive.
(vi) $\chi(\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)) \cong \delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \chi\sigma)$, $\chi \in (F^\times)\tilde{}$.
(vii) $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)^- \cong \delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \overline{\sigma})$.
(viii) $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)\tilde{} \cong \delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \omega^{-1}\tilde{\sigma})$ *where ω is the central character of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m)$.*
(ix) We have $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma) \cong \delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \omega\sigma)$ for $\omega \in X$. Besides permutations of the segments, these are the only non-trivial equivalences among these representations.

Proof. Let $\pi_i, i = 1, \ldots, t$, be irreducible cuspidal representations of $\mathrm{GL}(d_i, F)$ respectively ($d_i \geq 1,\ i = 1, \ldots, t$). Set $d = d_1 + \cdots + d_t$. First observe that

$$\text{s.s.}(s_{(d)}(\pi_1 \times \cdots \times \pi_t \rtimes \sigma)) = \sum_{(\varepsilon_i) \in \{\pm 1\}^t} \pi_1^{\varepsilon_1} \times \cdots \times \pi_t^{\varepsilon_t} \otimes \left(\prod_{i=1}^{t} \omega_{\pi_i}^{(1-\varepsilon_i)/2} \right) \sigma \tag{3-2}$$

where π_i^1 denotes π_i and π_i^{-1} denotes $\tilde{\pi}_i$. One proves the above formula directly by induction, using (1-4).

Suppose that $\pi_1 \times \cdots \times \pi_t \rtimes \sigma$ is not regular. Let π be a representation of $\mathrm{GL}(c_i, F)$, $i = 1, \ldots, t$. Denote $\alpha = (c_1, c_2, \ldots, c_t)$. Then the formula (3-2) and the transitivity of Jacquet modules imply that non-regularity can happen only in the following two ways:

(a) There exists at least one $(\varepsilon_i) \in \{\pm 1\}^t$ such that the Jacquet module of $\pi_1^{\varepsilon_1} \times \cdots \times \pi_t^{\varepsilon_t} \otimes \left(\prod_{i=1}^{t} \omega_{\pi_i}^{(1-\varepsilon_i)/2} \right) \sigma$ with respect to M_α is not a multiplicity one representation. Lemma 2.3 implies that there exist $1 \leq i < j \leq j$ such that

$$\pi_i^{\varepsilon_i} \cong \pi_j^{\varepsilon_j}.$$

Suppose that (a) does not happen. Then the formula (3-2) implies that the following possibility must happen:

(b) The elements of the family of representations $\{\pi_i^{\varepsilon_i}; 1 \leq i \leq t\}$ are mutually inequivalent for any $(\varepsilon_i) \in \{\pm 1\}^t$, and there exist two different $(\varepsilon_i), (\varepsilon_i') \in \{\pm 1\}^t$, such that Jacquet modules with respect to M_α of $\pi_1^{\varepsilon_1} \times \cdots \times \pi_t^{\varepsilon_t} \otimes \left(\prod_{i=1}^t \omega_{\pi_i}^{(1-\varepsilon_i)/2}\right) \sigma$ and $\pi_1^{\varepsilon'_1} \times \cdots \times \pi_t^{\varepsilon'_t} \otimes \left(\prod_{i=1}^t \omega_{\pi_i}^{(1-\varepsilon'_i)/2}\right) \sigma$ have a common irreducible subquotient. Note that the elements of the family of representations $\{\pi_i^{\varepsilon_i}; 1 \leq i \leq t\}$ are mutually inequivalent, as well as of $\{\pi_i^{\varepsilon'_i}; 1 \leq i \leq t\}$. The existence of a common irreducible subquotient implies the following two conditions

$$\{\pi_i^{\varepsilon_i}; 1 \leq i \leq t\} = \{\pi_i^{\varepsilon'_i}; 1 \leq i \leq t\},$$
$$\left(\prod_{i=1}^t \omega_{\pi_i}^{(1-\varepsilon_i)/2}\right) \sigma \cong \left(\prod_{i=1}^t \omega_{\pi_i}^{(1-\varepsilon'_i)/2}\right) \sigma.$$

The first condition comes looking at supports of representations of general linear groups (the supports must be the same by the Bernstein-Zelevinsky theory), and the equality in the first condition is assumed (only) in the sense of sets of equivalence classes.

Suppose that the representation (3-1) is not regular. From our assumptions on ρ_i's and τ_j's, and how we have formed (3-1), it is obvious that (a) can not happen. Thus we have (b). The first condition in (b) $\{\pi_i^{\varepsilon_i}; 1 \leq i \leq t\} = \{\pi_i^{\varepsilon'_i}; 1 \leq i \leq t\}$ and our assumptions on ρ_i's and τ_j's imply that if π_i is not unitarizable, then $\varepsilon_i = \varepsilon'_i$. Now the second condition in (b) $\left(\prod_{i=1}^t \omega_{\pi_i}^{(1-\varepsilon_i)/2}\right) \sigma \cong \left(\prod_{i=1}^t \omega_{\pi_i}^{(1-\varepsilon'_i)/2}\right) \sigma$ and the above remark $\varepsilon_i = \varepsilon'_i$ for non-unitarizable π_i's, imply

$$\left(\prod_{\substack{1 \leq i \leq t \\ \pi_i \text{ is unitarizable}}} \omega_{\pi_i}^{(1-\varepsilon_i)/2}\right) \sigma \cong \left(\prod_{\substack{1 \leq i \leq t \\ \pi_i \text{ is unitarizable}}} \omega_{\pi_i}^{(1-\varepsilon'_i)/2}\right) \sigma.$$

Thus

$$\left(\prod_{\substack{1 \leq i \leq t \\ \pi_i \text{ is unitarizable}}} \omega_{\pi_i}^{(\varepsilon_i-\varepsilon'_i)/2}\right) \sigma \cong \sigma.$$

The conditions $\operatorname{card} X = 2^n$ and $\omega\sigma \not\cong \sigma$ for $\omega \in X$, $\omega \neq 1_{F^\times}$, imply $\varepsilon_i = \varepsilon'_i$ for any unitarizable π_i. Thus, $\varepsilon_i = \varepsilon'_i$ for any $i \in \{1, 2, \ldots, t\}$. This is a contradiction. Therefore, (3-1) is regular.

The second statement in (i) follows from Lemma 2.1.

Note that $\delta(\Delta_i) \rtimes \sigma$ is a subrepresentation of $\nu^{p_i}\rho_i \times \nu^{p_i-1}\rho_i \times \cdots \times \nu\rho_i \times \rho_i \rtimes \sigma$, and $\delta(\Gamma_j) \rtimes \sigma$ is a subrepresentation of $\nu^{s_j+q_j}\tau_j \times \cdots \times \nu^{s_j}\tau_j \rtimes \sigma$. Therefore our notation in (ii) agrees with that in Propositions 3.1 and 3.2. Note that by Propositions 3.1 and 3.2 all statements of the theorem hold if $n + m = 1$. Further, one gets (vi) from the isomorphism $\chi(\delta(\Delta_1) \times \cdots \times \delta(\Gamma_m) \rtimes \sigma) \cong \delta(\Delta_1) \times \cdots \times \delta(\Gamma_m) \rtimes \chi\sigma$ (see (1-2)). Also, (vii) is obvious.

Note that $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes \sigma$ is a subrepresentation of (3-1). The regularity of (3-1) implies that (3-1) contains a unique irreducible subrepresentation, say π. Therefore $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes \sigma$ contains a unique irreducible subrepresentation, which is again π. This proves (ii). We denote π

by $\delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma)$. Note that π can be characterized as a unique irreducible subrepresentation of (3-1). Further (3-1) is isomorphic to $\left[\prod_{i=1}^{n}\left(\rho_i \times \nu\rho_i \times \cdots \times \nu^{p_i}\rho_i\right)\right] \times \left[\prod_{j=1}^{m} \nu^{s_j}\tau_j \times \nu^{s_j+1}\tau_j \times \cdots \times \nu^{s_j+q_j}\tau_j)\right] \rtimes \omega\sigma$ for any $\omega \in X$. The isomorphism follows from (1-1), (1-3) and the second statement of (i). One also uses the fact that representations $\rho_i \times \nu^{\alpha}\tau_j$, $1 \le i \le n, 1 \le j \le m$, $\alpha \in \mathbb{R}$, and $\rho_i \times \nu^{\alpha}\rho_j$, $1 \le i, j \le n, i \ne j$, $\alpha \in \mathbb{R}$, are irreducible. The above isomorphism implies $\pi \cong \delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_m, \omega\sigma)$ for $\omega \in X$. This proves the first statement in (ix). For the second statement in (ix), observe that the set (of classes of) irreducible cuspidal representations $\Delta_1 \cup \cdots \cup \Delta_n \cup \Gamma_1 \cup \cdots \cup \Gamma_m$ is determined by the representation $\delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma)$ (this follows from (3-2) and standard facts about Jacquet modules of general linear groups obtained in [BeZ] and [Z]). Also Δ_i's and Γ_j's may be characterized as segments in cuspidal representations of general linear groups in $\Delta_1 \cup \cdots \cup \Delta_n \cup \Gamma_1 \cup \cdots \cup \Gamma_m$, which have the maximal length. Thus Δ_i's and Γ_j's are determined by π. The formula in (iii) implies that σ is determined by π up to a multiple by $\omega \in X$ (note that (iii) is not yet proved).

Recall that $\pi \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes \sigma$. Now Frobenius reciprocity implies that $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \otimes \sigma$ is a quotient of $s_{(r)}(\pi)$. Since $\pi \cong \delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Delta_m, \omega\sigma)$ for any $\omega \in X$, we get that each $\delta(\Delta_1) \times \cdots \times \delta(\Gamma_m) \otimes \omega\sigma, \omega \in X$, is a quotient of $s_{(r)}(\pi)$. It remains to prove that there are no other subquotients in $s_{(r)}(\pi)$. We shall prove this by induction with respect to $n+m$, and this will complete the proof of (iii). We shall also prove (viii) by induction.

We have already mentioned that (iii) holds if $n+m=1$. Suppose that $n=m=1$. Note that $\delta(\Delta_1) \rtimes \delta(\Gamma_1, \sigma)$ and $\delta(\Gamma_1) \rtimes \delta(\Delta_1, \sigma)$ are subrepresentations of $\delta(\Delta_1) \times \delta(\Gamma_1) \rtimes \sigma$ (one gets this using (1-1), Propositions 3.1 and 3.2, and the fact that $\delta(\Delta_1) \times \delta(\Gamma_1)$ is irreducible). Thus $\delta(\Delta_1, \Gamma_1, \sigma)$ is a subrepresentation of $\delta(\Delta_1) \rtimes \delta(\Gamma_1, \sigma)$ and $\delta(\Gamma_1) \times \delta(\Delta_1, \sigma)$. Let ρ be an irreducible cuspidal representation of some general linear group and let ℓ be a positive integer. Then (2-2) implies that we can write

$$\mathfrak{M}^*(\delta([\rho, \nu^{\ell}\rho])) = A + \sum_{i=0}^{\ell+1} \delta([\rho, \nu^{\ell-i}\rho]) \otimes \delta([\nu^{\ell-i+1}\rho, \nu^{\ell}\rho]) \otimes 1 \tag{3-3}$$

where $A \in R \otimes R \otimes R(G) \backslash R \otimes R \otimes R_0(G)$. We get now from (1-4) the following equalities

$$\begin{aligned} &\text{s.s.}(s_{(r)}(\delta(\Delta_1) \rtimes \delta(\Gamma_1, \sigma))) \\ &= \sum_{i=0}^{p_1+1} \delta([\rho_1, \nu^{p_1-i}\rho_1])\tilde{} \times \delta([\nu^{p_1-i+1}\rho_1, \nu^{p_1}\rho_1]) \times \delta([\nu^{s_1}\tau_1, \nu^{s_1+q_1}\tau_1]) \otimes \omega_{\delta([\rho_1, \nu^{p_1-i}\rho_1])}\sigma, \end{aligned}$$

$$\text{s.s.}(s_{(r)}(\delta(\Gamma_1) \times \delta(\Delta_1, \sigma))) = \sum_{\omega \in X} \sum_{j=0}^{q_1+1}$$

$$\delta([\nu^{s_1}\tau_1, \nu^{s_1+q_1-j}\tau_1])\tilde{} \times \delta([\nu^{s_1+q_1-j+1}\tau_1, \nu^{s_1+q_1}\tau_1]) \times \delta([\rho_1, \nu^{p_1}\rho_1]) \otimes \omega_{\delta([\nu^{s_1}\tau_1, \nu^{s_1+q_1-j}\tau_1])}\omega\sigma.$$

The regularity and above two formulas imply that $\delta(\Gamma_1) \rtimes \delta(\Delta_1, \sigma)$ and $\delta(\Delta_1) \rtimes \delta(\Gamma_1, \sigma)$ have non-disjoint Jordan-Hölder series. Furthermore, common factor in the above semi simplifications of the Jacquet modules can come only from the last two terms of the sum in the first formula, and from the last term of the sum with respect to j in the second formula. From this one can easily get that the formula in (iii) holds when $n = m = 1$. Furthermore, one can also conclude that $\delta(\Delta_1, \Gamma_1, \sigma)$ can be characterized as a unique common irreducible subquotient of $\delta(\Gamma_1) \rtimes \delta(\Delta_1, \sigma)$ and $\delta(\Delta_1) \rtimes \delta(\Gamma_1, \sigma)$. Note that $\delta(\Gamma_1, \Delta_1, \sigma)\tilde{}$ is a common irreducible subquotient of $\delta(\Gamma_1)\tilde{} \rtimes \delta(\Delta_1, \sigma)\tilde{}$ and $\delta(\Delta_1)\tilde{} \rtimes \delta(\Gamma_1, \sigma)\tilde{}$. The formulas for contragredients in Propositions 3.1 and 3.2 imply the following equalities in the $R(G)$: $\delta(\Gamma_1)\tilde{} \rtimes \delta(\Delta_1, \sigma)\tilde{} = \delta(\Gamma_1)\tilde{} \rtimes \delta(\Delta_1, \omega^{-1}_{\delta(\Delta_1)}\tilde{\sigma}) = \delta(\Gamma_1) \rtimes \omega^{-1}_{\delta(\Gamma_1)}\delta(\Delta_1, \omega^{-1}_{\delta(\Delta_1)}\tilde{\sigma})) = \delta(\Gamma_1) \rtimes \delta(\Delta_1, \omega^{-1}_{\delta(\Delta_1)\times\delta(\Gamma_1)}\tilde{\sigma})$, and similarly $\delta(\Delta_1)\tilde{} \rtimes \delta(\Gamma_1, \sigma)\tilde{} = \delta(\Delta_1) \times \delta(\Gamma_1, \omega^{-1}_{\delta(\Delta_1)\times\delta(\Gamma_1)}\tilde{\sigma})$. Recall that $\delta(\Delta_1, \Gamma_1, \omega^{-1}_{\delta(\Delta_1)\times\delta(\Gamma_1)}\tilde{\sigma})$ is a unique common irreducible subquotient of $\delta(\Gamma_1) \rtimes \delta(\Delta_1, \omega^{-1}_{\delta(\Delta_1)\times\delta(\Gamma_1)}\tilde{\sigma})$ and $\delta(\Delta_1) \times \delta(\Gamma_1, \omega^{-1}_{\delta(\Delta_1)\times\delta(\Gamma_1)}\tilde{\sigma})$. Thus $\delta(\Gamma_1, \Delta_1, \sigma)\tilde{} \cong \delta(\Gamma_1, \Delta_1, \omega^{-1}_{\delta(\Delta_1)\times\delta(\Gamma_1)}\tilde{\sigma})$. Therefore (viii) holds if $n = m = 1$.

Suppose now that $n + m \geq 2$ and $(n, m) \neq (1, 1)$. Then $n \geq 2$ or $m \geq 2$. Consider first the case $m \geq 2$. Then (1-4), (3-3) and the inductive assumption imply

$$\begin{aligned}
&\text{s.s.}(s_{(r)}(\delta(\Gamma_m) \rtimes \delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_{m-1}, \sigma))) \\
&\quad = \sum_{\omega\in X} \sum_{i=0}^{q_m+1} \delta([\nu^{s_m}\tau_m, \nu^{s_m+q_m-i}, \tau_m])\tilde{} \times \delta([\nu^{s_m+q_m-i+1}\tau_m, \nu^{s_m+q_m}, \tau_m]) \\
&\quad\quad \times \delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_{m-1}) \otimes \omega_{\delta([\nu^{s_m}\tau_m, \nu^{s_m+q_m-i}, \tau_m])}\omega\sigma.
\end{aligned}$$

Similarly

$$\text{s.s.}(s_{(r)}(\delta(\Gamma_{m-1}) \rtimes \delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_{m-2}, \Gamma_m, \sigma))) = \sum_{\omega\in X} \sum_{i=0}^{q_{m-1}+1}$$

$$\delta([\nu^{s_{m-1}}\tau_{m-1}, \nu^{s_{m-1}+q_{m-1}-i}\tau_{m-1}])\tilde{} \times \delta([\nu^{s_{m-1}+q_{m-1}-i+1}\tau_{m-1}, \nu^{s_{m-1}+q_{m-1}}, \tau_{m-1}])$$

$$\times\delta(\Delta_1)\times\cdots\times\delta(\Delta_n)\times\delta(\Gamma_1)\times\cdots\times\delta(\Gamma_{m-2})\times\delta(\Gamma_m)\otimes\omega_{\delta([\nu^{s_{m-1}}\tau_{m-1}, \nu^{s_{m-1}+q_{m-1}-i}\tau_{m-1}])}\omega\sigma.$$

In the same way as in the case of $n = m = 1$ one gets from the above formulas that (iii) holds. Further, one gets that $\delta(\Gamma_m) \rtimes \delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_{m-1}, \sigma)$ and $\delta(\Gamma_{m-1}) \rtimes \delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_{m-2}, \Gamma_m, \sigma)$ have exactly one irreducible subquotient in common, which is $\delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma)$. From this characterization of that representation, one gets (viii) using the inductive assumption similarly as in the case of $n = m = 1$.

Suppose now $n \geq 2$. Let X_t denote the group generated by $\omega_{\rho_1}, \dots, \omega_{\rho_{t-1}}, \omega_{\rho_{t+1}}, \dots, \omega_{\rho_n}$. Again (1-4), (3-3) and the inductive assumption give

$$\begin{aligned}
\text{(3-4)}\quad &\text{s.s.}(s_{(r)}(\delta(\Delta_n) \rtimes \delta(\Delta_1, \dots, \Delta_{n-1}, \Gamma_1, \dots, \Gamma_m, \sigma))) \\
&\quad = \sum_{\omega\in X_n} \sum_{i=0}^{p_n+1} \delta([\rho_n, \nu^{p_n-i}\rho_n])\tilde{} \times \delta([\nu^{p_n-i+1}\rho_n, \nu^{p_n}\rho_n]) \\
&\quad\quad \times \delta(\Delta_1) \times \cdots \times \delta(\Delta_{n-1}) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \otimes \omega_{\delta([\rho_n, \nu^{p_n-i}\rho_n])}\omega\sigma.
\end{aligned}$$

Analogously

$$
\begin{aligned}
\text{(3-5)} \quad & s.s(s_{(r)}(\delta(\Delta_{n-1}) \rtimes \delta(\Delta_1, \dots, \Delta_{n-2}, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma))) \\
&= \sum_{\omega \in X_{n-1}} \sum_{i=0}^{p_{n-1}+1} \delta([\rho_{n-1}, \nu^{p_{n-1}-i}\rho_{n-1}])\tilde{} \times \delta([\nu^{p_{n-1}-i+1}\rho_{n-1}, \nu^{p_{n-1}}\rho_{n-1}]) \\
&\times \delta(\Delta_1) \times \dots \times \delta(\Delta_{n-2}) \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \dots \times \delta(\Gamma_m) \otimes \omega_{\delta([\rho_{n-1}, \nu^{p_{n-1}-i}\rho_{n-1}])}\omega\sigma.
\end{aligned}
$$

First observe that

$$\delta(\Delta_n) \rtimes \delta(\Delta_1, \dots, \Delta_{n-1}, \Gamma_1, \dots, \Gamma_m, \sigma) \text{ and } \delta(\Delta_{n-1}) \rtimes \delta(\Delta_1, \dots, \Delta_{n-2}, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma)$$

have non-disjoint Jordan-Hölder series and that $\delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma)$ is a common irreducible subquotient (one can see this from definition of $\delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma)$, or from (3-4) and (3-5)). Furthermore, if something is in common in the above two Jacquet modules, then it is coming from the last two terms of the sums with respect to i in (3-4) and (3-5). It is easy to see that if some irreducible π is a common subquotient of the Jacquet modules (3-4) and (3-5), then π is already one of the terms of the sum in (iii). This proves (iii) and shows that $\delta(\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma)$ is a unique common irreducible subquotient of $\delta(\Delta_n) \rtimes \delta(\Delta_1, \dots, \Delta_{n-1}, \Gamma_1, \dots, \Gamma_m, \sigma)$ and $\delta(\Delta_{n-1}) \rtimes \delta(\Delta_1, \dots, \Delta_{n-2}, \Delta_n, \Gamma_1, \dots, \Gamma_m, \sigma)$. Similarly as before, one gets from this characterization the formula for the contragredient.

We get (v) from the criterion (2-4) and (iii). Similarly we get (iv). The proof of the theorem is now complete. $\square$

An admissible representation π will be called selfdual (or selfcontragredient) if $\pi \cong \tilde{\pi}$. One can find in [A] a lot of information about such representations of $\mathrm{GL}(n, F)$ in the tame case. Some simple remarks about them are also contained in the following remarks.

3.4. Remarks. *(i) If ρ and σ are characters of $F^\times$ where ρ has order two, then $\rho \cong \tilde{\rho}$ and $\omega_\rho\sigma \ncong \sigma$. Therefore, the conditions of Proposition 3.2 are satisfied and the same proposition gives us essentially square integrable representations which are subquotients of the non-unitary principal series representations. These representations (corresponding to characters) follow from [Ro2] (we found these representations explicitly in [T3]).*

(ii) Suppose that σ is an irreducible cuspidal representation of $GSp(1, F) = GL(2, F)$. If the residual characteristic of F is odd, then σ corresponds to an admissible character θ of a quadratic extension E of F. With few exceptions ([MoSy], Corollary 2.16), if ρ is a quadratic character then $\rho\sigma \ncong \sigma$ unless ρ corresponds to E by the local class field theory. Therefore, Proposition 3.2 provides us with non-cuspidal irreducible essentially square integrable representations which are not supported in the minimal parabolic subgroups.

(iii) If a cuspidal representation ρ of $GL(2, F)$ corresponds to an admissible character θ of a quadratic extension E, such that the restriction of θ to F is trivial, then ρ is selfdual, and its central character ω_ρ corresponds to E by the local class field theory. In particular it is non-trivial. Taking now for σ any character of $F^\times$, one gets $\rho \cong \tilde{\rho}$, $\omega_\rho\sigma \ncong \sigma$. Therefore we again get non-cuspidal essentially square integrable representations which are not supported in the minimal parabolic subgroups.

(iv) There exist examples of fields F and odd $n \geq 3$, such that there is an irreducible selfdual cuspidal representation ρ of $GL(n, F)$ with a non-trivial central character. Then again $\omega_\rho \sigma \not\cong \sigma$ for any character σ of $F^\times$.

(v) From the above examples one can obtain an example of ρ and σ which satisfy conditions $\rho \cong \tilde{\rho}$ and $\omega_\rho \sigma \not\cong \sigma$ of Proposition 3.2, and neither ρ nor σ is a character.

(vi) Based on Shahidi's results in [Sd2] on reducibility of $\rho \rtimes \sigma$ where ρ and σ are irreducible cuspidal representations, Proposition 3.1 gives examples of irreducible non-cuspidal square integrable representations which do not need to be supported in the minimal parabolic subgroups.

The above discussion tells us that Proposition 3.2 produces a considerable number of new irreducible non-cuspidal essentially square integrable representations of GSp-groups which are not supported in the minimal parabolic subgroups. Note that we get them in a relatively simple way. Theorem 3.3 gives irreducible square integrable representations which are not supported in the minimal parabolic subgroups, which are "combinations" of the square integrable representations discussed in the above remarks.

4. Some general facts about square integrable representations of GSp-groups

Let π be an irreducible admissible representation of $\mathrm{GSp}(t, F)$. Then there exists a partition $\alpha = (l_1, l_2, \ldots, l_n)$ of some $t' \leq t$ such that $s_\alpha(\pi)$ has an irreducible cuspidal subquotient, say ρ. We can write

$$\rho = \rho_1 \otimes \cdots \otimes \rho_n \otimes \sigma$$

where ρ_i are irreducible cuspidal representations of $\mathrm{GL}(l_i, F)$ while σ is a similar representation of $\mathrm{GSp}(t - t', F)$. In that case all irreducible subquotients of $s_\alpha(\pi)$ are cuspidal. If ρ is a quotient of $s_\alpha(\pi)$, then π is isomorphic to a subrepresentation of $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$ (the converse is also true: if $\pi \hookrightarrow \rho_1 \times \cdots \times \rho_n \rtimes \sigma$, then $\rho_1 \otimes \cdots \otimes \rho_n \otimes \sigma$ is a quotient of $s_\alpha(\pi)$) There exists always an irreducible cuspidal ρ which is a quotient of $s_\alpha(\pi)$. If ρ is regular, then ρ is always a quotient of $s_\alpha(\pi)$ (see [C]).

We shall assume in the sequel that π is essentially square integrable. In the following lemmas of this section we shall find some of the conditions which representations ρ_i need to satisfy (recall that we assume that $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$ has an irreducible essentially square integrable subquotient). All the following lemmas have similar strategies of proofs.

We shall fix any $i_0 \in \{1, \ldots, n\}$. Denote by $Y_{i_0}^0$ (resp. $Y_{i_0}^1$) the set of all $i \in \{1, \ldots, n\}$ such that there exists $\alpha \in \mathbb{Z}$ so that $\rho_{i_0} \cong \nu^\alpha \rho_i$ (resp. $\tilde{\rho}_{i_0} \cong \nu^\alpha \rho_i$). Set $Y_{i_0} = Y_{i_0}^0 \cup Y_{i_0}^1$ and $Y_{i_0}^c = \{1, 2, \ldots, n\} \backslash Y_{i_0}$.

Recall that $\rho_i = \nu^{e(\rho_i)} \rho_i^u$ where $e(\rho_i) \in \mathbb{R}$ and ρ_i^u is unitarizable (these conditions characterize $e(\rho_i)$ and ρ_i^u uniquely).

4.1. Lemma. *We have $\rho_{i_0}^u \cong (\rho_{i_0}^u)\tilde{}$.*

Proof. First we shall prove the lemma under assumption that ρ is a quotient of $s_\alpha(\pi)$. Then π is a subrepresentation of $\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma$. Suppose that $\rho_{i_0}^u \not\cong (\rho_{i_0}^u)\tilde{}$. Then for any $j_0, j_0' \in Y_{i_0}^0, j_1, j_1' \in Y_{i_0}^1$ and $j_c \in Y_{i_0}^c$ we know that

$$
\begin{aligned}
&\rho_{j_0} \times \tilde{\rho}_{j_0'} \cong \tilde{\rho}_{j_0'} \times \rho_{j_0}, && \rho_{j_1} \times \tilde{\rho}_{j_1'} \cong \tilde{\rho}_{j_1'} \times \rho_{j_1},\\
&\rho_{j_0} \times \rho_{j_1} \cong \rho_{j_1} \times \rho_{j_0}, && \tilde{\rho}_{j_0} \times \tilde{\rho}_{j_1} \cong \tilde{\rho}_{j_1} \times \tilde{\rho}_{j_0},\\
&\rho_{j_0} \times \rho_{j_c} \cong \rho_{j_c} \times \rho_{j_0}, && \tilde{\rho}_{j_0} \times \rho_{j_c} \cong \rho_{j_c} \times \tilde{\rho}_{j_0},\\
&\rho_{j_1} \times \rho_{j_c} \cong \rho_{j_c} \times \rho_{j_1}, && \tilde{\rho}_{j_1} \times \rho_{j_c} \cong \rho_{j_c} \times \tilde{\rho}_{j_1},\\
&\rho_{j_0} \rtimes \sigma \cong \tilde{\rho}_{j_0} \rtimes \omega_{\rho_{j_0}}\sigma, && \rho_{j_1} \rtimes \sigma \cong \tilde{\rho}_{j_1} \rtimes \omega_{\rho_{j_1}}\sigma.
\end{aligned}
\tag{4-1}
$$

(the first two lines follow from [Z], the last line from Lemma 2.1). Write $Y_{i_0}^0 = \{a_1, \dots, a_{k_0}\}$ where $a_i < a_j$ for all $i < j$, $Y_{i_0}^1 = \{b_1, \dots, b_{k_1}\}$ where $b_i < b_j$ for $i < j$, and $Y_{i_0}^c = \{d_1, \dots, d_{k_c}\}$ where $d_i < d_j$ for all $i < j$. Using the relations (4-1), (1-1) and (1-3) one gets

$$
\begin{aligned}
\rho_1 \times \rho_2 \times \cdots \times & \rho_n \rtimes \sigma\\
&\cong \rho_{a_1} \times \rho_{a_2} \times \cdots \times \rho_{a_{k_0}} \times \rho_{d_1} \times \rho_{d_2} \times \cdots \times \rho_{d_{k_c}} \times \rho_{b_1} \times \cdots \times \rho_{b_{k_1}} \rtimes \sigma\\
&\cong \rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \rho_{d_1} \times \cdots \times \rho_{d_{k_c}} \times \rho_{b_1} \times \cdots \times \rho_{b_{k_1-1}} \times \tilde{\rho}_{b_{k_1}} \rtimes \omega_{\rho_{b_{k_1}}}\sigma\\
&\cong \rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \rho_1 \times \cdots \times \rho_{d_{k_c}} \times \tilde{\rho}_{b_{k_1}} \times \rho_{b_1} \times \cdots \times \rho_{b_{k_1-1}} \rtimes \omega_{\rho_{b_{k_1}}}\sigma.
\end{aligned}
$$

Continuing the process one gets

$$\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma \cong \rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \tilde{\rho}_{b_{k_1}} \times \cdots \times \tilde{\rho}_{b_1} \times \rho_{d_1} \times \cdots \times \rho_{d_{k_c}} \rtimes \omega'\sigma$$

for some character ω' of $F^\times$. Similarly, one gets

$$\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma \cong \rho_{b_1} \times \cdots \times \rho_{b_{k_1}} \times \tilde{\rho}_{a_{k_0}} \times \cdots \times \tilde{\rho}_{a_1} \times \rho_{d_1} \times \cdots \times \rho_{d_{k_c}} \rtimes \omega''\sigma$$

for some character ω'' of $F^\times$. The Frobenius reciprocity implies that both

$$
\begin{aligned}
\rho' &= \rho_{a_1} \otimes \cdots \otimes \rho_{a_{k_0}} \otimes \tilde{\rho}_{b_{k_1}} \otimes \cdots \otimes \tilde{\rho}_{b_1} \otimes \rho_{d_1} \otimes \cdots \otimes \rho_{d_{k_c}} \otimes \omega'\sigma,\\
\rho'' &= \rho_{b_1} \otimes \cdots \otimes \rho_{b_{k_1}} \otimes \tilde{\rho}_{a_{k_0}} \otimes \cdots \otimes \tilde{\rho}_{a_1} \otimes \rho_{d_1} \otimes \cdots \otimes \rho_{d_{k_c}} \otimes \omega''\sigma
\end{aligned}
$$

are quotients of corresponding Jacquet modules. Let $\rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \rho_{b_1} \times \cdots \times \rho_{b_{k_1}}$ be a representation of some $\mathrm{GL}(u, F)$. Note that $(\beta_u, e_*(\rho')) = -(\beta_u, e_*(\rho''))$ Then the square integrability criterion (2-4) tells that π can not be essentially square integrable. This is a contradiction. This proves the lemma if ρ is a quotient of $s_\alpha(\pi)$.

In general, there always exists an irreducible quotient $\rho''' = \rho_1''' \times \rho_2''' \times \cdots \times \rho_{n'''}''' \otimes \sigma'''$ of $s_\alpha(\pi)$. The first part of the proof applies to ρ'''. From the formula (3-2), using [Z] (Lemma 2.3 in our paper), we can easily get irreducible subquotients of $s_\alpha(\pi)$ (actually of $s_\alpha(\mathrm{Ind}_{P_\alpha}^{\mathrm{GSp}(t,F)}(\rho''')))$. According to [Z] and (3-2), one gets $\rho_1, \rho_2, \dots, \rho_n$ from $\rho_1''', \rho_2''', \dots, \rho_{n'''}'''$ by permutation (in particular, $n = n'''$), and putting $\tilde{}$ on some of ρ_i'''. This implies the lemma. $\square$

4.2. Lemma. *Suppose $\omega_{\rho_{i_0}^u}\sigma \not\cong \sigma$. Then $e(\rho_{i_0}) \in (1/2)\mathbb{Z}$.*

Proof. Suppose $e(\rho_{i_0}) \notin (1/2)\mathbb{Z}$. Since the relations (4-1) hold in this situation, one can repeat the proof of the preceding lemma and see that π is not essentially square integrable. This finishes the proof. $\square$

4.3. Lemma. *Suppose $\omega_{\rho_{i_0}^u}\sigma \not\cong \sigma$. Then the set $\{\rho_i, \tilde{\rho}_i;\ i \in Y_{i_0}\}$ is a segment in the cuspidal representations of general linear groups.*

Proof. An argument similar to the argument used in the proof of Lemma 4.1 shows that it is enough to prove the lemma when ρ is a quotient of $s_\alpha(\rho)$. Lemmas 4.1 and 4.2 imply $\rho_{i_0}^u \cong (\rho_{i_0}^u)\tilde{}$ and $e(\rho_{i_0}) \in (1/2)\mathbb{Z}$. Suppose that $Z = \{\rho_i, \tilde{\rho}_i, i \in Y_{i_0}\}$ is not a segment. Choose a segment $[\nu^{\alpha_1}\rho_{i_0}^u, \nu^{\alpha_2}\rho_{i_0}^u]$ contained in Z such that:
(i) α_2 is maximal possible,
(ii) α_1 is minimal possible (after α_2 was already fixed).
One can easily see that $\{\rho_{i_0}^u, \nu^{\pm 1/2}\rho_{i_0}^u\} \cap [\nu^{\alpha_1}\rho_{i_0}^u, \nu^{\alpha_2}\rho_{i_0}^u] = \emptyset$ (use the fact that α_2 is maximal, and that Z is symmetric, i.e. $Z = \{\tilde{\tau}; \tau \in Z\}$). We now consider $Y_{i_0}^+ = \{i \in Y_{i_0}; e(\rho_i) \in [\alpha_1, \alpha_2]\}$, $Y_{i_0}^- = \{i \in Y_{i_0}; e(\rho_i) \in [-\alpha_1, -\alpha_2]\}$, and $Y_{i_0}^r = Y_{i_0} \backslash \{Y_{i_0}^+ \cup Y_{i_0}^-\}$. We can now repeat with $Y_{i_0}^+$ and $Y_{i_0}^-$ the process done in the proof of Lemma 4.1, and get that π can not be essentially square integrable. $\square$

4.4. Lemma. *Suppose that ρ is a quotient of $s_\alpha(\pi)$ (which happens if and only if $\pi \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma$). Write $Y_{i_0}^0 = \{a_1, \dots, a_{k_0}\}$, $a_i < a_j$ for $i < j$, as before. Suppose that there exists $i \in Y_{i_0}^0$ with $e(\rho_i) \le 0$. Then there exists $1 \le j < i$ with $e(\rho_j) = e(\rho_i) + 1$. If a is minimal with $e(\rho_a) \le 0$, then $e(\rho_a) > -1$. If b is minimal with $e(\rho_b) < 0$, then $e(\rho_b) \ge -1$.*

Proof. Suppose $e(\rho_i) \le 0$. Take the minimal index $m_1 \le i$ such that there exists a non-negative integer ℓ so that $\rho_{m_1} = \nu^{-\ell}\rho_i$. Then $e(\rho_{m_1}) \le e(\rho_i) \le 0$ and $e(\rho_i) - e(\rho_{m_1}) \in \mathbb{Z}$. Further $\nu^{-1}\rho_{m_1} \not\cong \rho_j$ for any $1 \le j \le m_1$.Suppose that $\nu\rho_{m_1} \not\cong \rho_j$ for all $1 \le j < m_1$. This implies that we can bring ρ_{m_1} at the beginning, i. e. $\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma \cong \rho_{m_1} \times \rho_1 \times \rho_2 \times \cdots \times \rho_{m_1-1} \times \rho_{m_1+1} \times \cdots \times \rho_n \rtimes \sigma$ (use (1-1)). Then $\rho' = \rho_{m_1} \otimes \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{m_1-1} \otimes \rho_{m_1+1} \otimes \cdots \otimes \rho_n \otimes \sigma$ is a subquotient of the corresponding Jacquet module of π (this follows from Frobenius reciprocity). Now $(\beta_k, e(\rho')) \le 0$ if ρ_{m_1} is a representation of $\mathrm{GL}(k, F)$. This contradicts to the square integrability criterion (2-4). Thus $\nu\rho_{m_1} \cong \rho_{m_2}$ for some $1 \le m_2 < m_1$.

If $e(\rho_{m_2}) = e(\nu\rho_{m_1}) = 1 + e(\rho_{m_1}) \le 0$, then we can continue procedure and find $1 \le m_3 < m_2$ with $\rho_{m_3} \cong \nu\rho_{m_2}$. One can continue this procedure if $e(\rho_{m_3}) \le 0$. In this way one gets a sequence of positive integers $i \ge m_1 > m_2 > \cdots > m_{k-1} > m_k \ge 1$ such that $\rho_{m_{i+1}} \cong \nu\rho_{m_i}$ for all $1 \le i \le k-1$, $e(\rho_{m_{k-1}}) \le 0$ and $e(\rho_{m_k}) > 0$. One can now see easily that the lemma holds. $\square$

Let (τ, σ) be a pair consisting of an irreducible cuspidal representation τ of a general linear group and a similar representation σ of $\mathrm{GSp}(m, F)$. We shall now consider the following assumptions, which will be important if (τ, σ) satisfies (for our study of square integrable representations):

(R_G) if $\tau \rtimes \sigma$ reduces, then there exists $\alpha_0 \in \{0, 1/2, 1\}$ such that

$\nu^{\pm\alpha_0}\tau^u \rtimes \sigma$ reduce and $\nu^\beta\tau^u \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}, |\beta| \neq \alpha_0$;

$(R_{(1/2)\mathbb{Z}})$ if $\tau \rtimes \sigma$ reduces, then there exists $\alpha_0 \geq 0$ in $(1/2)\mathbb{Z}$ such that

$\nu^{\pm\alpha_0}\tau^u \rtimes \sigma$ reduce and $\nu^\beta\tau^u \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}, |\beta| \neq \alpha_0$.

Recall that if $\tau \rtimes \sigma$ reduces, then τ^u is selfdual, i.e. $(\tau^u)\tilde{} \cong \tau^u$.

4.5. Remark. *Suppose that F is a field of characteristic 0. Then F. Shahidi has shown that (R_G) holds if σ is generic. One case was earlier examined by Waldspurger ([W], he does not assume $\operatorname{char} F = 0$). In particular, it holds for any τ if σ is a character ([Sd2]). C. Mœglin has shown that there are examples where (R_G) does not hold ([Mœ]). Let us note that $(R_{(1/2)\mathbb{Z}})$ holds in all known examples (to this author) of reducibility of $\tau \rtimes \sigma$. Shahidi's Conjecture 9.4 in [Sd1] would imply that $(R_{(1/2)\mathbb{Z}})$ hold in general.*

We continue with the notation from the beginning of the section. In the same way as Lemma 4.2 we get the following:

4.6. Lemma. *Suppose that $(R_{(1/2)\mathbb{Z}})$ holds for (ρ_i, σ), $i = 1, 2, \ldots, n$. Then $e(\rho_{i_0}) \in (1/2)\mathbb{Z}$ for $i = 1, 2, \ldots, n$* □

4.7. Lemma. *Assume that (R_G) holds for (ρ_i, σ), $i = 1, 2, \ldots, n$.*
(i) Suppose that $\nu\rho_{i_0}^u \rtimes \sigma$ reduces. If $e(\rho_{i_0}) \in \mathbb{Z}$, then $\{\rho_i, \tilde{\rho}_i; i \in Y_{i_0}\} \cup \{\rho_{i_0}^u\}$ is a segment in cuspidal representations of general linear groups. If $e(\rho_{i_0}) \in 1/2 + \mathbb{Z}$, then $\{\rho_i, \tilde{\rho}_i; i \in Y_{i_0}\}$ is a segment in cuspidal representations of general linear groups.
(ii) If $\nu\rho_{i_0}^u \rtimes \sigma$ is irreducible, then $\{\rho_i, \tilde{\rho}_i; i \in Y_{i_0}\}$ is a segment in cuspidal representations of general linear groups.

Proof. Denote $X = \{\rho_i, \tilde{\rho}_i; i \in Y_{i_0}\}$. Clearly $X = \tilde{X}$, i.e. $X = \{\tilde{\tau}, \tau \in X\}$. Proceeds case by case, using the strategy of the proof of Lemma 4.3.

(i) We know from the previous lemma that $e(\rho_{i_0}) \in (1/2)\mathbb{Z}$. Also $(\rho_{i_0}^u)\tilde{} \cong \rho_{i_0}^u$ by Lemma 4.1. Suppose that $e(\rho_{i_0}) \in 1/2 + \mathbb{Z}$. Recall $X = \tilde{X}$. One proves that X is a segment in the same way as in the proof of Lemma 4.3 (note that in this situation $\tau \rtimes \sigma$ is irreducible for any $\tau \in X$). Suppose now $e(\rho_{i_0}) \in \mathbb{Z}$. Then in a similar way we get that $X \cup \{\rho_{i_0}^u\}$ is a segment (note that in the case of the Steinberg representation, X is not a segment).

(ii) One considers two cases. The first is when $\rho_{i_0}^u \rtimes \sigma$ reduces. If $e(\rho_{i_0}) \in 1/2 + \mathbb{Z}$ or $e(\rho_{i_0}) \in \mathbb{Z}$, then one gets that X is a segment similarly as before. Suppose now that $\nu^{1/2}\rho_{i_0}^u \rtimes \sigma$ reduces. Again we get in both cases $e(\rho_{i_0}) \in 1/2 + \mathbb{Z}$, $e(\rho_{i_0}) \in \mathbb{Z}$ that X is a segment. □

4.8. Remark. *We expect that if $\nu\rho_{i_0}^u \rtimes \sigma$ reduces, then $e(\rho_{i_0}) \in \mathbb{Z}$ (i.e. that the second possibility in (i) of the above lemma never happens).*

At the end of this section we shall summarize some of the main properties of the parabolically induced representations which have essentially square integrable subquotients, which we have proved.

4.9. Theorem. *Let $\rho_1, \rho_2, \dots, \rho_n$ be irreducible cuspidal representations of general linear groups and let σ be a similar representation of $GSp(m, F)$. Suppose that $\rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma$ has an essentially square integrable subquotient. Then*
(i) $(\rho_i^u)\tilde{} \cong \rho_i^u$ for any $1 \leq i \leq n$.
(ii) Suppose that $(R_{(1/2)\mathbb{Z}})$ holds for (ρ_j, σ), $1 \leq j \leq n$. Then $e(\rho_j) \in (1/2)\mathbb{Z}$ for $1 \leq j \leq n$.
(iii) Suppose that (R_G) holds for (ρ_j, σ), $1 \leq j \leq n$. Let $1 \leq i \leq n$. Denote by $Y(\rho_i)$ the set of all ρ_j, $1 \leq j \leq n$, such that $\rho_j \cong \nu^z \rho_i$ for some $z \in \mathbb{Z}$. Set $Y(\rho_i)\tilde{} = \{\tilde{\tau}; \tau \in Y(\rho_i)\}$. If $\nu\rho_{i_0}^u \rtimes \sigma$ reduces and $e(\rho_{i_0}) \in \mathbb{Z}$, then $Y(\rho_i) \cup Y(\rho_i)\tilde{} \cup \{\rho_i^u\}$ is a segment in cuspidal representations of general linear groups. In all other cases $Y(\rho_i) \cup Y(\rho_i)\tilde{}$ is a segment in cuspidal representations of general linear groups. $\square$

5. Exhaustion in the regular case (generic reducibilities)

In the following two lemmas we shall use the notation introduced at the beginning of the preceding section.

Recall that π was essentially square integrable. If $\rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma$ is regular, then (3-2) and [Z] imply that $\rho_i \not\cong \rho_j$ and $\rho_i \not\cong \tilde{\rho}_j$ for $i \neq j$.

5.1. Lemma. *Let ρ be a quotient of $s_\alpha(\pi)$ (i.e. $\pi \hookrightarrow \rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma$). Suppose that $\rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma$ is regular and $\omega_{\rho_{i_0}^u}\sigma \not\cong \sigma$. Write $Y_{i_0} = \{a_1, \dots, a_k\}$ where $a_i < a_j$ for $i < j$. Then $k \geq 2$ and we have $\rho_{a_k} = \rho_{i_0}^u$, $\rho_{a_{k-1}} = \nu\rho_{i_0}^u, \dots, \rho_{a_1} = \nu^{k-1}\rho_{i_0}^u$, i.e. $\rho_{a_i} = \nu^{k-i}\rho_{i_0}^u$ for $1 \leq i \leq k$.*

Proof. Suppose that $e(\rho_i) < 0$ for some $i \in Y_{i_0}$. Lemma 4.2 implies $e(\rho_{i_0}) \in (1/2)\mathbb{Z}$. By Lemmas 4.2 and 4.4 there exists $j \in Y_{i_0}$ with $e(\rho_j) = -1/2$ or -1. Suppose that $e(\rho_j) = -1/2$. Now Lemma 4.4 implies that there exists $j' \in Y_{i_0}$ with $e(\rho_{j'}) = 1/2$. This contradicts regularity. Thus $e(\rho_j) = -1$. Again Lemma 4.4 implies the existence of j' and $j'' \in Y_{i_0}$ such that $e(\rho_{j'}) = 0$ and $e(\rho_{j''}) = 1$. This contradicts regularity.

Lemma 4.3 and the regularity condition imply that the sequence $\rho_{a_1}, \rho_{a_2}, \dots, \rho_{a_k}$ is up to a permutation a sequence $\nu^{\alpha_0}\rho_{i_0}^u, \nu^{\alpha_0+1}\rho_{i_0}^u, \dots, \nu^{\alpha_0+k-1}\rho_{i_0}^u$, where $\alpha_0 = 0$ or $1/2$ (k is a positive integer). If $\alpha_0 = 1/2$, then using the fact that $\rho_{a_k} \times \rho_j$ and $\tilde{\rho}_{a_k} \times \rho_j$ are irreducible for $j \in Y_{i_0}^c$, and $\rho_{a_k} \rtimes \sigma \cong \tilde{\rho}_{a_k} \rtimes \omega_{\rho_{a_k}}\sigma$, we can bring $\tilde{\rho}_{a_k}$ at the beginning, i.e. $\rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma \cong \tilde{\rho}_{a_k} \times \rho_1 \times \rho_2 \times \dots \times \rho_{a_k-1} \times \rho_{a_k+1} \times \dots \times \rho_n \rtimes \sigma$. Thus $\pi \hookrightarrow \tilde{\rho}_{a_k} \times \rho_1 \times \rho_2 \times \dots \times \rho_{a_k-1} \times \rho_{a_k+1} \times \dots \times \rho_n \rtimes \sigma$. This, Frobenius reciprocity and criterion (2-4) imply that π is not essentially square integrable.

Now we want to prove that $\rho_{a_i} \cong \nu^{k-i}\rho_{i_0}^u$ for $i = 1, \dots, k$. Suppose that this is not the case. Choose the maximal index i such that $\rho_{a_i} \not\cong \nu^{k-i}\rho_{i_0}^u$. Then clearly $\rho_{a_i} \times \rho_{a_j} \cong \rho_{a_j} \times \rho_{a_i}$ for all $j > i$. Recall $\rho_{a_i} \rtimes \sigma \cong \tilde{\rho}_{a_i} \rtimes \omega_{\rho_{a_i}}\sigma$ (Lemma 2.1 and (1-3)). If $\rho_{a_i} \not\cong \nu\rho_{i_0}^u$, then we can bring $\tilde{\rho}_{a_i}$ to the beginning, i.e. $\rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma \cong \tilde{\rho}_{a_i} \times \rho_1 \times \rho_2 \times \dots \times \rho_{a_i-1} \times \rho_{a_i+1} \times \dots \times \rho_n \rtimes \sigma$. Therefore $\pi \hookrightarrow \tilde{\rho}_{a_i} \times \rho_1 \times \rho_2 \times \dots \times \rho_{a_i-1} \times \rho_{a_i+1} \times \dots \times \rho_n \rtimes \sigma$. This contradicts the assumption of essential square integrability of π (use Frobenius reciprocity and criterion (2-4)). Thus $\rho_{a_i} \cong \nu\rho_{i_0}^u$. The regularity and choice of i imply $i \geq k-1$. If $i = k-1$, then $\rho_{a_i} = \rho_{a_{k-1}} \not\cong \nu\rho_{i_0}^u$ by the choice of i. This contradicts $\rho_{a_i} = \rho_{a_{k-1}} \cong \nu\rho_{i_0}^u$. Therefore $i = k$. We have now $\rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma \cong \rho_1 \times \rho_2 \times \dots \times \rho_{a_k-1} \times \rho_{a_k+1} \times \dots \times \rho_n \times \rho_{a_k} \times \rtimes \sigma \cong \rho_1 \times \rho_2 \times \dots \times \rho_{a_k-1} \times \rho_{a_k+1} \times \dots \times \rho_n \times \tilde{\rho}_{a_k} \times \rtimes \omega_{\rho_{a_k}}\sigma$. Since $\tilde{\rho}_{a_k} \cong \nu^{-1}\rho_{i_0}^u$, Lemma 4.4 implies that there exist $i, j \in \{1, 2, \dots, n\} \backslash \{a_k\}$, $i < j$, such that $\rho_j \cong \rho_{i_0}^u$ and $\rho_i \cong \nu\rho_{i_0}^u$.

This contradicts to the regularity of $\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma$. Therefore, it can not happen that $\rho_{a_i} \cong \nu\rho_{i_0}^u$. This finishes the proof of the lemma. $\square$

5.2. Lemma. *Let $\pi \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma$. Suppose that (R_G) holds for (ρ_i, σ), $i = 1, 2, \ldots, n$, and suppose that $\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma$ is regular. Assume that $\sigma \cong \omega_{\rho_{i_0}^u}\sigma$. Write $Y_{i_0} = \{a_1, \ldots, a_k\}$ where $a_i < a_j$ for $i < j$. Then one of the following two possibilities holds:*
(i) $\nu^{1/2}\rho_{i_0}^u \rtimes \sigma$ reduces and $\rho_{a_k} = \nu^{1/2}\rho_{i_0}^u, \rho_{a_{k-1}} = \nu^{3/2}\rho_{i_0}^u, \ldots, \rho_{a_1} = \nu^{k-1/2}\rho_{i_0}^u$ (i.e. $\rho_{a_i} = \nu^{k-i+1/2}\rho_{i_0}^u$ for $1 \le i \le k$).
(ii) $\nu\rho_{i_0}^u \rtimes \sigma$ reduces and $\rho_{a_k} = \nu\rho_{i_0}^u, \rho_{a_{k-1}} = \nu^2\rho_{i_0}^u, \ldots, \rho_{a_1} = \nu^k\rho_{i_0}^u$ (i.e. $\rho_{a_i} = \nu^{k-i+1}\rho_{i_0}^u$ for $1 \le i \le k$).

Proof. Suppose that $e(\rho_{a_i}) \le 0$ for some $i \in Y_{i_0}$. Then Lemma 4.1 and the regularity imply $e(\rho_{a_i}) < 0$. Choose minimal i such that $e(\rho_{a_i}) < 0$. Lemma 4.4 implies $e(\rho_{a_i}) \ge -1$. The regularity condition implies $e(\rho_{a_i}) > -1$ (otherwise, Lemma 4.4 implies that $e(\rho_{a_j}) = 0$ for some $1 \le j < i$ and this contradicts regularity). Therefore $e(\rho_{a_i}) = -1/2$. Again Lemma 4.4 and the regularity condition imply that this is impossible. Thus all $e(\rho_{a_i})$ are strictly positive. This implies that $\rho_{a_k} \rtimes \sigma$ reduces. In particular, (R_G) implies that $e(\rho_{a_k}) = 1/2$ or 1. Now Lemma 4.7 and the regularity imply that the sequence $\rho_{a_1}, \ldots, \rho_{a_k}$ is up to a permutation a sequence $\nu^{1/2}\rho_{i_0}^u, \nu^{3/2}\rho_{i_0}^u, \ldots, \nu^{k-1/2}\rho_{i_0}^u$, or a sequence $\nu\rho_{i_0}^u, \nu^2\rho_{i_0}^u, \ldots, \nu^k\rho_{i_0}^u$. Further one concludes that $\rho_{a_k} \rtimes \sigma$ reduces (otherwise $\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma \cong \rho_1 \times \rho_2 \times \cdots \times \rho_{a_k - 1} \times \rho_{a_k + 1} \times \cdots \times \rho_n \times \tilde{\rho}_{a_k} \times \rtimes \omega_{\rho_{a_k}}\sigma$ which contradicts the first part of the proof).

Suppose $e(\rho_{a_k}) = 1/2$. Choose maximal i such that $\rho_{a_i} \not\cong \nu^{k-i+1/2}\rho_{i_0}^u$. Then $\rho_{a_i} \times \rho_j \cong \rho_j \times \rho_{a_i}$ for all $j > a_i$ by the choice of i. One concludes now that $\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma \cong \rho_1 \times \rho_2 \times \cdots \times \rho_{a_i - 1} \times \rho_{a_i + 1} \times \cdots \times \rho_n \times \tilde{\rho}_{a_i} \times \rtimes \omega_{\rho_{a_i}}\sigma$. This contradicts the first part of the proof. Therefore $\rho_{a_j} \not\cong \nu^{k-j+1/2}\rho_{i_0}^u$ for any $1 \le j \le k$. This completes the proof of (i).

One proceeds similarly in the case of $e(\rho_{a_k}) = 1$. $\square$

Now we can state the final result of this section.

5.3. Theorem. *Suppose that π is a regular irreducible essentially square integrable representation of some $GSp(m, F)$. Then π is a subquotient of $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$, where ρ_i are irreducible cuspidal representations of general linear groups and σ is a similar representation of a GSp-group. Suppose that (R_G) holds for (ρ_i, σ), $i = 1, \ldots, n$. Then π is equivalent to one of the essentially square integrable representations listed in Theorem 3.3.*

Proof. The proof of exhaustion claimed in the theorem is based on Lemmas 5.1 and 5.2 (roughly, the first lemma is directed to the "Rodier type situations", while the second lemma to the "Steinberg type situations").

Suppose that π is a regular irreducible essentially square integrable representation of some $\mathrm{GSp}(m, F)$, and that it is a subquotient of $\rho_1' \times \cdots \times \rho_n' \rtimes \sigma$, where ρ_i' are irreducible cuspidal representations of general linear groups and σ is a similar representation of a GSp-group. We shall assume that (R_G) holds for (ρ_i, σ), $i = 1, \ldots, n$. Then π embeds into $(\rho_{p(1)}')^{\varepsilon_1} \times \cdots \times (\rho_n')^{\varepsilon_{p(n)}} \rtimes \sigma$ for some permutation p of $\{1, \ldots, n\}$ and some choice of $\varepsilon_i \in \{\pm 1\}$ (this follows from Corollary 6.3.7 of [C], and the formula (3-2) and Lemma 2.3). Here $(\rho_{p(i)}')^{\varepsilon_i}$ denotes $\rho_{p(i)}'$ if $\varepsilon_i = 1$, and $(\rho_{p(i)}')\tilde{}$ if $\varepsilon_i = -1$. Denote $(\rho_{p(i)}')^{\varepsilon_i}$ by ρ_i.

Lemma 4.1 implies that $(\rho'_{p(i)})^u = \rho_i^u$. Therefore, all (ρ_i, σ) satisfy (R_G) and also

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_n \rtimes \sigma. \tag{5-1}$$

Fix any $i_1 \in \{1, \dots, n\}$ (similarly as we have fixed i_0 in the last and this section). Take $\epsilon_1 \in \{1/2, 1\}$ such that $\nu^{\epsilon_1}\rho_{i_1}^u \rtimes \sigma$ reduces, if such ϵ_1 exists. If it does not exist, put $\epsilon_1 = 0$. Now Lemmas 5.1 and 5.2 imply

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_n \rtimes \sigma \cong$$

$$(\nu^{\epsilon_1+k_1-1}\rho_{i_1}^u \times \nu^{\epsilon_1+k_1-2}\rho_{i_1}^u \times \cdots \times \nu^{\epsilon_1+1}\rho_{i_1}^u \times \nu^{\epsilon_1}\rho_{i_1}^u) \times \left(\prod_{i \in X} \rho_i\right) \rtimes \sigma,$$

where X denotes the subset of all $i \in \{1, \dots, n\}$ such that $\rho_i \not\cong \nu^\alpha \rho_{i_1}^u$ for any $\alpha \in (1/2)\,\mathbb{Z}$ (note that (R_G) and Lemmas 5.1 and 5.2 imply that this is equivalent to: $\rho_i \not\cong \nu^\alpha \rho_{i_1}^u$ for any $\alpha \in \mathbb{R}$). Now repeating the procedure several times we get

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_n \rtimes \sigma \cong \left(\prod_{j=1}^{l} (\nu^{\epsilon_j+k_j-1}\rho_{i_j}^u \times \nu^{\epsilon_j+k_j-2}\rho_{i_j}^u \times \cdots \times \nu^{\epsilon_j+1}\rho_{i_j}^u \times \nu^{\epsilon_j}\rho_{i_j}^u)\right) \rtimes \sigma,$$

where $\epsilon_j \in \{0, 1/2, 1\}$, and for $j_1 \neq j_2$, $\rho_{i_{j_1}} \not\cong \nu^\alpha \rho_{i_{j_2}}^u$ for any $\alpha \in (1/2)\mathbb{Z}$.

Denote $\Psi_j = [\nu^{\epsilon_j}\rho_{i_j}^u, \nu^{\epsilon_j+k_j-1}\rho_{i_j}^u]$. Since

$$\delta(\Psi_j) \hookrightarrow \nu^{\epsilon_j+k_j-1}\rho_{i_j}^u \times \nu^{\epsilon_j+k_j-2}\rho_{i_j}^u \times \cdots \times \nu^{\epsilon_j+1}\rho_{i_j}^u \times \nu^{\epsilon_j}\rho_{i_j}^u,$$

we have

$$\delta(\Psi_1) \times \cdots \times \delta(\Psi_l) \rtimes \sigma \hookrightarrow \rho_1 \times \cdots \times \rho_n \rtimes \sigma. \tag{5-2}$$

Since $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$ is regular by assumption, it has a unique irreducible subrepresentation (it is π by (5-1)). By (5-2) each irreducible subrepresentation of $\delta(\Psi_1) \times \cdots \times \delta(\Psi_l) \rtimes \sigma$ is an irreducible subrepresentation of $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$. Thus π is a subrepresentation of $\delta(\Psi_1) \times \cdots \times \delta(\Psi_l) \rtimes \sigma$. Now we shall prove that segments Ψ_i satisfy assumptions of Theorem 3.3 for the essentially square integrable case (see (v) of Theorem 3.3). This will make the proof complete, because then (ii) of Theorem 3.3 implies $\pi \cong \delta(\Psi_1, \dots, \Psi_n, \sigma)$.

We could make now our notations bellow exactly the same as in Theorem 3.3, denoting by Δ_j segments which have unitarizable beginning (i.e. for which $\epsilon_{i_j} = 0$) and the remaining segments by Γ_j, but we shall not do this here (it is not necessary, and for us is shorter to keep the notation that we had above).

First, the regularity of $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$ implies that beginnings $\nu^{\epsilon_j}\rho_{i_j}^u$ of the segments Ψ_j's are inequivalent. The regularity and (R_G) imply also that $\rho_{i_j}^u$, $j = 1, \dots, l$, must be inequivalent. It remains only to see that the condition on X holds (X is as in Theorem 3.3). Recall that X is the group of characters generated by central characters of all beginnings ρ_{i_j} of Ψ_j, which are unitarizable. Let p be a number of segments Ψ_j which start with a unitarizable representation ρ_{i_j}. If ρ_{i_j} is unitarizable, then Lemma 4.1 imply $\rho_{i_j} \cong (\rho_{i_j})\tilde{}$, which implies that the square of the central character of ρ_{i_j} is the trivial character. Thus $\operatorname{card}(X) \leq 2^p$. Suppose that

$$\sigma \cong \omega\sigma \text{ for some } \omega \in X\backslash\{1_{F^\times}\} \quad \text{or} \quad \operatorname{card}(X) < 2^p. \tag{5-3}$$

Denote by $\tau_1, \ldots \tau_p$ the set of all ρ_{i_j} which are unitarizable. Let $\tau_1 \times \cdots \times \tau_p$ be a representation of $\mathrm{GL}(a, F)$. Applying the formula (3-2) we get

$$\text{s.s.}(s_{(a)}(\tau_1 \times \cdots \times \tau_p \rtimes \sigma)) = \sum_{(\epsilon'_i) \in \{\pm 1\}^p} \tau_1 \times \cdots \times \tau_p \otimes \left(\prod_{i=1}^{t} \omega_{\tau_i}^{(1-\epsilon'_i)/2} \right) \sigma$$

since $\tau_i \cong (\tau_i)\tilde{}$. This and (5-3) imply that $s_{(a)}(\tau_1 \times \cdots \times \tau_p \rtimes \sigma)$ is not a multiplicity one representation (because $\prod_{i=1}^{t} \omega_{\tau_i}^{(1-\epsilon'_i)/2} \in X$, $\operatorname{card}(X) \leq 2^p$ and (5-3)). Thus, $\tau_1 \times \cdots \times \tau_p \rtimes \sigma$ is not regular. Denote by $Y = \{1 \leq i \leq n; \rho_i \not\cong \tau_j \text{ for any } j \in \{1, \ldots, p\}\}$, and let $\prod_{i \in Y} \rho_i$ be a representation of $\mathrm{GL}(b, F)$. Then obviously $\left(\prod_{i \in Y} \rho_i\right) \otimes \left(\prod_{j=1}^{p} \tau_j\right) \rtimes \sigma$ is a quotient of $s_{(b)}\left(\left(\prod_{i \in Y} \rho_i\right) \times \left(\prod_{j=1}^{p} \tau_j\right) \rtimes \sigma\right)$ by Frobenius reciprocity. Now irregularity of $\left(\prod_{j=1}^{p} \tau_j\right) \rtimes \sigma$ which we have shown, and the transitivity of Jacquet modules imply that $\left(\prod_{i \in Y} \rho_i\right) \times \left(\prod_{j=1}^{p} \tau_j\right) \rtimes \sigma$ is not regular. Thus $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$ is not regular (recall that $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$ and $\left(\prod_{i \in Y} \rho_i\right) \times \left(\prod_{j=1}^{p} \tau_j\right) \rtimes \sigma$ have the same Jordan-Hölder series). This contradicts to our assumption. Thus, $\sigma \not\cong \omega\sigma$ for any $\omega \in X$, $\omega \neq 1_{F^\times}$, and $\operatorname{card}(X) = 2^p$. Therefore, the conditions of Theorem 3.3 in the case of segments Ψ_j are satisfied. This ends the proof. □

6. Sp(n, F) and SO$(2n+1, F)$ (generic reducibilities)

We can get a wide family of regular irreducible square integrable representations of $\mathrm{Sp}(n, F)$ by studying restrictions of regular irreducible square integrable representations of $\mathrm{GSp}(n, F)$ constructed in Theorem 3.3 (see [T1]). For each regular irreducible square integrable representation σ of $\mathrm{Sp}(n, F)$ there exists a regular irreducible square integrable representation $\sigma^{\#}$ of $\mathrm{GSp}(n, F)$ such that σ is isomorphic to a subrepresentation of the restriction $\sigma^{\#}|\mathrm{Sp(n,F)}$. Note that for a regular irreducible square integrable representation $\sigma^{\#}$ of $\mathrm{GSp}(n, F)$, irreducible subrepresentations of $\sigma^{\#}|\mathrm{Sp(n,F)}$ do not need to be regular (they are square integrable).

We can also construct that family of regular irreducible square integrable representations of $\mathrm{Sp}(n, F)$ by repeating the process that we did in sections 3-5 in the case of $\mathrm{GSp}(n, F)$. This process is even more simple for $\mathrm{Sp}(n, F)$ than for $\mathrm{GSp}(n, F)$. There are two reasons for that. One is that the formula for $\mu^*(\pi \rtimes \sigma)$ is more simple in the case of $\mathrm{Sp}(n, F)$ than $\mathrm{GSp}(n, F)$ (see (6-3) bellow). The other simplification comes from the fact that $\mathrm{Sp}(n, F)$ has fewer regular irreducible square integrable representations then $\mathrm{GSp}(n, F)$.

Operations $\rtimes$ and μ^* will be defined bellow for $\mathrm{Sp}(n, F)$. If we construct regular irreducible square integrable representations of $\mathrm{Sp}(n, F)$ by repeating the process that we did for $\mathrm{GSp}(n,F)$, then their parameterization becomes a formal consequence of operations $\rtimes$ and μ^*. Operations $\rtimes$ and μ^* will be defined bellow also for groups $\mathrm{SO}(2n+1,F)$. Since they have the same formal properties as in the case of $\mathrm{Sp}(n,F)$ (see (6-1), (6-2) and (6-3) bellow), we shall get that the same results that hold for irreducible square integrable representations of $\mathrm{Sp}(n,F)$ will hold for $\mathrm{SO}(2n+1,F)$.

Now we shall recall notation for groups $\mathrm{Sp}(n,F)$ and $\mathrm{SO}(2n+1,F)$ introduced in [T4] (see also [T3]). More information regarding this notation can be found in that papers. Denote

$$\mathrm{Sp}(n,F) = \mathrm{GSp}(n,F) \cap \mathrm{SL}(2n,F),$$
$$\mathrm{SO}(2n+1,F) = \{g \in \mathrm{SL}(2n+1,F);\ {}^{\tau}g\ g = I_{2n+1}\},$$

where I_k denotes $k \times k$ identity matrix (recall that ${}^{\tau}g$ denotes the transposed matrix of g with respect to the second diagonal). In this section S_n denotes the group $\mathrm{Sp}(n,F)$ or $\mathrm{SO}(2n+1,F)$. We fix in S_n the minimal parabolic subgroup $P_{\min}$ consisting of all upper triangular matrices in S_n. Standard parabolic subgroups are those parabolic subgroups which contain $P_{\min}$.

Take an ordered partition $\alpha = (n_1, n_2, \ldots, n_k)$ of a non-negative integer $m \leq n$. We denote

$$M_\alpha = \{\text{q-diag}(g_1, g_2, \ldots, g_k, h,\ {}^{\tau}g_k^{-1},\ {}^{\tau}g_{k-1}^{-1}, \ldots,\ {}^{\tau}g_1^{-1}); g_i \in \mathrm{GL}(n_i,F), h \in S_{n-m}\}.$$

We identify M_α with $\mathrm{GL}(n_1,F) \times \cdots \times \mathrm{GL}(n_k,F) \times S_{n-m}$ using the isomorphism

$$\text{q-diag}(g_1, g_2, \ldots, g_k, h,\ {}^{\tau}g_k^{-1},\ {}^{\tau}g_{k-1}^{-1}, \ldots,\ {}^{\tau}g_1^{-1}) \mapsto \text{q-diag}(g_1, g_2, \ldots, g_k, h).$$

Set $P_\alpha = M_\alpha P_{\min}$. Then P_α is a standard parabolic subgroup in S_n. Let N_α be the unipotent radical of M_α.

Take an admissible representation π of $\mathrm{GL}(n,F)$ and a similar representation σ of S_m. We denote by $\pi \rtimes \sigma$ the representation of S_{n+m} parabolically induced by $\pi \otimes \sigma$ from $P_{(n)}$. If additionally π' is an admissible representation of $\mathrm{GL}(n',F)$, then

$$\pi' \rtimes (\pi \rtimes \sigma) \cong (\pi' \times \pi) \rtimes \sigma. \tag{6-1}$$

Denote by $R_n(S)$ the Grothendieck group of the category of all admissible representations of S_n of finite length. Let $R(S) = \oplus_{n\geq 0} R_n(S)$. Using $\rtimes$ we define $\mathbb{Z}$-bilinear map $\rtimes : R \times R(S) \to R(S)$ similarly as in the case of GSp-groups. In that way $R(S)$ becomes R-module. In this R-module holds

$$\pi \rtimes \sigma \cong \tilde{\pi} \rtimes \sigma. \tag{6-2}$$

For an ordered partition $\alpha = (n_1, n_2, \ldots, n_k)$ of a non-negative integer $m \leq n$ and an admissible representation σ of finite length of S_n we denote by $s_\alpha(\sigma)$ the Jacquet module of σ with respect to P_α. We can consider in natural way $\text{s.s.}(s_\alpha(\sigma)) \in R_{n_1} \otimes \ldots R_{n_k} \otimes R_{n-m}$. Set

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(s_{(k)}(\sigma)) \in R \otimes R(S).$$

We extend additively μ^* to $R(S)$. Define a structure of $R \otimes R$-module on $R \otimes R(S)$ in the following natural way: $(\sum_i r'_i \otimes r''_i) \rtimes (\sum_j r_j \otimes s_j) = \sum_{i,j} (r'_i \times r_j) \otimes (r''_i \rtimes s_j)$. Define $m : R \otimes R \to R$ by the formula $m(\sum_i r'_i \otimes r''_i) = \sum_i r'_i \times r''_i$. Let $\sim$ denote the contragredient mapping on R. Define $M^* : R \to R \otimes R$ by the formula $M^* = (m \otimes \mathrm{Id}_R) \circ (\sim \otimes m^*) \circ s \circ m^*$. Then by [T4]

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma), \tag{6-3}$$

for $\pi \in R$ and $\sigma \in R(S)$.

Now we shall recall a well-known

6.1. Lemma. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(n, F)$, let σ be an irreducible cuspidal representation of some $GSp(m, F)$ and let $\alpha \in \mathbb{R}$. If $(\nu^\alpha \rho) \rtimes \sigma$ reduces, then $\rho \cong \tilde{\rho}$. Further, $\nu^\alpha \rho \rtimes \sigma$ reduces if and only if $\nu^{-\alpha} \rho \rtimes \sigma$ reduces.* $\square$

Similarly as before, we shall consider for a pair (τ, σ) consisting of an irreducible cuspidal representation τ of a general linear group and a similar representation σ of S_m if it satisfies the following assumptions:

$(\mathcal{R}_G)$ if $\tau \rtimes \sigma$ reduces, then there exists $\alpha_0 \in \{0, 1/2, 1\}$ such that

$$\nu^{\pm\alpha_0} \tau^u \rtimes \sigma \text{ reduce and } \nu^\beta \tau^u \rtimes \sigma \text{ is irreducible for } \beta \in \mathbb{R}, |\beta| \neq \alpha_0;$$

$(\mathcal{R}_{(1/2)\mathbb{Z}})$ if $\tau \rtimes \sigma$ reduces, then there exists $\alpha_0 \geq 0$ in $(1/2)\mathbb{Z}$ such that

$$\nu^{\pm\alpha_0} \tau^u \rtimes \sigma \text{ reduce and } \nu^\beta \tau^u \rtimes \sigma \text{ is irreducible for } \beta \in \mathbb{R}, |\beta| \neq \alpha_0.$$

Again $(\mathcal{R}_G)$ holds for any τ if σ is generic, when $\mathrm{char}\, F = 0$ (in particular, if $\sigma = 1$; [Sd2]). This is the reason why we shall say that τ and σ have generic reducibility if they satisfy $(\mathcal{R}_G)$. If τ and σ satisfy $\mathcal{R}_{(1/2)\mathbb{Z}}$, then we shall say that they have reducibility in $(1/2)\mathbb{Z}$ (or $(1/2)\mathbb{Z}$-reducibility). There is no known to this author reducibility of τ and σ which is not in $(1/2)\mathbb{Z}$. C. Mœglin has obtained in the symplectic case ([Mœ]), while M. Reeder has obtained in the orthogonal case ([Ree4]) examples of reducibilities in $(1/2)\mathbb{Z}$, which are not generic. Shahidi's Conjecture 9.4 of [Sd1] would imply that $\mathcal{R}_{(1/2)\mathbb{Z}}$ holds in general.

Now Theorem 4.9 holds in this situation:

6.2. Theorem. *Let $\rho_1, \rho_2, \ldots, \rho_n$ be irreducible cuspidal representations of general linear groups and let σ be a similar representation of S_m. Assume that $\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma$ has an essentially square integrable subquotient. Then*
(i) $(\rho_i^u)\tilde{} \cong \rho_i^u$ for any $1 \leq i \leq n$.
(ii) If $(\mathcal{R}_{(1/2)\mathbb{Z}})$ holds for (ρ_j, σ), $1 \leq j \leq n$, then $e(\rho_j) \in (1/2)\mathbb{Z}$ for $1 \leq j \leq n$.
(iii) Assume that $(\mathcal{R}_G)$ holds for (ρ_j, σ), $1 \leq j \leq n$. Let $1 \leq i \leq n$. Denote by $Y(\rho_i)$ the set of all ρ_j, $1 \leq j \leq n$, such that $\rho_j \cong \nu^z \rho_i$ for some $z \in \mathbb{Z}$. Set $Y(\rho_i)\tilde{} = \{\tilde{\tau}; \tau \in Y(\rho_i)\}$. If $\nu \rho_{i_0}^u \rtimes \sigma$ reduces and $e(\rho_{i_0}) \in \mathbb{Z}$, then $Y(\rho_i) \cup Y(\rho_i)\tilde{} \cup \{\rho_i^u\}$ is a segment in cuspidal representations of general linear groups. In all other cases $Y(\rho_i) \cup Y(\rho_i)\tilde{}$ is a segment in cuspidal representations of general linear groups. $\square$

In a similar as Theorems 3.3 and 5.3 we get the following theorems:

6.3. Theorem. *Let τ_j, $j = 1, 2, \dots, m$ be mutually inequivalent irreducible unitarizable cuspidal representations of $GL(b_j, F)$, where $b_j \geq 1$. Suppose that σ is an irreducible cuspidal representation of $Sp(\ell, F)$. Assume that for any $j = 1, 2, \dots, m$, there exists $s_j > 0$ such that $\nu^{s_j}\tau_j \rtimes \sigma$ reduces. Take non-negative integers q_j, $j = 1, \dots, m$,. Set $\Gamma_j = [\nu^{s_j}\tau_j, \nu^{s_j+q_j}\tau_j]$. Let $\delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m)$ be a representation of $GL(r, F)$. Then:*
(i) The representation $\nu^{s_1}\tau_1 \times \nu^{s_1+1}\tau_1 \times \cdots \times \nu^{s_1+q_1}\tau_1 \times \nu^{s_2}\tau_2 \times \nu^{s_2+1}\tau_2 \times \cdots \times \nu^{s_{m-1}+q_{m-1}}\tau_{m-1} \times \nu^{s_m}\tau_m \times \cdots \times \nu^{s_m+q_m}\tau_m \rtimes \sigma$ is regular.
(ii) The representation $\delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes \sigma$ contains a unique irreducible subrepresentation. We denote it by $\delta(\Gamma_1, \dots, \Gamma_m, \sigma)$
(iii) $s_{(r)}(\delta(\Gamma_1, \dots, \Gamma_m, \sigma)) = \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \otimes \sigma$
(iv) The representation $\delta(\Gamma_1, \dots, \Gamma_m, \sigma)$ is square integrable.
(v) $\delta(\Gamma_1, \dots, \Gamma_m, \sigma)^- \cong \delta(\Gamma_1, \dots, \Gamma_m, \overline{\sigma})$.
(vi) $\delta(\Gamma_1, \dots, \Gamma_m, \sigma)\tilde{} \cong \delta(\Gamma_1, \dots, \Gamma_m, \tilde{\sigma})$.
(vii) Besides permutations of the segments, there are no other equivalences among representations $\delta(\Gamma_1, \dots, \Gamma_m, \sigma)$.
(viii) Denote $\tau = \tau_1$, $s = s_1$ and $q = q_1$. Then

$$\mu^*(\delta([\nu^s\tau, \nu^{s+q}\tau], \sigma)) = \sum_{k=-1}^{q} \delta([\nu^{s+k+1}\tau, \nu^{s+q}\tau]) \otimes \delta([\nu^s\tau, \nu^{s+k}\tau], \sigma),$$

where we assume $\delta(\emptyset, \sigma) = \sigma$ in the above formula. $\square$

6.4. Theorem. *Let π be a regular irreducible square integrable representation of S_q. Assume that π is a subquotient of some $\rho_1 \times \cdots \times \rho_m \rtimes \sigma$ where ρ_i are irreducible cuspidal representations of general linear groups and σ is a similar representation of some S_p. Assume that $(\mathcal{R}_G)$ holds for pairs (ρ_i, σ), $i = 1, \dots, n$. Then π is equivalent to a square integrable representation listed in Theorem 6.3.* $\square$

Let us note at the end that although the above expressions of parameterizations of regular irreducible square integrable representations of $\mathrm{Sp}(n, F)$ and $\mathrm{SO}(2n+1)$ related to the generic reducibilities are the same, the explicit pictures for fixed n may look very different. The reason is that reducibility of $\rho \rtimes \sigma$ is different for $\mathrm{Sp}(n, F)$ and $\mathrm{SO}(2n+1)$. Shahidi's paper [Sd2] contains interesting information about such differences.

7. Sp(n, F) and SO$(2n+1, F)$ (reducibilities in $(1/2)\mathbb{Z}$)

In this section we shall parameterize regular irreducible square integrable representations which are attached to reducibilities in $(1/2)\mathbb{Z}$. If it can be shown that there are no other reducibilities (which is not known), then these representations will be all regular irreducible square integrable representations of groups $\mathrm{Sp}(n, F)$ and $\mathrm{SO}(2n+1, F)$.

Recall that we have denoted by $\mathfrak{s}(\Delta)$ the Zelevinsky segment representation attached to a segment Δ in cuspidal representations of general linear groups (see the second section).

Before we proceed further, we shall recall Langlands classification in the case of general linear groups, which we need in study of regular representations attached to non-generic reducibilities. Denote by D the set of all equivalence classes of all irreducible essentially square integrable smooth representations of all $\mathrm{GL}(n, F)$, $n \geq 1$ (recall that for $\delta \in D$

there is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable). Let $M(D)$ be the set of all multisets in D. For each $a = (\delta_1, \dots, \delta_k) \in M(D)$, take a permutation p of $\{1, \dots, k\}$ such that $e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \cdots \geq e(\delta_{p(k)})$. The representation $\lambda(a) = \delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)}$ (whose equivalence class is independent of choice of p) has a unique irreducible quotient, which we denote by $L(a)$. This is Langlands classification for general linear groups. The representations $\lambda(a), a \in M(D)$ are called standard modules (of general linear groups).

We shall write $L(a) = L((\delta_1, \dots, \delta_k))$ also simply as $L(\delta_1, \dots, \delta_k)$.

7.1. Lemma. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(p, F)$ and σ a similar representation of S_q. Suppose that $\alpha > 1$ and $\nu^\alpha\rho \rtimes \sigma$ reduces. Fix $k \in \mathbb{Z}$ which satisfies $0 < \alpha - k \leq \alpha$. Then the representation*

$$\nu^{\alpha-k} \times \nu^{\alpha-k+1} \times \cdots \times \nu^\alpha\rho \rtimes \sigma$$

contains a unique irreducible subrepresentation, which we denote by $\delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma)$. We have

$$\mu^*(\delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma)) = \sum_{i=o}^{k+1} \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha-i}\rho]) \otimes \delta([\nu^{\alpha-i+1}\rho, \nu^\alpha\rho], \sigma). \tag{7-1}$$

The representation $\delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma)$ is a regular square integrable representation.

Proof. The formula (3-2) (written for S_m groups) and Lemma 2.3 imply that the representation $\delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma)$ is regular. The formula (7-1) implies square integrability of $\delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma)$. We shall prove that formula by induction with respect to k.

For $k = 0$ the claim of lemma hold obviously (see Proposition 3.1). Take $k \geq 0$ such that $\alpha - k - 1 > 0$ and suppose that the formula (7-1) holds for this k. Then

$$\delta([\nu^{\alpha-k-1}\rho, \nu^\alpha\rho], \sigma) \hookrightarrow \mathfrak{s}([\nu^{\alpha-k-1}\rho, \nu^\alpha\rho]) \rtimes \sigma, \tag{7-2}$$

since $s([\nu^{\alpha-k-1}\rho, \nu^\alpha\rho])$ is a subrepresentation of $\nu^{\alpha-k-1}\rho \times \nu^{\alpha-k}\rho \times \cdots \times \nu^\alpha\rho$. Further

$$\delta([\nu^{\alpha-k-1}\rho, \nu^\alpha\rho], \sigma) \hookrightarrow \nu^{\alpha-k-1}\rho \rtimes \delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma), \tag{7-3}$$

since $\delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma)$ is a subrepresentation of $\nu^{\alpha-k}\rho \times \nu^{\alpha-k+1}\rho \times \cdots \times \nu^\alpha\rho \rtimes \sigma$. Using (6-3) we get

$$\begin{aligned} &\text{s.s.}(s_{((k+2)p)}(\mathfrak{s}([\nu^{\alpha-k-1}\rho, \nu^\alpha\rho]) \rtimes \sigma)) \\ &\qquad = \sum_{i=-1}^{k+1} \mathfrak{s}([\nu^{-\alpha}\rho, \nu^{-\alpha+k-i}\rho]) \times \mathfrak{s}([\nu^{\alpha-k-1}\rho, \nu^{\alpha-k-1+i}\rho]) \otimes \sigma, \end{aligned} \tag{7-4}$$

$$\begin{aligned} &\text{s.s.}(s_{((k+2)p)}(\nu^{\alpha-k-1}\rho \rtimes \delta([\nu^{\alpha-k}\rho, \nu^\alpha\rho], \sigma))) \\ &\qquad = (\nu^{-(\alpha-k-1)}\rho \otimes 1 + \nu^{\alpha-k-1}\rho \otimes 1) \times \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^\alpha\rho]) \otimes \sigma. \end{aligned} \tag{7-5}$$

Now we shall find all common irreducible subquotients of (7-4) and (7-5). This reduces to finding of common irreducible subquotients of

$$\sum_{i=-1}^{k+1} \mathfrak{s}([\nu^{-\alpha}\rho, \nu^{-\alpha+k-i}\rho]) \times \mathfrak{s}([\nu^{\alpha-k-1}\rho, \nu^{\alpha-k-1+i}\rho]) \otimes \sigma \tag{7-6}$$

and

$$(\nu^{-(\alpha-k-1)}\rho \otimes 1 + \nu^{\alpha-k-1}\rho \otimes 1) \times \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho]) \tag{7-7}$$

(in other words, we can work without σ). To have common irreducible subquotient, we must have the same supports. Suppose that $\nu^{-(\alpha-k-1)}\rho$ is in the support of a common irreducible subquotient. Note that this representation shows up in (7-6) only in the support of $\mathfrak{s}([\nu^{-\alpha}\rho, \nu^{-(\alpha-k)+1}\rho])$, and in (7-7) only in the support of $\nu^{-(\alpha-k-1)}\rho \rtimes \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho])$. Obviously, these two representations have disjoint Jordan-Hölder series ($\nu^{\alpha}\rho$ is in the support of the second representation, but it is not in the support of the first one).

This and (7-7) imply that common irreducible subquotients must be subquotients of $\nu^{\alpha-k-1}\rho \rtimes \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho])$. Considering the supports (in particular $\nu^{-\alpha}\rho$), we see from (7-6) that this common irreducible subquotient must be a subquotient of $\mathfrak{s}([\nu^{\alpha-k-1}\rho, \nu^{\alpha}\rho])$. This implies that the only common irreducible subquotient of (7-4) and (7-5) is the representation $\mathfrak{s}([\nu^{\alpha-k-1}\rho, \nu^{\alpha}\rho]) \otimes \sigma$. Thus, $s_{((k+2)p)}((\delta([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho], \sigma)) = \mathfrak{s}([\nu^{\alpha-k-1}\rho, \nu^{\alpha}\rho]) \otimes \sigma$. Well-known properties of the representation theory of general linear groups, the transitivity of Jacquet modules and the inductive assumption imply now (7-1). This completes the proof of the lemma. $\square$

7.2. Proposition. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(p, F)$ and σ a similar representation of S_q. Suppose that $\alpha > 1$ and $\nu^{\alpha}\rho \rtimes \sigma$ reduces. Let $k, l \in \mathbb{Z}$ such that $0 < \alpha - k \leq \alpha \leq \alpha + l$. Then the representation*

$$(\nu^{\alpha+l} \times \nu^{\alpha+l-1} \times \cdots \times \nu^{\alpha+2}\rho \times \nu^{\alpha+1}\rho) \times (\nu^{\alpha-k} \times \nu^{\alpha-k+1} \times \cdots \times \nu^{\alpha-1}\rho \times \nu^{\alpha}\rho) \rtimes \sigma$$

has a unique irreducible subrepresentation. We denote by $\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+l}\rho], \sigma)$ that irreducible subrepresentation. We have

$$s_{((k+l+1)p)}(\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+l}\rho], \sigma)) \\ = L(\nu^{\alpha-k}\rho, \nu^{\alpha-k+1}\rho, \ldots, \nu^{\alpha-2}\rho, \nu^{\alpha-1}\rho, \delta([\nu^{\alpha}\rho, \nu^{\alpha+l}\rho])) \otimes \sigma.$$

Further, the representation $\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+l}\rho], \sigma)$ is a regular square integrable representation and it is a unique irreducible subrepresentation of

$$L(\nu^{\alpha-k}\rho, \nu^{\alpha-k+1}\rho, \ldots, \nu^{\alpha-2}\rho, \nu^{\alpha-1}\rho, \delta([\nu^{\alpha}\rho, \nu^{\alpha+l}\rho])) \rtimes \sigma.$$

Proof. Proposition 3.1 and the above lemma imply the proposition for $k = 0$ or $l = 0$. Therefore, we shall suppose that $k \geq 1$ and $i \geq 1$. The formula (3-2) (written for S_m

groups) implies that we are again in the regular situation. In a similar way as we got (7-2) and (7-3), using

$$
\begin{aligned}
&(\nu^{\alpha-k} \times \nu^{\alpha-k+1} \times \cdots \times \nu^{\alpha-1}\rho) \times (\nu^{\alpha+l} \times \nu^{\alpha+l-1} \times \cdots \times \nu^{\alpha+2}\rho \times \nu^{\alpha+1}\rho \times \nu^{\alpha}\rho) \rtimes \sigma \\
\cong&(\nu^{\alpha+l} \times \nu^{\alpha+l-1} \times \cdots \times \nu^{\alpha+2}\rho \times \nu^{\alpha+1}\rho) \times (\nu^{\alpha-k} \times \nu^{\alpha-k+1} \times \cdots \times \nu^{\alpha-1}\rho \times \nu^{\alpha}\rho) \rtimes \sigma,
\end{aligned}
$$

the above lemma and (ii) of Proposition 3.1, we get the following embeddings:

$$\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+l}\rho], \sigma) \hookrightarrow \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha-1}\rho]) \rtimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+l}\rho], \sigma), \tag{7-8}$$

$$\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+l}\rho], \sigma) \hookrightarrow \delta([\nu^{\alpha+1}\rho, \nu^{\alpha+l}\rho]) \rtimes \delta([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho], \sigma). \tag{7-9}$$

Write now

$$
\begin{aligned}
&\text{s.s.}(s_{((k+l+1)p)}(\mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha-1}\rho]) \rtimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+l}\rho], \sigma))) \\
&\sum_{i=-1}^{k-1} \mathfrak{s}([\nu^{-\alpha+1}\rho, \nu^{-\alpha+k-i-1}\rho]) \times \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha-k+i}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+l}\rho]) \otimes \sigma,
\end{aligned} \tag{7-10}
$$

$$
\begin{aligned}
&\text{s.s.}(s_{((k+l+1)p)}(\delta([\nu^{\alpha+1}\rho, \nu^{\alpha+l}\rho]) \rtimes \delta([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho], \sigma))) \\
&\sum_{j=0}^{l} \delta([\nu^{-\alpha-l+j}\rho, \nu^{-\alpha-1}\rho]) \times \delta([\nu^{\alpha+l-j+1}\rho, \nu^{\alpha+l}\rho]) \times \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho]) \otimes \sigma.
\end{aligned} \tag{7-11}
$$

Considering the supports (in the sense of general linear groups), one can see that a common irreducible subquotient of (7-10) and (7-11) must be a common irreducible subquotient of $\mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha-1}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+l}\rho]) \otimes \sigma$ and $\delta([\nu^{\alpha+1}\rho, \nu^{\alpha+l}\rho]) \times \mathfrak{s}([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho]) \otimes \sigma$. One can get now from the Bernstein-Zelevinsky theory that

$$L(\nu^{\alpha-k}\rho, \nu^{\alpha-k+1}\rho, \ldots, \nu^{\alpha-2}\rho, \nu^{\alpha-1}\rho, \delta([\nu^{\alpha}\rho, \nu^{\alpha+l}\rho])) \otimes \sigma$$

is the only common irreducible subquotient of (7-8) and (7-9). This proves the formula for $s_{((k+l+1)p)}(\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+l}\rho], \sigma))$. This formula, Lemma 2.3 and the square integrability criterion imply that $\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+l}\rho], \sigma)$ is square integrable. Frobenius reciprocity further implies that $\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+l}\rho], \sigma)$ embeds into

$$L(\nu^{\alpha-k}\rho, \nu^{\alpha-k+1}\rho, \ldots, \nu^{\alpha-2}\rho, \nu^{\alpha-1}\rho, \delta([\nu^{\alpha}\rho, \nu^{\alpha+l}\rho])) \rtimes \sigma.$$

Regularity implies the uniqueness of the irreducible subrepresentation. The proof is now complete. $\square$

7.3. Remark. *For questions of reducibility of parabolically induced representations, it is useful to know formulas for Jacquet modules of inducing representations with respect to all maximal parabolic subgroups. For the representations $\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho],\sigma)$ introduced in the above proposition, the formula should be*

$$\mu^*(\delta([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho],\sigma)) = \mathfrak{d}([\nu^{\beta-k}\rho,\nu^{\beta+l}\rho])\otimes\sigma \\ + \sum_{i=0}^{k}\sum_{j=0}^{l}\delta([\nu^{\beta+j+1}\rho,\nu^{\beta+l}\rho])\times\mathfrak{s}([\nu^{\beta-k}\rho,\nu^{\beta-i-1}\rho])\otimes\delta([\nu^{\beta-i}\rho,\nu^{\beta+j}\rho],\sigma).$$

Checking of this formula we leave for some other occasion (essentially, the proof should reduce to a question about permutations).

Similarly as we proved Theorems 3.3 and 5.3 (and how one proves Theorems 6.3 and 6.4), we prove the following theorem, using the above proposition (the construction of the square integrable representations in the theorem can be also derived from the above proposition, using [Jn]).

7.4. Theorem. *Let ρ_i, $i=1,\dots,n$ be mutually inequivalent irreducible unitarizable cuspidal representations of general linear groups, and let σ be an irreducible unitarizable cuspidal representations of S_q. Suppose that there exist $\alpha_i>0, i=1,\dots,n$, such that $\nu^{\alpha_i}\rho_i\rtimes\sigma$ reduces for $i=1,\dots,n$. Let $k_i,l_i\in\mathbb{Z}$, $i=1,\dots,n$, be such that*

$$0<\alpha_i-k_i\le\alpha_i\le\alpha_i+l_i,\quad i=1,\dots,n.$$

Denote

$$\Delta_i=[\nu^{\alpha_i-k_i}\rho_i,\nu^{\alpha_i+l_i}\rho_i],$$
$$\mathfrak{d}(\Delta_i)=L(\nu^{\alpha_i-k_i}\rho_i,\nu^{\alpha_i-k_i+1}\rho_i,\dots,\nu^{\alpha_i-2}\rho_i,\nu^{\alpha_i-1}\rho_i,\delta([\nu^{\alpha_i}\rho_i,\nu^{\alpha_i+l_i}\rho_i]))$$

(note that $\mathfrak{d}(\Delta_i)=\delta([\nu^{\alpha_i}\rho_i,\nu^{\alpha_i+l_i}\rho_i])$ if $\alpha_i\in\{1/2,1\}$). Suppose that $\delta(\Delta_1)\times\dots\delta(\Delta_n)$ is a representation of $GL(r,F)$.
(i) The representation

$$\mathfrak{d}(\Delta_1)\times\mathfrak{d}(\Delta_2)\times\dots\times\mathfrak{d}(\Delta_k)\rtimes\sigma$$

has a unique irreducible subrepresentation. We denote it by $\delta(\Delta_1,\Delta_2,\dots,\Delta_n,\sigma)$. Then $\delta(\Delta_1,\Delta_2,\dots,\Delta_n,\sigma)$ is a regular square integrable representation, and it satisfies

$$s_{(r)}(\delta(\Delta_1,\Delta_2,\dots,\Delta_n,\sigma))=\mathfrak{d}(\Delta_1)\times\mathfrak{d}(\Delta_2)\times\dots\times\mathfrak{d}(\Delta_n)\otimes\sigma.$$

(ii) If we suppose that $\mathcal{R}_{(1/2)\mathbb{Z}}$ holds in general, then each regular irreducible square integrable representation of symplectic or odd-orthogonal group over F is equivalent to one of the representations $\delta(\Delta_1,\Delta_2,\dots,\Delta_n,\sigma)$ defined in (i). $\square$

8. Non-degenerate standard modules with degenerate irreducible subrepresentations

First we shall recall definition of standard modules and Langlands classification in the case of symplectic and odd-orthogonal groups.

Set $D_+ = \{\delta \in D; e(\delta) > 0\}$. In the case of groups $\mathrm{GSp}(n, F)$ (resp. S_n) we denote by T the set of all equivalence classes of all irreducible essentially tempered smooth representations of all $\mathrm{GSp}(n, F)$, $n \geq 0$ (resp. S_n, $n \geq 0$). For $b = ((\delta_1, \ldots, \delta_k), \tau) \in M(D_+) \times T$ take a permutation p of $\{1, \ldots, n\}$ such that $e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \cdots \geq e(\delta_{p(k)})$. The representation $\lambda(b) = \delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)} \rtimes \tau$ (which is independent of choice of p, up to an equivalence) has a unique irreducible quotient, which is denoted by $L(b)$. This is Langlands classification for $\mathrm{GSp}(n, F)$, $\mathrm{Sp}(n, F)$ and $\mathrm{SO}(2n + 1, F)$ groups. Further, $\lambda(b), b = ((\delta_1, \ldots, \delta_k), \tau) \in M(D_+) \times T$, are called standard modules of these groups, and a standard module $\lambda(b)$ is called non-degenerate, if τ is non-degenerate for some non-degenerate character.

H. Jacquet and J.A. Shalika have proved (in the case of general linear groups) that each $\lambda(a), a \in M(D)$, has an injective Whittaker model ([JcSk]). Each representation $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)$ from Theorem 3.3 with $n > 0$, provides easily an example of non-degenerate standard module of symplectic group which does not have an injective Whittaker model (thus, Jacquet and Shalika's theorem does not generalize to other classical groups). Now we shall sketch how these representations provide counter examples for symplectic groups (for simplicity, we shall consider only the case when σ is a character of $\mathrm{GSp}(0, F) = F^\times$).

8.1. Proposition. *Let $\delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)$ be an essentially square integrable representation from Theorem 3.3, where σ is a character and $n \geq 1$. Then $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes 1$ is a non-degenerate standard module (of a symplectic group) which has no injective Whittaker model.*

Proof. Denote $\delta = \delta(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_m, \sigma)$. Assume that δ is a representation of $\mathrm{GSp}(l, F)$. Now (vi) and (ix) of Theorem 3.3 imply that $\phi\delta \cong \delta$ for $\phi \in X$ (X is as in Theorem 3.3). By Clifford theory for reductive p-adic groups ([GKn], see also [T1]), $\delta|\mathrm{Sp}(l, F)$ is a direct sum of at least two irreducible representations (since $\mathrm{card}(X) = 2^n > 1$). Recall that δ is a subrepresentation of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes \sigma$. Therefore, $\delta|\mathrm{Sp}(l, F)$ embeds into $\big(\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes \sigma\big)|\mathrm{Sp}(l, F) \cong \delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes 1$ ([T3], Proposition 4.3, (iv)). Note that $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes 1$ is a non-degenerate standard module for $\mathrm{Sp}(l, F)$ (recall that all p_i in Theorem 3.3 are positive, by (v) of that theorem). Suppose that $\delta(\Delta_1) \times \cdots \times \delta(\Delta_n) \times \delta(\Gamma_1) \times \cdots \times \delta(\Gamma_m) \rtimes 1$ has an injective Whittaker module. Then each irreducible subrepresentation π of it has a Whittaker module (for the same fixed non-degenerate character). Therefore, the space of Whittaker forms on π is non-trivial. This implies easily that the space of Whittaker forms on δ is at least two-dimensional. This contradicts to the uniqueness of the Whittaker model of irreducible representations (of δ in this case) of $\mathrm{GSp}(l, F)$ ([Ro1]). $\square$

Note that the first examples of non-degenerate standard modules without injective Whittaker models appear already for $\mathrm{Sp}(2, F)$. They are representations $\delta([\psi, \nu\,\psi]) \rtimes 1 =$

$\nu^{1/2}\psi\mathrm{St}_{\mathrm{GL}(2,F)} \rtimes 1$, where ψ is a character of order two (one can take ψ to be unramified).

In the above examples (when char $F = 0$), each irreducible subrepresentation of standard modules is non-degenerate for some non-degenerate character ([Mi3]). We shall present now for groups $\mathrm{Sp}(n, F)$, $\mathrm{GSp}(n, F)$ and $\mathrm{SO}(2n+1, F)$ simple examples of non-degenerate standard modules, which contain degenerate irreducible subrepresentations (by degenerate, we shall mean degenerate with respect to any non-degenerate character). Clearly, these standard modules also do not have injective Whittaker models. We shall also give such examples with Iwahori fixed vectors (such examples usually appear first, i.e. in the lowest semi simple ranks). Whittaker models in this particular case have attracted considerable interest ([BaMo], [Li], [Ree1] and [Ree2] are some of the recent papers in this direction).

In the Grothendieck group of the category of smooth representations of finite length of a reductive group G over F, there is a natural partial order (the cone of positive elements is formed by the semi simplifications of representations of finite length, i.e. the irreducible representations generate the cone as an additive semi-group). This partial order will be denoted by $\leq$.

The following family of examples is the simplest from the point of Jacquet modules. The beginning case of $n = 1$ is particularly simple.

8.2. Lemma. *Let ρ be a selfdual irreducible cuspidal representation of $GL(\ell, F)$ and let σ be an irreducible cuspidal representation of $GSp(m, F)$. Suppose that $\rho \rtimes \sigma$ reduces. Let n be a non-negative integer. The representation $\delta([\rho, \nu^n\rho]) \rtimes \sigma$ contains exactly two irreducible subrepresentations, and they are not isomorphic.*

For the purpose of our examples of non-degenerate standard modules with degenerate irreducible subrepresentations, it is not necessary to prove that the irreducible subrepresentations in the lemma are not isomorphic (but it follows almost automatically from the rest of the proof).

Proof. Note first that $\omega_\rho\sigma \cong \sigma$ by Lemma 2.2. Since after twist with a suitable character, $\rho \rtimes \sigma$ is unitarizable, $\rho \rtimes \sigma$ is a direct sum of at least two irreducible representations. Therefore, $\dim_{\mathbb{C}} \mathrm{End}_{\mathrm{GSp}(\ell+m,F)} (\rho \rtimes \sigma) > 1$. Since $\mu^*(\rho \rtimes \sigma) = 1 \otimes \rho \rtimes \sigma + 2\rho \otimes \sigma$ by (1-4), Frobenius reciprocity implies $\dim_{\mathbb{C}} \mathrm{End}_{\mathrm{GSp}(\ell+m,F)} (\rho \rtimes \sigma) = 2$. This implies that $\rho \rtimes \sigma$ is a sum of two inequivalent irreducible representations, which we denote by τ_1 and τ_2. Further, Frobenius reciprocity and $\mu^*(\rho \rtimes \sigma) = 1 \otimes \rho \rtimes \sigma + 2\rho \otimes \sigma$ imply $\mu^*(\tau_i) = 1 \otimes \tau_i + \rho \otimes \sigma$.

We shall now first prove the lemma in the case $n = 1$ (the general case goes by the same type of argument). First we get directly using (1-4)

$$\text{s.s.}\left(s_{(2\ell)}\left(\nu\rho \rtimes \tau_i\right)\right) = \nu\rho \times \rho \otimes \sigma + \nu^{-1}\rho \times \rho \otimes \omega_{\nu\rho}\sigma, \quad i = 1, 2; \tag{8-1}$$

$$\text{s.s.}\left(s_{(2\ell)}\left(\delta([\rho, \nu\rho]) \rtimes \sigma\right)\right) = \delta([\nu^{-1}\rho, \rho]) \otimes \omega_{\nu\rho}\sigma + \rho \times \nu\rho \otimes \sigma + \delta([\rho, \nu\rho]) \otimes \sigma; \tag{8-2}$$

$$\text{s.s.}\left(s_{(2\ell)}(\nu\rho \times \rho \rtimes \sigma)\right) = 2\ \text{s.s.}\left(s_{(2\ell)}\left(\nu\rho \rtimes \tau_i\right)\right). \tag{8-3}$$

Lift embeddings $\tau_i \hookrightarrow \rho \rtimes \sigma$, $i = 1, 2$, to embeddings of induced representations $\nu\rho \rtimes \tau_i \hookrightarrow \nu\rho \times \rho \rtimes \sigma$, $i = 1, 2$. We have an obvious embedding $\delta([\rho, \nu\rho]) \rtimes \sigma \hookrightarrow \nu\rho \times \rho \rtimes \sigma$. Fix three such embeddings into $\nu\rho \times \rho \rtimes \sigma$, and identify domains of these embeddings with the images in $\nu\rho \times \rho \rtimes \sigma$. Clearly, $(\nu\rho \rtimes \tau_1) \cap (\nu\rho \rtimes \tau_2) = \{0\}$ since $\tau_1 \cap \tau_2 = \{0\}$.

Further, $(\nu\rho \rtimes \tau_i) \cap (\delta[\rho, \nu\rho] \rtimes \sigma) \neq \{0\}$, for $i = 1, 2$ (if it would be equal $\{0\}$, then s.s. $\big(s_{(2\ell)}\,(\nu\rho \rtimes \tau_i)\big) +$s.s. $\big(s_{(2\ell)}\,(\delta([\rho, \nu\rho]) \rtimes \sigma)\big) \leq$ s.s. $\big(s_{(2\ell)}(\nu\rho \times \rho \rtimes \sigma)\big)$, which implies by (8-3) s.s. $\big(s_{(2\ell)}\,(\delta([\rho, \nu\rho]) \rtimes \sigma)\big) \leq$ s.s. $\big(s_{(2\ell)}\,(\nu\rho \rtimes \tau_i)\big)$, and this contradicts to (8-1) and (8-2)).

Therefore, $\delta([\rho, \nu\rho]) \rtimes \sigma$ has at least two (different) irreducible subrepresentations. Let π be any irreducible subrepresentation of $\delta([\rho, \nu\rho]) \rtimes \sigma$. Frobenius reciprocity implies that $\delta([\rho, \nu\rho]) \otimes \sigma$ is a quotient of $s_{(2\ell)}(\pi)$. Now (8-2) implies that there are exactly two irreducible subrepresentations, and that $\delta([\rho, \nu\rho]) \otimes \sigma$ has multiplicity one in $s_{(2\ell)}(\pi)$. Further, Frobenius reciprocity implies $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GSp}(2\ell+m,F)}(\pi, \delta([\rho, \nu\rho]) \rtimes \sigma) \leq 1$. Therefore, these irreducible subrepresentations are not isomorphic.

For the general case of $n \geq 1$, one checks directly that the multiplicity of $\delta([\rho, \nu^n\rho]) \otimes \sigma$ in $s_{((n+1)\ell)}\,(\delta([\nu\rho, \nu^n\rho]) \rtimes \tau_i)$ is 1 for $i = 1, 2$, and 2 in $s_{((n+1)\ell)}\,(\delta([\rho, \nu^n\rho]) \rtimes \sigma)$. Now repeating the above argument (in the representation $\delta([\nu\rho, \nu^n\rho]) \times \rho \rtimes \sigma$), one gets the lemma. $\square$

The Steinberg representation of a connected reductive group G will be denoted by St_G.

8.3. Corollary. *Let σ be an irreducible cuspidal representation of $GSp(1, F)$=$GL(2, F)$, and let m be a positive integer. Then*

$$\delta([1_{F^\times}, \nu^m\, 1_{F^\times}]) \rtimes \sigma = \nu^{m/2} St_{\mathrm{GL}(m+1,F)} \rtimes \sigma$$

is a non-degenerate standard module which contains exactly two irreducible subrepresentations, and they are not isomorphic. At least one of these subrepresentations is degenerate.

Proof. J.-L. Waldspurger proved in [W] that $1_{F^\times} \rtimes \sigma$ reduces (there is another proof of it in [Sd1] when $\mathrm{char}\, F = 0$). It is well known that σ is non-degenerate. Now the above lemma implies that $\delta([1_{F^\times}, \nu^m\, 1_{F^\times}]) \rtimes \sigma$ has two (non-isomorphic) irreducible subrepresentations. Denote them by π_1 and π_2.

Fix a non-degenerate character θ of the subgroup U of all upper triangular unipotent matrices in $\mathrm{GSp}(m+2, F)$. Consider the functor $r_{U,\theta}$ defined in 1.8 of [BeZ] (take $M = \{1\}$ to be the trivial subgroup there). Now [Ro1] and (b) of Proposition 1.9 in [BeZ] imply $\dim_{\mathbb{C}} r_{U,\theta}\,(\delta([1_{F^\times}, \nu^m\, 1_{F^\times}]) \rtimes \sigma) = 1$ (we can use [CaSk] instead of [BeZ] for our purpose). Since $r_{U,\theta}$ is exact functor ([BeZ], Proposition 1.9, (a)), $r_{U,\theta}(\pi_i) = \{0\}$ for some $i \in \{1, 2\}$. Now (b) of Proposition 1.9 in [BeZ] implies that π_i is degenerate with respect to θ. Recall that $\mathrm{GSp}(m + 2, F)$ has only one orbit of non-degenerate characters. Therefore, π_i is degenerate for any non-degenerate character. This is what we wanted to prove. $\square$

Note that one of the first non-degenerate standard modules that we considered in the above corollary is the representation $\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2,F)} \rtimes \sigma$ of $\mathrm{GSp}(3, F)$.

8.4. Remark. *Let σ be an irreducible cuspidal representation of $GSp(1, F) = GL(2, F)$. Write the restriction $\sigma|Sp(1, F) = \oplus_{i=1}^{c}\sigma_i$ as a sum of irreducible representations of $Sp(1, F) = SL(2, F)$. Then each σ_i is non-degenerate for some non-degenerate character. Therefore standard modules $\delta([1_{F^\times}, \nu^m\, 1_{F^\times}]) \rtimes \sigma_i = \nu^{m/2} St_{\mathrm{GL}(m+1,F)} \rtimes \sigma_i$, $m \geq 1$, of $Sp(m + 2, F)$ are non-degenerate. From the above corollary easily follows that for some $1 \leq i \leq c$, $\delta([1_{F^\times}, \nu^m\, 1_{F^\times}]) \rtimes \sigma_i = \nu^{m/2} St_{\mathrm{GL}(m+1,F)} \rtimes \sigma_i$ contains a degenerate irreducible* *subrepresentation. Thus, there exist also non-degenerate standard modules of symplectic groups with degenerate irreducible subrepresentations.*

The one-parameter family of examples of standard modules in the above corollary belongs to a wider two-parameter family of such examples. Instead of going into details of construction of this two-parameter family, we shall now present another two-parameter family of examples of non-degenerate standard modules with degenerate irreducible subrepresentations. This family provides also unramified examples (among others).

8.5. Lemma. *Let ρ be a selfdual irreducible cuspidal representation of $GL(\ell, F)$, and let σ be an irreducible cuspidal representation of S_m. Let $k, l \in \mathbb{Z}, 0 \leq k < l$. Suppose that $\nu^{1/2}\rho \rtimes \sigma$ reduces. Then $\delta([\nu^{-k-1/2}\rho, \nu^{l+1/2}\rho]) \rtimes \sigma$ contains exactly two irreducible subrepresentations. They are not isomorphic.*

Proof. We shall first give details in the case $k = 0$ and $l = 1$. We get directly from (6-3)

$$\text{(8-4)} \qquad \text{s.s.}\left(s_{(2\ell)}\left(\nu^{1/2}\rho \times \nu^{-1/2}\rho \rtimes \sigma\right)\right) = \sum_{\epsilon_i \in \{\pm 1\}} \nu^{\epsilon_1/2}\rho \times \nu^{\epsilon_2/2}\rho \otimes \sigma;$$

$$\text{(8-5)} \quad \text{s.s.}\left(s_{(2\ell)}\left(\delta([\nu^{-1/2}\rho \times \nu^{1/2}\rho]) \rtimes \sigma\right)\right) = 2\delta([\nu^{-1/2}\rho \times \nu^{1/2}\rho]) \otimes \sigma + \nu^{1/2}\rho \times \nu^{1/2}\rho \otimes \sigma;$$

$$\text{(8-6)} \qquad \text{s.s.}\left(s_{(2\ell)}\left(\nu^{1/2}\rho \times \delta(\nu^{1/2}\rho, \sigma)\right)\right) = \nu^{1/2}\rho \times \nu^{1/2}\rho \otimes \sigma + \nu^{-1/2}\rho \times \nu^{1/2}\rho \otimes \sigma.$$

Note that $\delta([\nu^{-1/2}\rho \times \nu^{1/2}\rho]) \rtimes \sigma \leq \nu^{1/2}\rho \times \nu^{-1/2}\rho \rtimes \sigma$ and $\nu^{1/2}\rho \times \delta(\nu^{1/2}\rho, \sigma) \leq \nu^{1/2}\rho \times \nu^{-1/2}\rho \rtimes \sigma$ (inequalities are in the Grothendieck group). Now (8-4) - (8-6) imply that $\delta([\nu^{-1/2}\rho \times \nu^{1/2}\rho]) \rtimes \sigma$ is a sum of two inequivalent representation, which we can denote by τ_i, $i = 0, 1$, in a such a way that

$$\text{(8-7)} \qquad \text{s.s.}\left(s_{(2\ell)}(\tau_i)\right) = \delta([\nu^{-1/2}\rho \times \nu^{1/2}\rho]) \otimes \sigma + i\ \nu^{1/2}\rho \times \nu^{1/2}\rho \otimes \sigma \qquad \text{for} \qquad i = 1, 2.$$

Note that $\mu^*(\nu^{1/2}\rho \times \nu^{1/2}\rho \rtimes \sigma) = (1 \otimes \nu^{1/2}\rho + \nu^{1/2}\rho \otimes 1 + \nu^{-1/2}\rho \otimes 1)^2 \rtimes (1 \otimes \sigma)$. Since $M^*(\nu^{3/2}\rho) = 1 \otimes \nu^{3/2}\rho + \nu^{3/2}\rho \otimes 1 + \nu^{-3/2}\rho \otimes 1$, the preceding formula and (8-7) imply that the multiplicity of $\delta([\nu^{-1/2}\rho, \nu^{3/2}\rho]) \otimes \sigma$ in s.s. $\left(s_{(3\ell)}(\nu^{3/2}\rho \rtimes \tau_i)\right)$ is one for $i = 0, 1$. One directly computes that the multiplicity of $\delta([\nu^{-1/2}\rho, \nu^{3/2}\rho]) \otimes \sigma$ in s.s. $\left(s_{(3\ell)}\left(\delta([\nu^{-1/2}\rho, \nu^{3/2}\rho]) \rtimes \sigma\right)\right)$ is two. Now embeddings $\tau_i \hookrightarrow \delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \rtimes \sigma$, $i = 0, 1$, and $\delta([\nu^{-1/2}\rho, \nu^{3/2}\rho]) \hookrightarrow \nu^{3/2}\rho \rtimes \delta([\nu^{-1/2}\rho, \nu^{1/2}\rho])$ induce embeddings of $\nu^{3/2}\rho \rtimes \tau_i$, $i = 0, 1$, and $\delta([\nu^{-1/2}\rho, \nu^{3/2}\rho]) \rtimes \sigma$ into $\nu^{3/2}\rho \times \delta([\nu^{-1/2}\rho, \nu^{1/2}\rho]) \rtimes \sigma$.

We can now repeat the argument from the last lemma to get the claim of the lemma in this case.

For the general case, using $\delta([\nu^{1/2}\rho, \nu^{k+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{k+1/2}\rho], \sigma)$ and looking at Jacquet modules $s_{((2k+2)\ell)}$, one first proves that $\delta([\nu^{-k-1/2}\rho, \nu^{k+1/2}\rho]) \rtimes \sigma$ reduces. The formula for $s_{((2k+2)\ell)}\left(\delta([\nu^{-k-1/2}\rho, \nu^{k+1/2}\rho]) \rtimes \sigma\right)$ implies that $\delta([\nu^{-k-1/2}\rho, \nu^{k+1/2}\rho]) \rtimes \sigma$

reduces into a sum of two non-equivalent irreducible pieces which we denote by τ_1 and τ_2. We can denote them in such a way that they satisfy

$$\delta([\nu^{-1/2-k}\rho, \nu^{k+1/2}\rho]) \otimes \sigma \leq s_{((2k+2)\ell)}(\tau_i) \leq \delta([\nu^{-1/2-k}\rho, \nu^{k+1/2}\rho]) \otimes \sigma + c\sum_{p=0}^{k} \delta([\nu^{1/2-p}\rho, \nu^{k+1/2}\rho]) \times \delta([\nu^{1/2+p}\rho, \nu^{k+1/2}\rho]) \otimes \sigma,$$

for some natural number c. From this estimate, one gets easily that the multiplicity of $\delta([\nu^{-1/2-k}\rho, \nu^{l+1/2}\rho]) \otimes \sigma$ in $s_{((k+l+2)\ell)}\left(\delta([\nu^{3/2+k}\rho, \nu^{l+1/2}\rho]) \rtimes \tau_i\right)$ is 1, for $i = 0, 1$. From the other side, direct computation gives that multiplicity of $\delta([\nu^{-1/2-k}\rho, \nu^{l+1/2}\rho]) \otimes \sigma$ in $s_{((k+l+2)\ell)}\left(\delta([\nu^{-1/2-k}\rho, \nu^{l+1/2}\rho]) \rtimes \sigma\right)$ is 2. Now one proceeds in the same way as before. □

8.6. Corollary. *For $k, l \in \mathbb{Z}$ such that $0 \leq k < l$, the representation*

$$\delta([\nu^{-k-1/2}1_{F^\times}, \nu^{l+1/2}\, 1_{F^\times}]) \rtimes 1 = (\nu^{(l-k)/2}\ St_{\mathrm{GL}(k+l+2,F)}) \rtimes 1$$

of $SO(2(k+l+2)+1, F)$ is a non-degenerate standard module which contains exactly two irreducible subrepresentations. They are inequivalent, and they have Iwahori fixed vectors. At least one of these representations is degenerate.

Proof. It is well known that $\mathrm{SO}(3, F)$-representation $\left(\nu^{1/2}\, 1_{F^\times}\right) \rtimes 1$ reduces. Further, in the case of $\mathrm{SO}(2(k+l+2)+1, F)$, one has only one orbit of non-degenerate characters. Now the proof goes in the same way as proof of Corollary 8.3 (using the above lemma). □

The first example of standard modules from the above corollary is the representation $\left(\nu^{1/2}\ \mathrm{St}_{\mathrm{GL}(3,F)}\right) \rtimes 1$ of $\mathrm{SO}(7, F)$.

8.7. Remarks. *(i) The degenerate irreducible subrepresentation π in the above corollary is characterized with the condition $\delta([\nu^{1/2}1_{F^\times}, \nu^{k+1/2}\, 1_{F^\times}]) \times \delta([\nu^{1/2}1_{F^\times}, \nu^{l+1/2}\, 1_{F^\times}]) \otimes 1 \not\leq s_{(2(l+k+2)+1)}(\pi)$. The other irreducible subrepresentation is non-degenerate.*

(ii) Let ψ be a character of $F^\times$ of order 2 (one can chose ψ to be unramified). It is not hard to show that standard module $\left(\nu^{1/2}\ St_{\mathrm{GL}(3,F)}\right) \times \left(\nu^{1/2}\psi\ St_{\mathrm{GL}(3,F)}\right) \rtimes 1$ has 4 inequivalent irreducible submodules (if ψ is unramified, they all have Iwahori fixed vectors). Therefore, this provides an example of a non-degenerate standard module with at least 3 degenerate irreducible subrepresentations.

(iii) For any $p \geq 1$, generalizing our construction, one can construct examples of standard modules with 2^p inequivalent irreducible submodules (with Iwahori fixed vectors, if one wants).

(iv) Representations $\left(\nu^{(l-k)/2}\ St_{\mathrm{GL}(l+k+1,F)}\right) \rtimes 1$ $(k, l \in \mathbb{Z},\ 0 < k < l)$ of symplectic groups, provide examples of non-degenerate standard modules with degenerate irreducible subrepresentations with Iwahori fixed vectors. Such examples for $GSp(l+k+1, F)$ are $\left(\nu^{(l-k)/2}\ St_{\mathrm{GL}(l+k+1,F)}\right) \rtimes 1_{F^\times}$ $(k, l \in \mathbb{Z},\ 0 < k < l)$.

(v) Using parabolic induction one can easily generate from examples in Corollaries 8.3 and 8.6 many new examples of non-degenerate standard modules with degenerate irreducible

subrepresentations. For example, take any $\delta_1, \dots, \delta_k \in M(D)$ such that $e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_k) \geq 1/2$. Then $\delta_1 \times \delta_2 \times \cdots \times \delta_k \times \big(\nu^{1/2}\, St_{\mathrm{GL}(3,F)}\big) \rtimes 1$ is a non-degenerate standard module, and it contains a degenerate irreducible subrepresentation.

(vi) All irreducible subrepresentations of non-degenerate standard modules that show up in Corollaries 8.3 and 8.6 are essentially square integrable (this is not quite simple to prove). Examples that we have generated in (v) have degenerate irreducible subrepresentations which are not square integrable.

(vii) In the case of G_2, the module $I_\beta(1/2, \delta(1))$ in (ii) of Proposition 4.3 of [Mi2] is an example of non-degenerate standard module with degenerate irreducible subrepresentation.

(viii) In [T7] we constructed a wide family of (non-regular) square integrable representations. There is also a plenty of non-degenerate standard modules with degenerate irreducible subrepresentations. Those examples are obtained there in a very indirect and much more complicated way than examples here (our primary objet of study in that paper is not the structure of standard modules).

We shall end this section with one simple result (Corollary 8.9) about reducibility points of representations parabolically induced by non-degenerate irreducible square integrable representations. This result (and much more) can also be proved in other ways (for example, [Mi3] contains explicitly computed reducibility points of the representations that we shall consider). The proof that we present here is probably the most elementary one (more details about methods of proving irreducibility of parabolically induced representations using Jacquet modules, one can find in [T6]). This result is of interest regarding connection of Shahidi's Conjecture 9.4 in [Sd1] with $\mathcal{R}_{(1/2)\mathbb{Z}}$ and $R_{(1/2)\mathbb{Z}}$.

First we have

8.8. Lemma. *Let char $F = 0$. Suppose that $\rho, \rho_1, \rho_2, \dots, \rho_n$ are irreducible cuspidal representations of general linear groups, and that σ is a non-degenerate irreducible cuspidal representation of S_q. Suppose that π is a subquotient of $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$. If $\rho \rtimes \pi$ reduces and $e(\rho_i) \in (1/2)\mathbb{Z}$ for $i = 1, \dots, n$, then $e(\rho) \in (1/2)\mathbb{Z}$.*

Proof. Suppose $e(\rho) \notin (1/2)\mathbb{Z}$ By (6-2) it is enough to prove the irreducibility of $\rho \rtimes \pi$ for $e(\rho) > 0$. By Shahidi's results on reducibility when σ is non-degenerate (see Remark 4.5), we know that $\rho \rtimes \sigma$ is irreducible. We know

$$\mu^*(\pi) \leq \left(\prod_{i=1}^{n} \big(1 \otimes \rho_i + (\rho_i \otimes 1 + \tilde{\rho}_i \otimes 1)\big) \right) \otimes \sigma, \tag{8-8}$$

$$\mu^*(\rho \rtimes \pi) = \big(1 \otimes \rho + (\rho \otimes 1 + \tilde{\rho} \otimes 1)\big) \rtimes \mu^*(\pi). \tag{8-9}$$

Now we shall draw some consequences of above two formulas.

Assume that ρ is a representation of $\mathrm{GL}(p, F)$. Then

$$\mathrm{s.s.}(s_{(p)}(\rho \rtimes \pi)) = \rho \otimes \pi + \tilde{\rho} \otimes \pi + \sum_i \tau_i' \otimes \tau_i'', \tag{8-10}$$

where τ_i', τ_i'' are irreducible, and $\tau_i' \not\cong \rho$, $\tau_i' \not\cong \tilde{\rho}$ for any i (clearly, ρ or $\tilde{\rho}$ are not in the support of any τ_i'). From this follows that $\rho \otimes \pi$ and $\tilde{\rho} \otimes \pi$ have multiplicity one $s_{(p)}(\rho \rtimes \pi)$.

Each irreducible subrepresentation of $\rho \rtimes \pi$ has $\rho \otimes \pi$ in the Jacquet module. Therefore, there is a unique irreducible subrepresentation, and this subrepresentation must have $\rho \otimes \pi$ in the Jacquet module. In a similar way, one see that $\rho \rtimes \pi$ has a unique irreducible quotient (pass to contragredients). Further, it is not hard to get that the irreducible quotient has $\tilde{\rho} \otimes \pi$ for a subquotient of a Jacquet module (use Corollary 4.2.5 of [C]).

Assume that $\rho_1 \times \cdots \times \rho_n$ is a representation of $\mathrm{GL}(a, F)$. Then

$$\text{s.s.}(s_{(p+a)}(\rho \rtimes \pi)) = \sum_i \rho \times \tau_i''' \otimes \sigma + \sum_j \tilde{\rho} \times \tau_j'''' \otimes \sigma, \tag{8-11}$$

where τ_i''', τ_j'''' are irreducible, ρ or $\tilde{\rho}$ is not in the support of any τ_i''', τ_j'''', and both $\rho \times \tau_i'''$, $\tilde{\rho} \times \tau_j''''$ are irreducible.

We shall also need one additional Jacquet module. We have

$$\text{s.s.}(s_{(a)}(\rho \rtimes \pi)) = \sum_i \pi_i' \otimes \pi_i'' + \sum_j \pi_j''' \otimes \rho \rtimes \sigma, \tag{8-12}$$

where π_i', π_i'', π_j''' are irreducible and each π_i'' is a subquotient of some $(\prod_{i \in X} \rho_i^{\epsilon_i}) \rtimes \sigma$ for some $X \subseteq \{1, \ldots, n\}$ and $\epsilon_i \in \{\pm 1\}$, $i \in X$ (in the formula $(\prod_{i \in X} \rho_i^{\epsilon_i}) \rtimes \sigma$, ρ_i^1 denotes ρ_i, while ρ_i^{-1} denotes $\tilde{\rho}_i$). Note that $\rho \times \sigma$ is irreducible by the Shahidi's results (see Remark 4.5).

Let τ be an irreducible subrepresentation of $\rho \rtimes \pi$. Then it has $\rho \otimes \pi$ for a subquotient of $s_{(p)}(\tau)$. Now (8-11) (and transitivity of Jacquet modules) implies that some $\rho \times \tau_i''' \otimes \sigma$ must be a subquotient of a Jacquet module of τ. This implies that $\tau_i''' \otimes \rho \otimes \sigma$ is a subquotient of a Jacquet module of τ. Now (8-12) implies that some $\pi_j''' \otimes \rho \rtimes \sigma$ is a subquotient of a Jacquet module of τ. This implies that $\pi_j''' \otimes \tilde{\rho} \rtimes \sigma$ is a subquotient of a Jacquet module of τ (use (6-2) and the irreducibility of $\rho \rtimes \sigma$). From (8-10) we see that some $\tilde{\rho} \times \tau_j'''' \otimes \sigma$ must be a subquotient of a Jacquet module of τ. This implies that $\tilde{\rho} \otimes \tau_j'''' \otimes \sigma$ must be a subquotient of a Jacquet module of τ. Now (8-10) implies that $\tilde{\rho} \otimes \pi$ must be a subquotient of $s_{(p)}(\tau)$. This implies the irreducibility. $\square$

8.9. Corollary. *Let ρ be an irreducible cuspidal representation of $GL(p, F)$, and let π be a non-degenerate irreducible square integrable representation of S_q. Suppose char $F = 0$. If $e(\rho) \notin (1/2)\mathbb{Z}$, then $\rho \rtimes \pi$ is irreducible.*

Proof. Shahidi has proved this when π is cuspidal. Suppose that π is not cuspidal. Then we can chose irreducible cuspidal representations $\rho_1, \ldots, \rho_n$, and an irreducible cuspidal representation σ of $S_{q'}$ such that π is a subquotient of $\rho_1 \times \cdots \times \rho_n \rtimes \sigma$. Since π is non-degenerate, σ must be generic. By (ii) of Theorem 4.9 and the Shahidi's results that we have mentioned about reducibilities when σ is non-degenerate (see Remark 4.5), we have $e(\rho_i) \in (1/2)\mathbb{Z}$, $i = 1, \ldots, n$. Now the above lemma implies the corollary. $\square$

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