June 20, 2003

Notes of the lectures at IMS, NUS, Singapore, at Lie Group Program, summer 2002.

ON CLASSIFICATION OF SOME CLASSES OF IRREDUCIBLE REPRESENTATIONS OF CLASSICAL GROUPS

Marko Tadić

Contents

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

1

INTRODUCTION

In these notes of the lectures given during the special period on representation theory of Lie groups in IMS, NUS, Singapore, we shall discuss the problem of classification of some important series of irreducible representations of general linear and classical groups, having in mind unitary representations. We shall discuss more p-adic groups, but a part of notes deals also with real groups.

One of the main goals of the notes is to give an introduction to the classification modulo cuspidal data, of irreducible square integrable representations of classical p-adic groups. After that we shall describe unitary duals of general linear groups over local fields, and describe the proof of the classification theorem in the case of complex general linear groups. We shall finish the notes with a series of questions regarding unitary representations of classical p-adic groups.

We are thankful to the organizers of the special period for providing a very stimulating atmosphere in which we had opportunity to present the lectures.

1. HARMONIC ANALYSIS AND UNITARY DUALS

1.1. One can interpret classical harmonic analysis in terms of unitary representations of $\mathbb{R}^n$ and $(\mathbb{R}/\mathbb{Z})^n$. This point of view opens a possibility of generalizing classical harmonic analysis, and building such a type of theory for a general locally compact group G (in general, neither compact, nor commutative). We shall briefly describe the main problems of harmonic analysis on such a group G. First, we shall introduce a few notions which we shall need for this description.

1.2. A **representation** (π, V) (or simply π or V) of a group G is a group homomorphism π from the group G to the group of all invertible linear operators on a complex vector space V (there is no requirement on continuity in this definition). A representation π on a non-zero vector space V is called **irreducible** (or **algebraically irreducible**) if $\{0\}$ and V are the only vector subspaces of V which are invariant for all $\pi(g), g \in G$.

A representation (π, H) is called unitary if H is a Hilbert space and:

(1) the mapping
$$(g, v) \mapsto \pi(g)v, \quad G \times H \to H$$
   is continuous;
(2) each operator $\pi(g), g \in G$ is unitary.

If we omit the second requirement in the above definition, then the representation defined in this way will be called **continuous** (one can consider much more general continuous representations, but we shall not need them in these notes).

A unitary (or only continuous) representation (π, G) is called **irreducible** (or **topologically irreducible**) if $\{0\}$ and H are the only closed subspaces of H which are invariant for all $\pi(g), g \in G$.

1.3. Now we can describe the main goals of harmonic analysis on a locally compact group G (which satisfies some technical requirements, which we shall not discuss here, but which are satisfied for the groups that we shall consider in these notes, i.e. for general linear and classical groups over local fields).

The first problem is to

(1) understand in a convenient way (possibly classify) the set of all the equivalence classes of irreducible unitary representations of G. This set is called the **unitary dual** of G, and it is denoted by

$$\hat{G}.$$

The second problem is to

(2) interpret other important unitary representations of G in terms of $\hat{G}$.

Such important unitary representations are usually given on functional spaces. The most important examples of such representations include representations of G on spaces of square integrable functions (with respect to an invariant measure, assuming that it exists) on a space X where G acts transitively. Then $X \cong H\backslash G$ for some closed group H of G and G acts by right translations on the space $L^2(H\backslash G)$ of the square integrable functions on $H\backslash G$ (in this case $H\backslash G$ carries an invariant measure for right translations of G).

The first example of such representation would be when H is the trivial subgroup of G, i.e. the representation of G on the space $L^2(G)$ of the square integrable functions on G with respect to right invariant measure on G. This representation is very important. A significant portion of Harish-Chandra's work is closely related to this representation in the case of semi simple real Lie groups (among others, he described the representation from $\hat{G}$ necessary for decomposing $L^2(G)$, and found Plancherel measure by which one decomposes $L^2(G)$ in terms of these irreducible representations).

In this lectures we shall be more related to the problem (1) of harmonic analysis, although we shall be also related to the problem (2). Irreducible square integrable representations, which are one of the main topics of our notes, are part of both problems, (1) and (2). They are subrepresentations of $L^2(G)$ if the center of G is compact.

Remark: Some of the most important parts of the Langlands program can be considered as a kind of problems from harmonic analysis on groups in the above sense. For example. the origin of the Langlands program one can view as a kind of problem of harmonic analysis. The program started as a strategy for proving the Artin's conjecture that Artin's L-functions are entire. Roughly, Langlands proposed a strategy that irreducible representations of the absolute Galois group of a number field would parameterize irreducible subrepresentations of adelic general linear groups on the spaces of cuspidal automorphic forms (which are unitary representations on functional spaces), in a way that corresponding L-functions match (this can clearly be regarded as a kind of problem of type (2) of harmonic analysis on groups). Realization of this strategy would imply the Artin's conjecture.

The above philosophy has its local counterpart (with corresponding parameterizations). In the local case, we are more related to the problem of type (1) of harmonic analysis on groups.

One can extend the above considerations to other reductive groups, and one can consider also different fields. A question may be, can one do the above mentioned parameterizations in a naturally compatible way. Such a question is related to the functoriality problem.

Above we gave only a very rough comments regarding the Langlands program. Details regarding this program can be found in [Gb] or [Kn].

2. Non-discrete locally compact fields, classical groups, reductive groups

2.1. Let F be a non-discrete locally compact field. Such field will be called **local field**. If F is connected, then the field is called archimedean. Otherwise, it is called non-archimedean. Non-archimedean fields are totally disconnected. They contain a basis of neighborhoods of 0 consisting of open compact subrings.

If a local field is archimedean, then it is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

Let p be a prime integer. Ideals $p^k\mathbb{Z}$, $k \in \mathbb{Z}_+$, define a basis of neighborhoods of 0 in $\mathbb{Z}$. The completion of Z with respect to this topology (more precisely, uniform structure defined by this topology) is denoted by $\mathbb{Z}_p$. The field of fractions of $\mathbb{Z}_p$ is denoted by $\mathbb{Q}_p$. This is the field of p-adic numbers.

We can introduce $\mathbb{Q}_p$ also as a completion of $\mathbb{Q}$ with respect to the absolute value

$$\left| \frac{a}{b} p^k \right|_p = k^{-k}, \quad a, b, k \in \mathbb{Z}, b \neq 0, p \nmid ab.$$

Any finite extension F of $\mathbb{Q}_p$ is in a natural way a topological space, and with this topology, F is a local non-archimedean field of characteristic 0. One gets each non-archimedean field of characteristic 0 in this way.

Let $\mathbb{F}_q[[X]]$ be the ring of all formal power series $\sum_{k=0}^{\infty} a_n X^n$ over a finite field $\mathbb{F}_q$ (with q elements), and let $\mathbb{F}_q((X))$ be the field of all Laurent power series $\sum_{k=-\infty}^{\infty} a_n X^n$ over $\mathbb{F}_q$ for which there exists $n_0 \in \mathbb{Z}$ such that $a_n = 0$ for all $n \leq n_0$. Then the powers of the ideal $X\mathbb{F}_q[[X]]$ in $\mathbb{F}_q[[X]]$ define a basis of neighborhoods of 0 in $\mathbb{F}_q((X))$, and therefore a topology on $\mathbb{F}_q((X))$. In this way $\mathbb{F}_q((X))$ becomes a local non-archimedean field of positive characteristic. One gets each local non-archimedean field of positive characteristic in this way.

Topology on a local field can be always defined using an absolute value. Moreover, there exists a unique absolute value $|\ |_F$ on a local non-discrete field F such that

$$\int_F f(x)dx = |a|_F^{-1} \int_F f(ax)dx,$$

for any $a \in F^\times$ and for any continuous, compactly supported function f on G, where dx denotes an invariant (for translations) measure on F. We shall always fix such an absolute value on F. Let us note that for $\mathbb{C}$, this absolute value is a square of the standard one.

2.2. We shall recall now of a definition of the classical groups. A **classical groups** over a local field F is the group of isomorphisms of either symplectic, or orthogonal or unitary space over F (of finite dimension). For the study of representations of classical groups, it is important to understand the representation theory of general linear groups $GL(n, F)$'s,

i.e. of the groups of all isomorphisms of finite dimensional vector spaces over F (soon it will become clear why this is important).

In the study of classical groups, we shall use very convenient language of structure theory of reductive groups without going into this theory. In general, we shall try to keep the technicalities as low as possible.

For the simplicity, we shall consider in these lectures two series of classical groups (these series consist of split and connected groups). The reason for this is only to simplify the notation.

2.3. The first is the series of symplectic groups:

Denote
$$
J_n = \begin{bmatrix} 00 & \dots & 01 \\ 00 & \dots & 10 \\ \vdots & & \\ 10 & \dots & 0 \end{bmatrix} \in GL(n, F).
$$

Then the symplectic group is

$$
Sp(2n, F) = \left\{ S \in GL(2n, F); \ {}^t S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} S = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\}.
$$

Here ${}^t S$ denotes the transposed matrix of S.

2.4. The second series consists of split odd-orthogonal groups:

Denote by I_n the identity matrix in $GL(n, F)$. Let

$$
SO(2n + 1, F) = \{ S \in SL(2n + 1, F); \ {}^\tau S\, S = I_{2n+1} \}.
$$

Here $SL(n, F) = \{ g \in GL(n, F); \det(g) = 1 \}$ and ${}^\tau S$ denotes the transposed matrix of S with respect to the other diagonal.

We could work also with $O(2n + 1, F)$ instead of $SO(2n + 1, F)$.

In these notes, we shall always deal with matrix forms of classical groups.

2.5. The above groups are connected, split, semi simple algebraic groups over F. They are topological groups in a natural way. In the case when F is a non-archimedean field, these groups are totally disconnected. Then one can write a basis of neighborhoods of identity which consists of open (and closed) compact subgroups. If F is archimedean, then symplectic and odd-orthogonal groups are connected semi simple Lie groups.

If G is $GL(n, F)$, or $Sp(2n, F)$ or $SO(2n+1, F)$, then we shall denote by $P_\emptyset$ the subgroup of all the upper triangular matrices in G. Then $P_\emptyset$ is called standard minimal parabolic subgroup of G. Any subgroup of G containing $P_\emptyset$ is called standard parabolic subgroup of G. There are finitely many of them, and we shall describe them precisely. Any subgroup conjugate to a standard parabolic subgroup is called parabolic subgroup.

Let
$$\alpha = (n_1, ..., n_k)$$
be an ordered partitions of n into positive integers. Consider matrices of $GL(n, F)$ as block matrices with blocks of sizes $n_i \times n_j$. Let P_α^{GL} (resp. M_α^{GL}), be the upper block-triangular matrices (resp. block-diagonal matrices) in $GL(n, F)$. Denote by N_α^{GL} the (block) matrices in P_α^{GL} which have identity matrices on the block-diagonal. Now
$$\alpha \mapsto P_\alpha^{GL}$$
is one-to-one mapping of the set of all ordered partitions of n onto the set of all standard parabolic subgroups of $GL(n, F)$. We have Levi decomposition $P_\alpha^{GL} = M_\alpha^{GL} N_\alpha^{GL}$. This means that P_α^{GL} is a semi direct product of M_α^{GL} and N_α^{GL}, where N_α^{GL} is a normal subgroup in P_α^{GL}, i.e.
$$P_\alpha^{GL} = M_\alpha^{GL} \ltimes N_\alpha^{GL}.$$
This decomposition of P_α^{GL} is called the **standard Levi decomposition** of P_α^{GL}, where M_α^{GL} is called the **standard Levi factor** of P_α^{GL} and N_α^{GL} is called the **unipotent radical** of P_α^{GL}.

2.6. Standard parabolic subgroups of $Sp(2n, F)$ **and** $SO(2n+1, F)$ are parameterized by ordered partitions $\alpha = (n_1, \ldots, n_k)$ of integers m, where $0 \le m \le n$. If we consider the group $G = Sp(2n, F)$ set
$$\alpha' = (n_1, ..., n_k, 2n - 2m, n_k, ..., n_1),$$
while in the case of the group $G = SO(2n + 1, F)$ set
$$\alpha' = (n_1, ..., n_k, 2n + 1 - 2m, n_k, ..., n_1).$$
Then
$$\alpha \mapsto P_\alpha = P_{\alpha'}^{GL} \cap G$$
gives a parameterization of standard parabolic subgroups in G. In similar way as in the case of general linear groups, one defines standard Levi decompositions in this case, using standard Levi decompositions of $P_{\alpha'}^{GL}$.

In the sequel, we shall denote by G one of the groups $GL(n, F), Sp(2n, F)$ or $SO(2n + 1, F)$.

We shall denote by $A_\emptyset$ the subgroup of all diagonal matrices in G. This is a maximal split (over F) torus in G (it is also a maximal torus in G).

2.7. We shall denote by K_0 a maximal compact subgroup of G.

If F is non-archimedean, let
$$\mathcal{O}_F = \{x \in F; |x|_F \le 1\},$$
$$\mathfrak{p}_F = \{x \in F; |x|_F < 1\}.$$

In the non-archimedean case one can take

$$K_0 = GL(n, \mathcal{O}_F) \cap G.$$

For $F = \mathbb{R}$ (resp. $F = \mathbb{C}$), one can define K_0 in a similar way as above, taking the group $O(n)$ (resp. $U(n)$) of orthogonal matrices in $GL(n, \mathbb{R})$ (resp. unitary matrices in $GL(n, \mathbb{C})$) instead of $GL(n, \mathcal{O}_F)$.

It is important to note that K_0 is an open subgroup if F is non-archimedean field, which is not the case (in general) in the archimedean case.

3. K_0-FINITE VECTORS

3.1. Let $(\pi, H) \in \hat{G}$. For $\tau \in \hat{K}_0$ denote by $m(\tau : \pi)$ the multiplicity of τ in π.

The basic property of K_0 is that it is a large subgroup of G (this was proved by Harish-Chandra in the archimedean case, and by J. Bernstein in the non-archimedean case). It means the that the function

$$\pi \mapsto m(\tau : \pi)$$

is a bounded function on $\hat{G}$, for any fixed $\tau \in \hat{K}_0$. This fact has a number of important consequences. Among others, it enables algebraization of the problem of determining of the unitary dual $\hat{G}$ of G.

3.2. Let $(\pi, H) \in \hat{G}$. Denote by H^∞ the set of all vectors $v \in H$ such that

$$\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{C}} \pi(K_0)v < \infty.$$

Then H^∞ is a dense K_0-invariant vector subspace of H^∞.

Suppose that F is non-archimedean. Since for each $g \in G$ the group

$$gK_0g^{-1} \cap K_0$$

has finite index in K_0, H^∞ is G-invariant. It follows easily that the following property holds for H^∞:

> For any $v \in H^\infty$ there exists an open subgroup K of G such that $\pi(k)v = v$ for any $k \in K$.

This follows from the fact that each continuous representation of K_0 is trivial on an open subgroup (since open subgroups in K_0 form a basis of neighborhoods of identity, and $GL(n, \mathbb{C})$ does not contain small (non-trivial) subgroups).

3.3. In the archimedean case, $gK_0g^{-1} \cap K_0$ is not (in general) of finite index in K_0. Because of this, H^∞ is not (in general) G-invariant. But then one can prove that it is invariant for the natural action of Lie algebra $\mathfrak{g}$ of G. Moreover, the action of $\mathfrak{g}$ and K_0 satisfies a natural condition. Such a structure is called $(\mathfrak{g}, K_0)$-module.

3.4. At this point usually archimedean and non-archimedean theory continue to develop separately. In the sequel, we shall more discuss the non-archimedean theory, but a number of topics hold for both theories (these will be examples of Lefschetz principle). We shall usually comment the results which hold in both theories.

4. Smooth representations

We shall assume in the sequel that F is a local non-archimedean field (if it is not otherwise specified).

4.1. A representation (π, V) of G is called **smooth** if it satisfies the following condition:

> For any $v \in V$ there exists an open subgroup K of G such that $\pi(k)v = v$ for any $k \in K$.

Denote by

$$\tilde{G}$$

the set of all equivalence classes of irreducible smooth representations of G. This set is called **non-unitary dual of** G, or **admissible dual of** G.

4.2. The mapping

$$(\pi, H) \mapsto (\pi^\infty, H^\infty); \qquad \hat{G} \to \tilde{G}$$

is injective (here π^∞ denotes the restriction of π to H^∞). Therefore, the unitary dual can be identified with a subset of $\tilde{G}$. We shall assume this identification in further. It can be shown that in this way the unitary dual is identified with the subset of all $(\pi, V) \in \tilde{G}$ such that on V there exists an inner product which is invariant for the action of G.

The problem of classification of $\tilde{G}$ has appeared much more manageable then the problem of classification of $\hat{G}$.

4.3. The problem of classification of unitary dual of G now breaks into two parts:

> problem of classification of $\tilde{G}$, which is called the **problem of non-unitary dual**;

> problem of determining the subset $\hat{G}$ of $\tilde{G}$ (in other words, the problem of identifying unitarizable classes in $\tilde{G}$), which is called the **unitarizability problem**.

We shall discuss both problems in these lectures.

4.4. Regarding the problem of non-unitary duals, let us note that there is Langlands classification of non-unitary duals, which reduces the problem of classification of non-unitary duals to the problem of classification of a special kind of irreducible representations of Levi subgroups, namely to the problem of classification of tempered representations, which will be introduced later. In the moment, let us just note that these tempered representations are unitarizable. The problem of classifying of irreducible tempered representations is very far from being easy.

Before we describe Langlands classification, we shall recall of a more simple (and less precise) reduction of the non-unitary duals.

We shall need to have a tool by which we shall be able to produce new representations. This tool is provided by parabolic induction, a construction which generalizes in a natural way induction studied already by Schur and Frobenius in the case of finite groups. Further, we shall need a tool for analyzing induced representations. Jacquet modules will be of great help for this.

5. Parabolically induced representations

5.1. Smooth representations of G and intertwinings form an Abelian category, which will be denoted by $Alg(G)$.

Let (π, V) be a representation of G. Denote

$$V^\infty = \{v \in V; \text{ there exists an open subgroup } K \text{ such that } \pi(k)v = v \text{ for } k \in K\}.$$

The space V^∞ is a G-subrepresentation of G, and it is called the **smooth part** of V. For a compact subgroup K of G let

$$V^K = \{v \in V; \pi(k)v = v \text{ for any } k \in K\}.$$

This vector space is called the space of K-**invariants** of V. Further,

$$(\pi, V) \mapsto V^K$$

is an exact functor on the category $Alg(G)$.

5.2. If (π, V) is a smooth representation of G, then there is a natural representation π' on the space of all linear forms V' on V defined by $(\pi'(g)v')(v) = v'(\pi(g^{-1})v)$. The smooth part of this representation is called the **contragredient** of (π, V). This representation is denoted by

$$(\tilde{\pi}, \tilde{V})$$

(recall $(\tilde{\pi}(g)\tilde{v})(v) = \tilde{v}(\pi(g^{-1})v)$). Then the mapping

$$(v, \tilde{v}) \mapsto < v, \tilde{v} >=: \tilde{v}(v), \qquad V \times \tilde{V} \to \mathbb{C}$$

is called canonical bilinear form. This form is G-invariant.

A function

$$g \mapsto < \pi(g)v, \tilde{v} >$$

is called a **matrix coefficient** of π.

Further,

$$(\pi, V) \mapsto (\tilde{\pi}, \tilde{V})$$

extends to a contravariant functor in a natural way. This functor is exact.

For a representation (π, V) of G, the representation on the complex conjugate vector space $\bar{V}$ of V will be denoted by

$$(\bar{\pi}, \bar{V}).$$

A smooth representation (π, V) will be called **Hermitian** if

$$(\pi, V) \cong (\bar{\tilde{\pi}}, \bar{\tilde{V}}).$$

5.3. A smooth representation (π, V) of G is called **admissible** if

$$\dim_{\mathbb{C}} V^K < \infty$$

for any open compact subgroup K of G.

For a smooth representation (π, V) of G we have always a natural intertwining of V into $\tilde{\tilde{V}}$. If the representation is admissible, then this is an isomorphism. The converse also holds, i.e. if V and $\tilde{\tilde{V}}$ are isomorphic, then (π, V) is admissible.

It is easy to show that each unitarizable admissible representation of G is Hermitian.

5.4. We shall fix the group G of rational points of a connected reductive group defined over a local non-archimedean field F. One of the main examples for us are general linear groups and classical groups. We shall fix a maximal split torus $A_{\emptyset}$ in G and a minimal parabolic subgroup $P_{\emptyset}$ of G which contains $A_{\emptyset}$. Standard parabolic subgroups of G are subgroups of G which contain $P_{\emptyset}$. For a standard parabolic subgroup P of G, a Levi decomposition of P into semi direct product of a reductive subgroup M and a normal unipotent subgroup N will be called standard if $A_{\emptyset} \subseteq M$. For standard parabolic subgroups we shall always assume that Levi decompositions are standard.

Parabolic subgroups and their Levi decompositions one gets from standard parabolic subgroups and their standard decompositions by conjugation with elements of G.

We shall fix a maximal compact subgroup K_0 of G for which Iwasawa decomposition

$$G = P_{\emptyset} K_0$$

holds (such a maximal compact subgroup always exists).

5.5. Let for a moment $\mathcal{G}$ be a locally compact group. Then there always exists a positive measure which is invariant for right translations. Such a measure will be denoted by dg. Right invariance means that

$$\int_{\mathcal{G}} f(gx)\, dg \;=\; \int_{\mathcal{G}} f(g)\, dg$$

for any continuous compactly supported function f on $\mathcal{G}$ and any $x \in \mathcal{G}$. This measure is unique up to a multiplication by a constant, and it is called a **right Haar measure** on $\mathcal{G}$.

A right Haar measure does not need to be left invariant (if it is, then the group is called **unimodular**; reductive groups are unimodular), but there exists a character $\Delta_{\mathcal{G}}$ of $\mathcal{G}$ (which is called the **modular function** or **modular character** of $\mathcal{G}$), such that holds

$$\int_{\mathcal{G}} f(xg)\, dg \;=\; \Delta_{\mathcal{G}}(x)^{-1} \int_{\mathcal{G}} f(g)\, dg$$

holds for any f and x as above.

5.6. Let us return back to the case of a connected reductive group G over a non-archimedean field F. Fix a parabolic subgroup P of G with a Levi decomposition $P = MN$

(more precisely, the group of rational points). Let (σ, U) be a smooth representation of M. Denote by $\underline{\mathrm{Ind}}_P^G(\sigma)$ the space of all functions $f \colon G \to U$ which satisfy

$$f(nmg) = \Delta_P(m)^{1/2}\sigma(m)f(g)$$

for each $m \in M$, $n \in N$, $g \in G$. Then G acts on $\underline{\mathrm{Ind}}_P^G(\sigma)$ by right translations $(R_g f)(x) = f(xg)$, $x, g \in G$. The smooth part of the representation $\underline{\mathrm{Ind}}_P^G(\sigma)$ is denoted by

$$\mathrm{Ind}_P^G(\sigma)$$

and called a **parabolically induced representation** of G from P by σ.

Parabolic induction becomes in an obvious way a functor from $Alg(M)$ into $Alg(G)$. The functor of parabolic induction is exact.

5.7. If σ is unitarizable, then $\mathrm{Ind}_P^G(\sigma)$ is also unitarizable. The inner product $(\ ,\ )$ on $\mathrm{Ind}_P^G(\sigma)$ is given by

$$(f_1, f_2) = \int_{K_0} (f_1(k), f_2(k))dk.$$

Further,

$$\mathrm{Ind}_P^G(\sigma)\tilde{\ } \cong \mathrm{Ind}_P^G(\tilde{\sigma})$$

The canonical bilinear form is given by the same formula as the above inner product:

$$< f_1, f_2 >= \int_{K_0} < f_1(k), f_2(k) > dk.$$

5.8. Suppose that $P = MN$ is a standard parabolic subgroup of G and $P' = M'N'$ another standard parabolic subgroup of G (the above decompositions are considered to be standard Levi decompositions). Let

$$P \subseteq P'.$$

Then

$$\mathrm{Ind}_P^G(\sigma) \cong \mathrm{Ind}_{P'}^G(\mathrm{Ind}_{P \cap M'}^{M'}(\sigma)).$$

This fact is called **induction by stages** (which gives the same result as the original, direct parabolic induction). It is easy to prove it (one writes an explicit isomorphism).

5.9. Iwasawa decomposition implies that $\mathrm{Ind}_P^G(\sigma)$ is an admissible representation if σ is admissible. It is less obvious to prove that if σ is a representation of finite length, then $\mathrm{Ind}_P^G(\sigma)$ is also a representation of finite length (of G).

5.10. Suppose that we have a parabolic subgroup P with Levi decompositions $P = MN$ and $P = M'N'$, which do not need to be the standard one (in the case that really interests us, at least one Levi decomposition should not be the standard one). Suppose

$$M = M'.$$

Let σ be a smooth finite length representation of M. Then

$$\mathrm{Ind}_P^G(\sigma) \text{ and } \mathrm{Ind}_{P'}^G(\sigma) \text{ have the same Jordan-Hölder series.}$$

This is an important fact, called **induction from associate parabolic subgroups**. It is not quite simple to prove it. It relies on the theory of characters. Since we shall not introduce characters in these notes, we shall not comment the proof here.

6. Jacquet modules

In this section we shall introduce a functor which is left adjoint to the functor of parabolic induction.

6.1. Suppose that (π, V) is a smooth representation of G and let $P = MN$ be a parabolic subgroup of G (actually, it is enough to assume that (π, V) is a smooth representation of P only). Let

$$V(N) = \operatorname{span}_{\mathbb{C}} \{\pi(n)v - v; n \in N, v \in V\}.$$

Since N is normal in P, $V(N)$ is P-invariant. In particular, it is M-invariant. We have a natural quotient action of M on

$$r_M^G(V) = V/V(N).$$

We shall consider the action of M on $r_M^G(V)$ which is the quotient action of the action of M (through π) on V, twisted with $\Delta_P^{-1/2}$. This action will be denoted by

$$r_M^G(\pi).$$

The representation

$$(r_M^G(\pi), r_M^G(V))$$

is called the **Jacquet module** of (π, V) with respect to $P = MN$. One defines in a natural way **Jacquet functor** from $Alg(G)$ into $Alg(M)$. Jacquet functor is exact.

6.2 If $P = MN$ and $P' = M'N'$ are standard parabolic subgroup, with standard Levi decompositions, such that

$$P \subseteq P',$$

then

$$r_{M'}^G(r_M^{M'}(\pi)) \cong r_M^G(\pi).$$

This fact will be called **transitivity of Jacquet modules**.

6.3 The fact that Jacquet functor is left adjoint to the functor of parabolic induction means that we have a natural isomorphism

$$\operatorname{Hom}_G\left(\pi, \operatorname{Ind}_P^G(\sigma)\right) \cong \operatorname{Hom}_M\left(r_M^G(\pi), \sigma\right).$$

The above isomorphism is called **Frobenius reciprocity**. One constructs this isomorphism using evaluation of $f \in \operatorname{Hom}_G\left(\pi, \operatorname{Ind}_P^G(\sigma)\right)$ at 1.

6.4 A smooth irreducible representation (π, V) of G is called **cuspidal (or supercuspidal)** if all the Jacquet modules for proper parabolic subgroups are trivial modules. It is natural to distinguish these representations, as will become clear very soon. Actually, in the definition of cuspidal representations, it is enough to require triviality Jacquet modules only of proper standard parabolic subgroups.

Cuspidal representations are very special representation, as we shall see later.

6.5. One easily sees that if (π, V) is a finitely generated smooth representation of G, then $r_M^G(\pi)$ is finitely generated representation of M. From this follows that it has an irreducible quotient.

Let (π, V) be an irreducible smooth representation of G. Among all the parabolic subgroups $P = MN$ which satisfy $r_M^G(\pi) \neq \{0\}$, chose a minimal one. Then by the above observation, $r_M^G(\pi)$ has an irreducible quotient, say σ. Minimality of P and transitivity of Jacquet modules (together with exactness) imply that σ is cuspidal. Now Frobenius reciprocity implies that π embeds into $\mathrm{Ind}_P^G(\sigma)$. In this way we have obtained a simple but important

Theorem. *An irreducible smooth representation π of G is cuspidal, or there exists a proper parabolic subgroup $P = MN$ of G and an irreducible cuspidal representation σ of M such that π is isomorphic to a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$.*

In this way the problem of classification of non-unitary dual $\tilde{G}$ breaks into two problems. One problem is to classify cuspidal representations of Levi factors, and the other one is to classify irreducible subrepresentations of representations parabolically induced by cuspidal representations.

From the above theorem it is not clear at all how to classify irreducible subrepresentations of representations parabolically induced by cuspidal representations. Langlands classification will provide a strategy for it. There is another way to describe irreducible cuspidal representations. They can be characterized as those representations which never show us as subquotients of representations parabolically induced from proper parabolic subgroups.

6.6. Remarks: (i) Let G be $GL(2, F)$ and π be an irreducible representation of G which is not cuspidal. Then Schur lemma and the above theorem imply that π is isomorphic to a subrepresentation of $\mathrm{Ind}_{P_\emptyset}^G(\chi)$, for a character χ of the Levi factor $M_\emptyset$ of $P_\emptyset$ (note that $M_\emptyset$ is commutative).

(ii) There is an archimedean version of the above theorem (see [CsMi]).

6.7. One important property of cuspidal representations is that their matrix coefficients are functions which are compactly supported modulo center. W. Casselman has shown that this property characterizes cuspidal representations.

H. Jacquet has proved that each cuspidal representation is admissible (a nice argument for this can be found in [Sav]). Now 5.9 and Theorem 6.5 imply that each irreducible smooth representation is admissible. This is the reason that $\tilde{G}$ is also called admissible dual.

6.8. Let us note that Jacquet functor carries admissible representations to admissible ones (this is not quite easy to prove). Further, it carries smooth representations of finite length of G to smooth representations of finite length of M. For proofs one can consult [Cs].

7. Filtrations of Jacquet modules

Jacquet modules are very important in analysis if admissible representations, in particular of the induced ones. In general, it is not easy to determine their structure. There is **geometric lemma**, obtained independently by J. Bernstein and A. V. Zelevinsky, and by W. Casselman, which describes filtrations of Jacquet modules of $\mathrm{Ind}_P^G(\sigma)$ in terms of representations parabolically induced by Jacquet modules of σ. We shall illustrate this lemma on the example of $G = GL(2, F)$. Later we shall describe how one can realize the geometric lemma as a part of an algebraic structure in the case of general linear and classical groups.

In this section we assume that

$$G = GL(2, F)$$

7.1. For $G = GL(2, F)$ we have $P_\emptyset = P_{(1,1)}^{GL}$. Irreducible smooth representations of $M_{(1,1)}^{GL}$ are one dimensional, i.e. characters. Since $M_{(1,1)}^{GL}$ is naturally isomorphic to $F^\times \times F^\times$, each irreducible smooth representations of $M_{(1,1)}^{GL}$ can be written as

$$\chi_1 \otimes \chi_2 \,,$$

where χ_1 and χ_2 are characters of $F^\times$.

We shall consider

$$\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2).$$

Denote

$$w_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

It is not hard to show that the following obvious sequence

$$(7\text{-}1\text{-}1) \quad 0 \to \{f \in \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2); \mathrm{supp}(f) \subseteq P_\emptyset w_0 P_\emptyset\} \hookrightarrow \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$$

$$\xrightarrow{\text{restriction}} \{f|P_\emptyset; f \in \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)\} \to 0.$$

is exact. Further, considering the mapping $f \mapsto f(1)$, one gets easily that

$$r_{M_\emptyset}^G(\{f|P_\emptyset; f \in \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)\}) \cong \chi_1 \otimes \chi_2.$$

It requires a little bit more efforts to show that

$$r_{M_\emptyset}^G(\{f \in \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2); \mathrm{supp}(f) \subseteq P_\emptyset w_0 P_\emptyset\}) \cong \chi_2 \otimes \chi_1.$$

7.2. Applying Jacquet functor to the exact sequence (7-1-1) (recall that the Jacquet functor is exact), we get the following exact sequence

$$0 \to \chi_2 \otimes \chi_1 \to r_{M_\emptyset}^G(\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)) \to \chi_2 \otimes \chi_1 \to 0.$$

As a consequence of this exact sequence, we can conclude that $\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$ has at most length 2.

8. Square integrable and tempered representations

8.1. Let (π, V) be an irreducible smooth representation of G. Then Schur lemma implies that the center $Z(G)$ of G acts by scalars. Corresponding character will be denoted by

$$\omega_\pi$$

and called the **central character of** π.

A smooth representation does not need to be irreducible, but the center can act by scalars. Then we shall say that the representation has a central character.

8.2. An admissible representation (π, V) of G will be called **square integrable** (more precisely, **square integrable modulo center**) if it has central character, if the central character is unitary, and if absolute values of all the matrix coefficients

$$g \mapsto | < \pi(g)v, \tilde{v} > |, \quad v \in V, \tilde{v} \in \tilde{V},$$

are square integrable functions modulo center (i.e. square integrable functions on $G/Z(G)$).

In this notes we shall consider only irreducible square integrable representations.

An admissible representation (π, V) of G will be called **essentially square integrable** (or **essentially square integrable modulo center**) if there exists a character χ of G such that $\chi\pi$ is square integrable.

8.3. Each (irreducible) square integrable representation (π, V) of G is unitarizable. To see this, take $\tilde{v}_o \in \tilde{V}$ different from 0. Now for $u, v \in V$ set

$$(u, v) = \int_{G/Z(G)} \tilde{v}_o(\pi(g)u)\overline{\tilde{v}_o(\pi(g)v)}dg.$$

One sees directly that this is a G-invariant inner product on V (if π is not irreducible, one proceeds similarly; see [Wd]).

8.4. Irreducible square integrable representations are very important. First, they are (very distinguish) elements of the unitary dual. Then, via Langlands classification, they are crucial in the parameterization of non-unitary duals. Further, using matrix coefficients one gets that they are (irreducible) subrepresentations of $L^2(G)$ if G has compact center (in the non-compact case, we have similar situation when one fixes central character). Therefore, they are very important for understanding decomposition of $L^2(G)$.

8.5. An irreducible smooth (which implies admissible) representation (π, V) of G is called **tempered** if there exists a parabolic subgroup $P = MN$ of G (not necessarily proper) and an irreducible square integrable representation δ of M such that π is isomorphic to a subrepresentation of

$$\text{Ind}_P^G(\delta).$$

An irreducible smooth representation (π, V) of G is called **essentially tempered** if there exists a character χ of G such that $\chi\pi$ is tempered.

One usually defines tempered representations without irreducibility requirement, but since we shall work only with irreducible tempered representations in these notes, the above definition is not restrictive for us.

8.6. We shall now introduce notation for general linear groups. The character

$$g \mapsto |\det(g)|_F, \quad GL(n, F) \to \mathbb{R}^\times$$

will be denoted by

$$\nu.$$

If τ is an (irreducible) essentially tempered representation of $GL(n, F)$, then one easily sees there exists a unique

$$e(\tau) \in \mathbb{R}$$

and a unique tempered representation τ^u such that

$$\tau = \nu^{e(\tau)}\tau^u.$$

This requirement uniquely defines $e(\tau)$.

8.7. There are a very useful criteria of Casselman for checking if an irreducible admissible representation is square irreducible or tempered. We shall explain this criterion on the case of $G = GL(2, F)$.

Let (π, V) be an irreducible essentially square integrable (resp. essentially tempered) representation of $GL(2, F)$. Then for any irreducible subquotient $\chi = \chi_1 \otimes \chi_2$ of $r_{P_\emptyset}^{GL(2,F)}(\pi)$ (χ_1 and χ_2 are characters of $F^\times$) we have

$$e(\chi_1) > e(\chi_2) \qquad (\text{resp.} e(\chi_1) \geq e(\chi_2)).$$

Moreover, the converse also holds for an irreducible admissible representation (π, V).

9. LANGLANDS CLASSIFICATION

9.1. Langlands classification parameterizes representations of $\tilde{G}$ by (irreducible) essentially tempered representations of Levi factors of standard parabolic subgroups. These essentially tempered representations need to satisfy certain positiveness condition (which will be discussed later).

Langlands classification claims the following: For an irreducible essentially tempered representation τ of Levi factor M of standard parabolic subgroup P of G, which satisfy the above mentioned positiveness condition, the representation $\text{Ind}_P^G(\tau)$ has a unique irreducible quotient. This irreducible quotient is called the **Langlands quotient** ($\text{Ind}_P^G(\tau)$ is called a **standard module** of G). Each $\pi \in \tilde{G}$ is isomorphic to some Langlands quotient,

and moreover π determines uniquely the standard parabolic subgroup P and essentially tempered representation τ.

We shall now explain the positiveness condition for the groups that we consider in these notes. We shall start with general linear groups.

9.2. Each Levi factor of a general linear group is a direct product of general linear groups. Because of this, each irreducible essentially tempered representation of a Levi factor of a general linear group is a tensor product of such representations of general linear groups. Therefore, the essentially tempered representations of a Levi factors of a general linear groups are of the form

$$\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_k,$$

where $\tau_1, \tau_2, \ldots, \tau_k$ are essentially tempered representations of general linear groups. The positiveness condition here is simply

$$e(\tau_1) > e(\tau_2) > \cdots > e(\tau_k).$$

9.3. We shall see how one gets the Langlands parameterization in the case of the simplest possible example, in the case of $G = GL(2, F)$. Let $\pi \in \tilde{G}$. If π is essentially tempered, then it is its own Langlands parameter. Suppose therefore that π is not essentially tempered. Then, in particular, it is not cuspidal. Therefore

$$\tilde{\pi} \quad \text{is a subquotient of} \quad \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$$

by Theorem 6.5, for some characters χ_1 and χ_2 of $F^\times$ (see also Remarks 6.6, (i)). Since π is not essentially tempered, $\tilde{\pi}$ is also not essentially tempered. Now by 8.7

$$e(\chi_1) \neq e(\chi_2).$$

Therefore by 7.2

(9-3-1) $$r_{P_\emptyset}^G(\tilde{\pi}) \hookrightarrow \chi_1 \otimes \chi_2 \ \oplus \ \chi_2 \otimes \chi_1.$$

Without lost of generality we can assume

$$e(\chi_1) > e(\chi_2),$$

since $\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$ and $\mathrm{Ind}_{P_\emptyset}^G(\chi_2 \otimes \chi_1)$ have the same Jordan-Hölder series (one sees this using induction from associate parabolic subgroups; see 5.10). Since $\tilde{\pi}$ is not essentially square integrable (recall that π it is not essentially tempered), from criterion for essentially square integrability follows that $\chi_2 \otimes \chi_1$ must be a subquotient of $r_{P_\emptyset}^G(\tilde{\pi})$. Now from (9-3-1) we see that there exits a non-trivial homomorphism

$$r_{P_\emptyset}^G(\tilde{\pi}) \longrightarrow \chi_2 \otimes \chi_1.$$

Frobenius reciprocity implies

$$\tilde{\pi} \hookrightarrow \mathrm{Ind}_{P_{\emptyset}}^{G}(\chi_2 \otimes \chi_1).$$

Passing to contragredients we get an epimorphism

$$\mathrm{Ind}_{P_{\emptyset}}^{G}(\chi_2^{-1} \otimes \chi_1^{-1}) \to \pi.$$

Note that

$$e(\chi_2^{-1}) = -e(\chi_2) > -e(\chi_1) = e(\chi_1^{-1}).$$

This implies that we have shown the existence of Langlands parameters for irreducible representations of $GL(2, F)$. Their uniqueness follow from the filtration of Jacquet modules (see 7.2).

Thus, we have "proved" Langlands classification for $GL(2, F)$. This case is too simple to illustrate the proof of the Langlands classification in general, but one can get at least some idea from this simplest case how proof goes in general. In any case, we see the importance of the Geometric lemma.

9.4. Since we have defined tempered representations by square integrable representations, it is natural to try to express Langlands classification in terms of square integrable representations, if this is possible.

In the study the representation theory of general linear groups, it is convenient to use notation that was used by Bernstein and Zelevinsky in their work on the representation theory of general linear groups. We shall now recall of (a very small part of) it.

For smooth representations π_1 and π_2 of $GL(n_1, F)$ and $GL(n_2, F)$ denote

$$\pi_1 \times \pi_2 = \mathrm{Ind}_{P^{GL}_{(n_1, n_2)}}^{GL(n_1 + n_2, F)}(\pi_1 \otimes \pi_2).$$

Then

(9-4-1) $$\pi_1 \times (\pi_2 \times \pi_3) \cong (\pi_1 \times \pi_2) \times \pi_3.$$

This follows from induction by stages. Further, for smooth representation π_1 and π_2 of finite length,

(9-4-2) $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ have the same Jordan-Hölder series.

This follows from induction from associate parabolic subgroups.

Remark: In the case of an archimedean field F, using parabolic induction we define multiplication $\times$ between $(\mathfrak{g}, K_0)$-modules of general linear groups (over F) in the same way as above. Then (9-4-1) and (9-4-2) hold also in this case.

9.5. A principal result regarding tempered induction for general linear groups is that this induction is irreducible (this fact holds for all the local fields). This fact has been proved independently at several places, but it seems that the first proof in this setting belongs to H. Jacquet, who proved it is for all the local fields (see [Jc1]).

This principal result claims the following:

> If $\pi_1, \pi_2, \ldots, \pi_k$ are (unitarizable) irreducible square integrable representations of general linear groups, then $\pi_1 \times \pi_2 \times \cdots \times \pi_k$ is irreducible.

Either from general facts regarding tempered representations, or from A. V. Zelevinsky paper [Ze1], follows that the tempered representation $\pi_1 \times \pi_2 \times \cdots \times \pi_k$ determines irreducible square integrable representations $\pi_1, \pi_2, \ldots, \pi_k$ up to a permutation.

9.5. Using 9.4 and the fact that for a character χ of $F^\times$ we have

$$[(\chi \circ \det)\pi_1] \times [(\chi \circ \det)\pi_2] \cong (\chi \circ \det)(\pi_1 \times \pi_2)$$

(which one proves directly), we can reformulate the Langlands classification for general linear groups in the following way.

Denote by D the set of all the irreducible essentially square integrable representations of $GL(n, F)$'s for all $n \geq 1$. Let $M(D)$ be the set of all finite multisets in D. These are functions from D into $\mathbb{Z}_+$ with finite supports. We shall write them similar as sets, but repetitions of elements will be allowed. We shall write them as

$$(\delta_1, \delta_2, \ldots, \delta_k), \quad \text{where} \quad \delta_i \in D.$$

Take any $d = (\delta_1, \delta_2, \ldots, \delta_k) \in M(D)$. Take a permutation p of $\{1, \ldots, k\}$ such that

$$e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \cdots \geq e(\delta_{p(k)}).$$

Now the representation

$$\delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)},$$

which will be denoted by

$$\lambda(d),$$

has a unique irreducible quotient (the representation $\lambda(d)$ is determined by $d \in M(d)$ up to an isomorphism.). This quotient will be denoted by

$$L(d).$$

In this way one gets bijection between $M(D)$ and the set of all the irreducible smooth representations of all general linear groups over F.

This is just a reformulation of the Langlands classification for general linear groups.

The Langlands classification has a number of natural properties. Let us mention three:

(9-5-1) $$L(\delta_1, \delta_2, \ldots, \delta_k)^\sim \cong L(\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_k),$$

(9-5-2) $$L(\delta_1, \delta_2, \ldots, \delta_k)^- \cong L(\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_k),$$

(9-5-3) $$\chi L(\delta_1, \delta_2, \ldots, \delta_k) \cong L(\chi\delta_1, \chi\delta_2, \ldots, \chi\delta_k)$$

(χ is a multiplicative character of the field, and further, for a representation π of $GL(n, F)$, $\chi\pi$ denotes the representation $(\chi \circ \det)\pi$).

Remark: The Langlands classification for general linear groups holds also if the field is archimedean F. In this case the non-unitary dual $GL(n, F)^\sim$ of $GL(n, F)$ is the set

of all the equivalence classes of irreducible $(\mathfrak{g}, K_0)$-modules of $GL(n, F)$. The irreducible representations (i.e. non-unitary duals) are classified by $M(D)$, where D is the set of all the equivalence classes of irreducible essentially square integrable $(\mathfrak{g}, K_0)$-modules of all $GL(n, F)$'s, $n \geq 1$ (if $F = \mathbb{C}$, then $D = (\mathbb{C})\tilde{}$, while for $F = \mathbb{R}$ we have $D \subseteq (\mathbb{R})\tilde{} \cup GL(2, \mathbb{R})\tilde{}$).

9.6. Now we shall describe the Langlands classification for symplectic and odd-orthogonal groups.

It is convenient to introduce for classical groups the following notation, which simplifies notation when one works with parabolically induced representations. Let π be a smooth representation of $GL(n, F)$ and let σ be a smooth representation of $Sp(2m, F)$ (resp. $SO(2m + 1, F)$). Denote

$$\pi \rtimes \sigma = \text{Ind}_{P_{(n)}}^{Sp(2(n+m), F)}(\pi \otimes \sigma)$$

$$\left(\text{resp. } \pi \rtimes \sigma = \text{Ind}_{P_{(n)}}^{SO(2(n+m)+1, F)}(\pi \otimes \sigma) \right).$$

From induction by stages follows

$$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma.$$

Further, for smooth representations π and σ of finite length,

(9-6-1) $\pi \rtimes \sigma$ and $\tilde{\pi} \rtimes \sigma$ have the same Jordan-Hölder series.

This follows from induction from associate parabolic subgroups.

9.7. Regarding the Langlands classification for symplectic and odd-orthogonal groups, one can first describe it in terms of essentially tempered representations, and after that pass to a description which include only essentially square integrable representations of general linear groups (and tempered representations of symplectic or odd-orthogonal groups), similarly as we did in the case of Langlands classification for general linear groups. Instead of this, we shall skip over the first description and go directly to the second description.

Set
$$D_+ = \{\delta \in D; e(\delta) > 0\}.$$

Denote by T the set of all equivalence classes of tempered representations of all $Sp(2m, F)$ (resp. $SO(2m + 1, F)$), for all $m \geq 0$. Take

$$((\delta_1, \delta_2, \ldots, \delta_k), \tau) \in M(D_+) \times T.$$

After a renumeration, we can assume

$$\delta_1 \geq \delta_2 \geq \cdots \geq \delta_k$$

Then the representation

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$$

has a unique irreducible quotient, which will be denoted by

$$L(\delta_1, \delta_2, \ldots, \delta_k; \tau).$$

Now the mapping

$$((\delta_1, \delta_2, \ldots, \delta_k), \tau) \mapsto L(\delta_1, \delta_2, \ldots, \delta_k; \tau)$$

defines a bijection from the set $M(D_+) \times T$ onto the set of all the equivalence classes of irreducible smooth representations of all $Sp(2m, F)$ (resp. $SO(2m + 1, F)$), $m \geq 0$.

This is the Langlands classification for (these) classical groups.

Remark: One can define $\rtimes$ also for the case of archimedean fields. The Langlands classification holds here in the same form.

10. Geometric lemma and algebraic structures

Geometric lemma, which is a technical result describing filtrations of Jacquet modules of induced representations in terms of representations induced by Jacquet modules of inducing representations, is very important for number of purposes. For general linear and classical groups we can "incorporate" it into an algebraic structures on the representations of these groups.

10.1. Let for a moment G be a reductive group over a non-archimedean field F. The Grothendieck group of the category of all smooth G-representations of finite length will be denoted by

$$\mathcal{R}(G).$$

This is just a free $\mathbb{Z}$-module over basis $\tilde{G}$ (it is isomorphic to the group of virtual characters of G). For a finite length representation π, let

$$\text{s.s.}(\pi) = \sum_{\tau \in \tilde{G}} m(\tau : \pi)\tau.$$

This is called semi simplification of π. We consider it as an element of $\mathcal{R}(G)$. There is a natural order on $\mathcal{R}(G)$.

We have

$$(10\text{-}1\text{-}1) \qquad\qquad \mathcal{R}(G_1 \times G_2) \cong \mathcal{R}(G_1) \otimes \mathcal{R}(G_2).$$

Further, r_M^G factors in a natural way to a homomorphism from $\mathcal{R}(G)$ into $\mathcal{R}(M)$, which is denoted again by r_M^G. This is a homomorphism of ordered groups (i.e. it respects also orders).

We have analogous definition for the parabolic induction: $\text{Ind}_P^G : \mathcal{R}(M) \to \mathcal{R}(G)$, which is again a morphism of ordered groups.

10.2. Set

$$R = \underset{n \in \mathbb{Z}_+}{\oplus} \mathcal{R}(GL(n, F)).$$

Then one lifts $\times$ to a multiplication on R in a natural way. The mapping $\times : R \times R \to R$ factors in a natural way through a mapping

$$m : R \otimes R \to R.$$

Let $\pi \in GL(n, F)\tilde{}$. We will consider

$$\text{s.s.} \left(r_{M_{(k,n-k)}^{GL}}^{GL(n,F)}(\pi) \right) \in R_k \otimes R_{n-k}$$

using the isomorphism (10-1-1). Define

$$m^*(\pi) = \sum_{k=0}^{n} \text{s.s.} \left(r_{M_{(k,n-k)}^{GL}}^{GL(n,F)}(\pi) \right) .$$

We can (and will) consider

$$m^*(\pi) \in R \otimes R ,$$

since each $R_k \otimes R_{n-k} \hookrightarrow R \otimes R$. We can lift m^* to an additive mapping

$$m^* : R \to R \otimes R .$$

This mapping is called comultiplication.

With the multiplication and comultiplication, R is a commutative Hopf algebra (over $\mathbb{Z}$). This algebra was constructed by A. V. Zelevinsky.

The most important part of this Hopf algebra structure is the formula

$$m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \times m^*(\pi_2),$$

which explains how to get composition factors of Jacquet modules (for maximal parabolic subgroups) of induced representations, by induction from Jacquet modules of inducing representations.

10.3. Suppose (only) in this section that F is an archimedean field. One defines $\mathcal{R}(G)$ in the same way as in the non-archimedean case, considering the category of $(\mathfrak{g}, K)$-modules of finite length (G is a connected reductive group over F). This is a free $\mathbb{Z}$-module over basis $\tilde{G}$ (it is isomorphic to the group of virtual characters of G). Now for general linear groups over F one defines R in the same way as in the non-archimedean case. By the same formula as in the non-archimedean case, one defines multiplication $\times$ on R (using parabolic induction). In this way R becomes a commutative ring with identity. In the archimedean case, there is no comodule structure on R as in the non-archimedean case.

Remark: The Kazhdan-Patterson lifting for $GL(n, \mathbb{C})$ has a very nice and natural description in terms of this algebra (see [T5]).

10.4. Assume in this section that F is any local field (archimedean or non-archimedean). For $a = (\delta_1, \ldots, \delta_k) \in M(D)$ consider

$$\text{s.s}(\lambda(a)) = \delta_1 \times \ldots \delta_k \in R.$$

Then a simple fact regarding Langlands classification implies that

$$\text{s.s}(\lambda(a)), \quad a \in M(D),$$

form a $\mathbb{Z}$-basis of R (this simple fact is usually expressed in the following form: characters of standard modules form a $\mathbb{Z}$-basis of $\mathcal{R}(G)$). In other words, for any local field F holds the following

Proposition. *The ring R is a polynomial $\mathbb{Z}$-algebra over D.*

10.5. We shall denote
$$R(S) = \bigoplus_{n \in \mathbb{Z}_+} \mathcal{R}(Sp(2n, F))$$
if we consider symplectic groups, and
$$R(S) = \bigoplus_{n \in \mathbb{Z}_+} \mathcal{R}(SO(2n+1, F))$$
if we consider odd-orthogonal groups.

In this setting, one again lifts $\rtimes$ in a natural way to a multiplication $R \times R(S) \to R(S)$. This multiplication factors through a mapping
$$\mu : R \otimes R(S) \to R(S).$$

For $\pi \in Sp(2n, F)\tilde{\ }$ (resp. $\pi \in SO(2n+1, F)\tilde{\ }$) set

$$\mu^*(\pi) = \sum_{k=0}^{n} \text{s.s.} \left(r_{M_{(k)}}^{Sp(2n,F)}(\pi) \right)$$

$$\left(\text{resp. } \mu^*(\pi) = \sum_{k=0}^{n} \text{s.s.} \left(r_{M_{(k)}}^{SO(2n+1,F)}(\pi) \right) \right).$$

Consider s.s. $\left(r_{M_{(k)}}^{Sp(2n,F)}(\pi) \right)$ $\left(\text{resp. s.s. } \left(r_{M_{(k)}}^{SO(2n+1,F)}(\pi) \right) \right)$ as an element of $R_k \otimes R_{n-k}(S)$ using (10-1-1), and consider further
$$\mu^*(\pi) \in R \otimes R(S) .$$

Lift μ^* to an additive mapping
$$\mu^* \colon R(S) \to R \otimes R(S) ,$$
which will be called comultiplication on $R(S)$.

With the above multiplication and complication, $R(S)$ is a module and a comodule over R. It is not a Hopf module over R, but is also far from this structure as we shall explain now.

Define
$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^* : R \to R \otimes R,$$
where 1 denotes the identity mapping, $\sim$ the contragredient mapping and s the transposition mapping $\sum x_i \otimes y_i \mapsto \sum y_i \otimes x_i$. Then
$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma)$$

$(R \otimes R(S)$ is a $R \otimes R$-module in an obvious way). We say that $R(S)$ is an M^*-Hopf module over R.

This is again a (combinatorial) formula from which we can again in a simple way get compositions factors of Jacquet modules of parabolically induced representations for classical group.

11. SQUARE INTEGRABLE REPRESENTATIONS OF p-ADIC GENERAL LINEAR GROUPS

11.1. Denote by

$$\mathcal{C}$$

the set of all equivalence classes of irreducible cuspidal representations of all $GL(n, F)$, $n \geq 1$.

A **segment in** $\mathcal{C}$ is the set of form

$$\Delta = [\rho, \nu^k \rho] = \{\rho, \nu\rho, \ldots, \nu^k \rho\},$$

where $\rho \in \mathcal{C}, k \in \mathbb{Z}_+$. Denote the set of all such a segments by

$$\mathcal{S}.$$

For a segment $\Delta = [\rho, \nu^k \rho] = \{\rho, \nu\rho, \ldots, \nu^k \rho\} \in \mathcal{S}$, the representation

$$\nu^k \rho \times \nu^{k-1} \rho \times \cdots \times \nu\rho \times \rho$$

contains a unique irreducible subrepresentation, which will be denoted by

$$\delta(\Delta).$$

Then $\delta(\Delta)$ is essentially square integrable representation, and in this way one gets a bijection from $\mathcal{S}$ onto D (which is the set of all the irreducible essentially square integrable representations of general linear groups $GL(n, F)$, $n \geq 1$). This is one of the consequences of Bernstein and Zelevinsky theory, which is based on Gelfand-Kazhdan theory of derivatives. One can obtain these results also by different methods.

In applications of square integrable representations of general linear groups, it is important to know what are Jacquet modules of these representations. This tells the following simple formula

$$(11\text{-}2\text{-}1) \qquad m^*(\delta([\rho, \nu^k \rho])) = \sum_{i=-1}^{k} \delta([\nu^{i+1}\rho, \nu^k \rho]) \otimes \delta([\rho, \nu^i \rho])$$

(see [Ze1]).

11.2. As we have seen, each segment of $\mathcal{S}$ determines uniquely essentially square integrable representation. Let us explain how to "read" corresponding segment from an essentially square integrable representation

$$\delta = \delta(\Delta), \Delta \in \mathcal{S}.$$

For this, we shall introduce two natural invariants of δ.

There exists exactly two (inequivalent) $\rho_1, \rho_2 \in \mathcal{C}$ such that $\rho_1 \times \delta$ and $\rho_2 \times \delta$ reduce. We can, after a possible renumeration, assume

$$e(\rho_1) \leq e(\rho_2).$$

Representations ρ_1 and ρ_2 will be called **cuspidal reducibilities of** δ (ρ_1 the lower one and ρ_2 the upper one).

Then

$$\Delta = [\nu\rho_1, \nu^{-1}\rho_2].$$

Thus cuspidal reducibilities of δ determine completely the segment in $\mathcal{C}$ corresponding to δ.

11.3. Let $\delta_1 \in D$ have cuspidal reducibilities ρ_1, and ρ_2. Take $\delta_2 = \delta(\Delta_2) \in D$. Then

$$\delta_1 \times \delta_2$$

reduces if and only if

(1) $\mathrm{card}(\{\rho_1, \rho_2\} \cap \Delta_2) = 1$;
(2) neither ρ_1 nor ρ_2 is a cuspidal reducibility of $\delta(\Delta_2)$.

12. Two simple examples of square integrable representations of classical p-adic groups

In the rest of these notes, we shall fix one of the series of classical groups, symplectic or odd-orthogonal, and denote by S_n either $Sp(2n, F)$ or $SO(2n + 1, F)$.

Before we proceed further with description of general square integrable representations, we shall give two examples of square integrable representations of classical groups.

The trivial one-dimensional representation of a group G will be denoted by 1_G.

12.1. Example: In this example we shall describe Steinberg representations for symplectic groups. Steinberg representation can be constructed for any reductive group.

Here we consider the series $S_n = Sp(2n, F)$.

An easy computation of modular character of $P_\emptyset$ in $S_1 = Sp(2, F) = SL(2, F)$ implies that

$$1_{S_1} \hookrightarrow \nu^{-1}\, 1_{F^\times} \rtimes 1_{S_0},$$

since $\nu^{-1}1_{F^\times} \rtimes 1_{S_0}$ contains constant functions. The length of the Jacquet module of $\nu^{-1}\, 1_{F^\times} \rtimes 1_{S_0}$ for the (standard) minimal parabolic subgroup is 2 (and irreducible subquotients are not isomorphic). This implies that $\nu^{-1}\, 1_{F^\times} \rtimes 1_{S_0}$ is a length two representation. Further, Frobenius reciprocity implies that $\nu^{-1}\, 1_{F^\times} \rtimes 1_{S_0}$ is not completely reducible (i.e. it is not a sum of irreducible subrepresentations). Now passing to contragredients we see that

$$\nu\, 1_{F^\times} \rtimes 1_{S_0}$$

contains a unique irreducible subrepresentation. This representation will be denoted by St_{S_1}, and called Steinberg representation of S_1. We can see (from the algebraic structure of

$R(S)$ over R) that the Jacquet module for the minimal parabolic subgroup is $\nu\, 1_{F^\times} \otimes 1_{S_0}$. Now Casselman's square integrability criterion in this situation implies that St_{S_1} is square integrable.

Define St_{S_0} to be 1_{S_0}. Now both

$$\nu^2\, 1_{F^\times} \rtimes \mathrm{St}_{S_1} \quad \text{and} \quad \delta([\nu\, 1_{F^\times}, \nu^2\, 1_{F^\times}]) \rtimes \mathrm{St}_{S_0}$$

embed into

$$\nu^2\, 1_{F^\times} \times \nu\, 1_{F^\times} \rtimes \mathrm{St}_{S_0}.$$

Analyzing Jacquet modules, we would see that these two subrepresentations have exactly one irreducible subquotient in common, and that this subquotient is square integrable. We shall denote it by St_{S_2}. It is a unique irreducible subrepresentation of $\nu^2\, 1_{F^\times} \times \nu\, 1_{F^\times} \rtimes \mathrm{St}_{S_0}$.

Continuing recursively in the above way, we define the **Steinberg representation**

$$\mathrm{St}_{Sp_n}$$

for any S_n. It is a unique irreducible subrepresentations of

$$\nu^n\, 1_{F^\times} \times \nu^{n-1}\, 1_{F^\times} \times \cdots \times \nu^2\, 1_{F^\times} \times \nu\, 1_{F^\times} \rtimes \mathrm{St}_{S_0}$$

(this defines St_{Sp_n}). It is again easy to write what are the Jacquet modules of these representations:

$$\mu^*(\mathrm{St}_{S_n}) = \sum_{k=0}^{n} \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \mathrm{St}_{S_k}.$$

12.2. Example: Let we now consider the series $S_n = SO(2n+1, F)$.

An easy computation of modular character of $P_\emptyset$ in $S_1 = SO(3, F)$ implies that

$$1_{S_1} \hookrightarrow \nu^{-1/2}\, 1_{F^\times} \rtimes 1_{S_0}$$

(since $\nu^{-1/2}\, 1_{F^\times} \rtimes 1_{S_0}$ contains constant functions).

Consider the representation

$$\nu^{1/2}\, 1_{F^\times} \times \nu^{-1/2}\, 1_{F^\times} \rtimes 1_{S_0}.$$

Here

$$\delta([\nu^{-1/2}\, 1_{F^\times}, \nu^{1/2}\, 1_{F^\times}]) \rtimes 1_{S_0} \quad \text{and} \quad \nu^{1/2}\, 1_{F^\times} \rtimes 1_{S_1}$$

are subrepresentations. Looking at Jacquet modules, one sees that

(12-2-1) $$\delta([\nu^{-1/2}\, 1_{F^\times}, \nu^{1/2}\, 1_{F^\times}]) \rtimes 1_{S_0}$$

reduces. This follows from

$$\mathrm{s.s.}\left(\delta([\nu^{-1/2}\, 1_{F^\times}, \nu^{1/2}\, 1_{F^\times}]) \rtimes 1_{S_0}\right) + \mathrm{s.s.}\left(\nu^{1/2}\, 1_{F^\times} \rtimes 1_{S_1}\right)$$

$$\not\leq \mathrm{s.s.}\left(\nu^{1/2}\, 1_{F^\times} \times \nu^{-1/2}\, 1_{F^\times} \rtimes 1_{S_0}\right)$$

and

$$\text{s.s.} \left(\delta([\nu^{-1/2} \, 1_{F^\times}, \nu^{1/2} \, 1_{F^\times}]) \rtimes 1_{S_0} \right) \not\leq \text{s.s.} \left(\nu^{1/2} \, 1_{F^\times} \rtimes 1_{S_1} \right),$$

what one checks using the structure of $R(S)$.

The representation (12-2-1) is unitarizable, so it is completely reducible. Considering the Jacquet module of this representation for $P_{(2)}$, and applying Frobenius reciprocity, we get that the representation $\delta([\nu^{-1/2} \, 1_{F^\times}, \nu^{1/2} \, 1_{F^\times}]) \rtimes 1_{S_0}$ reduces into a sum of two inequivalent irreducible subrepresentations, say T_1 and T_2. Thus

$$\delta([\nu^{-1/2} \, 1_{F^\times}, \nu^{1/2} \, 1_{F^\times}]) \rtimes 1_{S_0} = T_1 \oplus T_2.$$

Now

$$\delta([\nu^{-1/2} \, 1_{F^\times}, \nu^{3/2} \, 1_{F^\times}]) \rtimes 1_{S_0}$$
$$\hookrightarrow \nu^{3/2} \, 1_{F^\times} \times \delta([\nu^{-1/2} \, 1_{F^\times}, \nu^{1/2} \, 1_{F^\times}]) \rtimes 1_{S_0}$$
$$= \nu^{3/2} \, 1_{F^\times} \times (T_1 \oplus T_2)$$
$$\cong \nu^{3/2} \, 1_{F^\times} \rtimes T_1 \oplus \nu^{3/2} \, 1_{F^\times} \rtimes T_2.$$

Now the multiplicity of $\nu^{3/2} \, 1_{F^\times} \otimes T_i$ in corresponding Jacquet module of $\nu^{3/2} \, 1_{F^\times} \rtimes T_i$ is one. This implies that $\nu^{3/2} \, 1_{F^\times} \rtimes T_i$ has a unique irreducible subrepresentation. These two irreducible subrepresentations (for $i = 1, 2$) are square integrable.

One can show that $\nu^{3/2} \, 1_{F^\times} \rtimes T_i$ are subquotients of corresponding Jacquet module of $\delta([\nu^{-1/2} \, 1_{F^\times}, \nu^{3/2} \, 1_{F^\times}]) \rtimes 1_{S_0}$. From this we see that

$$\delta([\nu^{-1/2} \, 1_{F^\times}, \nu^{3/2} \, 1_{F^\times}]) \rtimes 1_{S_0}$$

has exactly two irreducible subrepresentations. They are inequivalent and square integrable.

Next question is how to distinguish these two irreducible square integrable subrepresentations. One can show that they have Jacquet modules of different length. This is a one possible way to distinguish them.

13. Invariants of square integrable representations of classical p-adic groups

13.1. Let π be an irreducible square integrable representation of a classical group S_q. C. Mœglin has attached to it a triple

$$(Jord(\pi), \pi_{cusp}, \epsilon_\pi).$$

Each of these three parameters was considered earlier (at least in some form), but C. Mœglin was the first who considered them in this form.

We shall describe each of these parameters. Our goal will be to explain their meaning from the point of harmonic analysis (they have a clear meaning from the point of view of Langlands program, what will be discussed later).

13.2. Jordan block of π: This is probably the most important of the three parameters (this parameter should determine the L-packet in which the representation lies). As we shall see later, the definition of $Jord(\pi)$ is very natural from the point of harmonic analysis (and can be given completely in terms of harmonic analysis).

For $\rho \in \mathcal{C}$ and $a \in \mathbb{N}$ denote

$$\delta(\rho, a) = \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]).$$

We shall start with the first definition of $Jord(\pi)$ (this is a little bit modified definition, to avoid L-functions). $Jord(\pi)$ is called the Jordan block of π and it consists of all $(\rho, a) \in \mathcal{C} \times \mathbb{N}$ such that

(1) ρ is selfdual (i.e. $\tilde{\rho} \cong \rho$; then ρ is unitarizable) and
(2) if $\nu^{1/2}\rho \rtimes 1_{S_0}$ is reducible (resp. irreducible), then a is even (resp. odd) and

$$\delta(\rho, a) \rtimes \pi$$

is irreducible

Although the above definition is simple, the clear meaning and importance of Jordan blocks is not evident from it. This is the reason that we shall give another description of Jordan blocks, from which will be much more clear importance of Jordan blocks for the harmonic analysis.

13.3. As we already have mentioned, there is a very natural way to come to Jordan blocks from the point of view of harmonic analysis, and we shall explain it now. Besides, because of the importance of Jordan blocks, they deserve to be understood better.

Before we start to explain it, let us note that the classification of irreducible square integrable representation of classical groups is done under a natural assumption, which will be explained later. This assumption shows up in proofs, not in the expression of the parameterization of irreducible square integrable representations. We shall assume that it holds in further.

Now we are going to explain the importance f $Jord(\pi)$ for harmonic analysis.

Once we have an irreducible square integrable representation π of a classical group, having in mind classification of the non-unitary dual via the Langlands classification, the first question that arises is:

Which irreducible tempered representations can be obtained from this π.

In other words, we would like to understand how a representation of the form

(13-3-1) $$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \pi$$

reduces, when δ_i are (unitarizable) irreducible square integrable representations of general linear groups.

If we would know the answer to this question, we would have a reduction of understanding of irreducible tempered representations of the classical groups (and in this way also of all the irreducible representations) to the problem of understanding of irreducible square integrable representations of the classical groups. Therefore, understanding of such a reduction would be of the first class importance.

The theory of R-groups reduces this question to the question when

$$(13\text{-}3\text{-}2) \qquad\qquad \delta \rtimes \pi$$

reduces, for δ an irreducible (unitarizable) square integrable representation of a general linear group.

Remark: For further discussion of Jordan blocks, one does not need to understand this reduction. But for the classification of the non-unitary duals, one needs it. Therefore, we shall explain the reduction that gives the theory of R-groups, without going deeper in this theory. Consider representation from (13-3-1). Denote by ℓ the number of inequivalent δ_i among $\delta_1, \delta_2, \ldots, \delta_k$ such that

$$\delta_i \rtimes \pi$$

reduces. Then $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \pi$ is a multiplicity one representation and it reduces into a direct sum of

$$2^\ell$$

irreducible (tempered) representations.

If p is a permutation of $\{1, 2, \ldots, k\}$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in \{\pm 1\}$, then representations $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \pi$ and $\delta_{p(1)}^{\epsilon_1} \times \delta_{p(2)}^{\epsilon_2} \times \cdots \times \delta_{(k)}^{\epsilon_k} \rtimes \pi$ are equivalent, where $\delta_{p(i)}^{\epsilon_i}$ denotes $\delta_{p(i)}$ if $\epsilon_i = 1$ and it denotes $\tilde{\delta}_{p(i)}$ if $\epsilon_i = -1$.

Let $\delta_1' \times \delta_2' \times \cdots \times \delta_{k'}' \rtimes \pi'$ be another representation, such that δ_j' are (unitarizable) irreducible square integrable representations of general linear groups and π' is an irreducible square integrable representation of a classical group $S_{q'}$. Suppose that $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \pi$ and $\delta_1' \times \delta_2' \times \cdots \times \delta_{k'}' \rtimes \pi'$ have an irreducible subquotient in common. Then $\pi \cong \pi'$ (and therefore $q = q'$), $k = k'$ and there exists a permutation p of $\{1, 2, \ldots, k\}$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in \{\pm 1\}$ such that

$$\delta_i' \cong \delta_{p(i)}^{\epsilon_i}$$

for all $i = 1, 2, \ldots, k$.

13.4. As we already have mentioned, to understand irreducible tempered representations, we need to understand when representations $\delta \rtimes \pi$ (from (13-3-2)) reduce. Having in mind the classification of irreducible square integrable representations of general linear groups, one needs to understand when

$$\delta(\rho, a) \rtimes \pi$$

reduces, for unitarizable $\rho \in \mathcal{C}$ and for $a \in \mathbb{N}$.

When we fix ρ, the reducibility of these representations can be described in a very nice way (a crucial role in this is played by $Jord(\pi)$).

Frobenius reciprocity implies irreducibility if ρ is not selfdual. Therefore, it remains to understand the reducibility for selfdual ρ's.

The following two examples are very simple but nice examples, from which one can get an an idea what happens regarding these reducibilities in general.

13.5. Examples: Let $S_n = Sp(2n, F)$ and $\pi = 1_{S_0}$.

(1) Suppose ψ is a character of order two of $F^\times = GL(1, F)$. Then

$$\delta(\psi, a) \rtimes 1_{S_0} \text{ is irreducible for all even } a;$$
$$\delta(\psi, a) \rtimes 1_{S_0} \text{ is reducible for all odd } a.$$

We see that understanding of reducibility in this case is very simple. One needs only to know the parity of $\mathbb{N}$ for which we have reducibility. Unfortunately, this is not always the case for other square integrable representations.

Now we shall give a simple example of a situation of a slightly different type.

(2) We shall consider now instead of ψ the trivial representation $1_{F^\times}$ of $GL(1, F)$ (on one-dimensional space). Then

$$\delta(1_{F^\times}, a) \rtimes 1_{S_0} \text{ is irreducible for all even } a;$$
$$\delta(1_{F^\times}, a) \rtimes 1_{S_0} \text{ is reducible for all odd } a, \textbf{ except for } a = 1.$$

Now we shall explain what happens in general regarding such reducibility.

Fix selfdual $\rho \in \mathcal{C}$. Then for exactly one parity in $\mathbb{N}$ holds

(1) $\delta(\rho, a) \rtimes \pi$ is reducible for all a from that parity, with possibly finitely many exceptions;

(2) $\delta(\rho, a) \rtimes \pi$ is irreducible for all a from the other parity.

The parity of $\mathbb{N}$ for which holds (1) will be called the **parity of reducibility of ρ and σ** (note that for this parity we can have finitely many exceptions of reducibility), and the other parity will be called the **parity of irreducibility of ρ and σ** (in this parity we have always irreducibility).

Therefore, for understanding tempered representations we need to know which is the parity of reducibility for selfdual $\rho \in \mathcal{C}$, and what are exceptions (if there are exceptions, then clearly they determine the parity of reducibility). Therefore, it is very important to know these exceptions. This is just $Jord(\pi)$:

New definition. $Jord(\pi)$ *is the set of all exceptions (ρ, a) in (1), when ρ runs over all selfdual representations in $\mathcal{C}$ $(a \in \mathbb{N})$.*

Suppose that π is an irreducible square integrable representation of S_n. C. Mœglin has proved that

$$\sum_{(\rho,a)\in Jord(\pi)} a\, d_\rho \ \leq\ \begin{cases} 2n & \text{if } S_n = SO(2n+1), \\ 2n+1 & \text{if } S_n = Sp(2n), \end{cases}$$

where d_ρ is determined by the fact that ρ is a representation of $GL(d_\rho, F)$. The above inequality clearly implies that $Jord(\pi)$ is finite.

The above inequality is expected to turn to be an equality.

13.6. Partial cuspidal support of π: In general, (conjugacy class of) an irreducible cuspidal representation τ of a Levi factor M of a parabolic subgroup P in a reductive group G is called **cuspidal support** of π, if π is a subquotient of $\mathrm{Ind}_P^G(\tau)$.

For classical groups, Levi factor M is a direct product of general linear groups and a classical group. Therefore,

$$\tau \cong \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_l \otimes \sigma,$$

where $\rho_i \in \mathcal{C}$ and σ is an irreducible cuspidal representation of a classical group.

Now the definition of partial cuspidal support of π is

$$\pi_{cusp} = \sigma.$$

We can define partial cuspidal support of π also in the following way: an irreducible cuspidal representation σ of a classical groups is called partial cuspidal support of π (and denoted by π_{cusp}) if there exists a smooth representation π' of a general linear group, such that

$$\pi \hookrightarrow \pi' \rtimes \sigma.$$

13.7. Partially defined function ϵ_π on $Jord(\pi)$: As we could see already from the simple Example 12.2, the construction of irreducible square integrable representations of classical groups involve reducibility of tempered induction, and thus R-groups (which for classical groups are sums of $\mathbb{Z}/2\mathbb{Z}$). This is roughly behind the fact that parameters of irreducible square integrable representations of classical groups will involve functions with values in $\{\pm 1\}$.

The definition of the domain of partially defined functions ϵ_π on $Jord(\pi)$ is quite technical. Because of this, we shall not give a complete definition of the partially defined functions (besides, from the general definition of partially defined functions, it is not quite easy to understand what are these functions). Rather, we shall try only to explain the main properties of these functions. One can understand pretty well the classification of irreducible square integrable representation of classical groups without knowing all the details of the definition of partially defined functions ϵ_π. Later, we shall give a constructive definition of these functions.

Let X be a free $\mathbb{Z}/2\mathbb{Z}$-module with basis $Jord(\pi)$. We shall note operation in the module X multiplicatively. Characters of this group are in a natural bijection with functions

$$Jord(\pi) \rightarrow \{\pm 1\}.$$

We can think of ϵ_π as of a function on a subset of X, in our case on a subset of

$$Jord(\pi) \cup \{x_1 x_2 \; ; \; x_1, x_2 \in Jord(\pi), x_1 \neq x_2\}.$$

Further, for $x_1, x_2 \in Jord(\pi)$ we shall write $\epsilon(x_1 x_2)$, when it is defined, also as

$$\epsilon_\pi(x_1)\epsilon_\pi(x_2)$$

even if $\epsilon_\pi(x_1)$ and $\epsilon_\pi(x_2)$ are not defined.

The fact $\epsilon(x_1 x_2) = 1$ (resp. $\epsilon(x_1 x_2) = -1$) will be written also as

$$\epsilon_\pi(x_1) = \epsilon_\pi(x_2) \quad (\text{resp. } \epsilon_\pi(x_1) \neq \epsilon_\pi(x_2))$$

even if $\epsilon_\pi(x_1)$ and $\epsilon_\pi(x_2)$ are not defined.

13.8. To give an idea of definition of ϵ_π, we shall describe one important case. The function ϵ_π is always defined on $(\rho, a)(\rho', a')$ if $\rho = \rho'$, $a \neq a'$ (and both $(\rho, a), (\rho', a')$ are from $Jord(\pi)$).

Suppose $(\rho, a_-), (\rho, a)$ are in $Jord(\pi)$, $a_- < a$, and

$$(\rho, a') \notin Jord(\pi) \text{ for any } a_- < a' < a.$$

Then $\epsilon_\pi((\rho, a_-)(\rho, a)) = \epsilon_\pi(\rho, a_-)\epsilon_\pi(\rho, a)$ is defined and

$$\epsilon_\pi(\rho, a_-)\epsilon_\pi(\rho, a) = 1$$

if and only if there exists a smooth representation π' of a classical group such that

$$\pi \hookrightarrow \delta([\nu^{(a_-+1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'.$$

13.9. In general, $\epsilon_\pi(\rho, a)$ is not always defined for $(\rho, a) \in Jord(\pi)$.

It is always defined if a is even. The definition in this case is the following:

Suppose $(\rho, a') \in Jord(\pi)$ with a' even. Chose a minimal a such that $(\rho, a) \in Jord(\pi)$. Then
$$\epsilon_\pi(\rho, a) = 1$$

if and only if there exists a smooth representation π' of a classical group such that

(13-9-1) $$\pi \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'.$$

If a is odd, then $\epsilon_\pi(\rho, a)$ is not always defined. It is not defined if and only if

$$(\rho, b) \in Jord(\pi_{cusp})$$

for some $b \in \mathbb{N}$.

From the above condition for $\epsilon_\pi(\rho, a)$ to be defined in the case of odd a, one can show that if $\epsilon_\pi(\rho, a)$ is defined (for odd a), then

(13-9-2) $$\rho \rtimes \pi_{cusp}$$

reduces (this is related to basic assumption under which we consider the classification of irreducible square integrable representations of classical groups; this assumption will be

explain later). C. Mœglin has used normalized intertwining operators to define $\epsilon_\pi(\rho, a)$ in this situation to define $\epsilon_\pi(\rho, a)$.

In the case when $\epsilon_\pi(\rho, a)$ is defined for odd a, as we have mentioned already $\rho \rtimes \pi_{cusp}$ reduces. It reduces into a sum of two inequivalent irreducible subrepresentations. One can chose one of these subrepresentations and attach to it 1, and to the other attach -1. Then one needs to extend in a natural way this choice to other tempered representations coming from inducing representations including $\delta(\rho, a)$ as a factor. One can do this using intertwining operators, but one can also do it without them.

Now we have almost complete definition of ϵ_π.

13.10. C. Mœglin has shown that for an irreducible square integrable representation π of a classical group, the triple

$$(Jord(\pi), \pi_{cusp}, \epsilon_\pi).$$

satisfies some technical conditions. The triples that satisfy these technical conditions she called **admissible triples**. We shall not give in the moment this technical definition. We shall give later a different and more explicit description of admissible triples. Let us just say that admissible triples are combinatorial objects modulo cuspidal data. It will become soon clear what we mean by cuspidal data.

C. Mœglin has proved that a mapping attaching an admissible triple to an irreducible square integrable representation a classical group, is an injective map from the set of all the equivalence classes of irreducible square integrable representations of classical groups (we fix a series of classical groups and a non-archimedean field F) into the set of all the admissible triples. Jointly, we have proved that this mapping is surjective. This means that we have a bijection between irreducible square integrable representations of classical groups and admissible triples. Since admissible triples are combinatorial objects modulo cuspidal data (as we mentioned already above), we have a classification of irreducible square integrable representations of classical groups modulo cuspidal data.

14. Reduction to cuspidal lines

The classification of irreducible square integrable representations of classical groups modulo cuspidal data will be easier to understand if we pass to cuspidal lines. We shall explain it in this section. This reduction could be is important also for some other purposes.

14.1. Fix an irreducible cuspidal representation σ of a classical group S_q and fix inequivalent selfdual irreducible cuspidal representations $\rho_1, \ldots, \rho_k$ of general linear groups. Denote by

$$\mathcal{D}(\rho_1, \ldots, \rho_k; \sigma)$$

the set of all equivalence classes of irreducible square integrable subquotients of representations

$$\nu^{\alpha_1} \tau_1 \times \nu^{\alpha_2} \tau_2 \times \cdots \times \nu^{\alpha_\ell} \tau_\ell \rtimes \sigma,$$

where $\alpha_i \in \mathbb{R}, \tau_i \in \{\rho_1, \ldots, \rho_k\}$. Then there is a natural bijection from $\mathcal{D}(\rho_1, \ldots, \rho_k; \sigma)$ into the Cartesian product

$$\mathcal{D}(\rho_1, \ldots, \rho_k; \sigma) \longrightarrow \left(\prod_{i=1}^{k} \mathcal{D}(\rho_i; \sigma) \right).$$

This bijection is given in the following way.

Fix $\pi \in \mathcal{D}(\rho_1, \ldots, \rho_k; \sigma)$ and $1 \leq j \leq k$. Then there exists an irreducible representation π_j of some S_{n_j} which is a subquotient of some $\nu^{\beta_1} \rho_j \times \nu^{\beta_2} \rho_j \times \ldots \nu^{\beta_{\ell_j}} \rho_j \rtimes \sigma$, and there exists an irreducible representation θ_j of a general linear group which is a subquotient of $\tau_1 \times \tau_2 \times \cdots \times \tau_{m_j}$ with $\tau_i \in \cup_{i=1, i \neq j}^{k} \{\nu^{\alpha} \rho_i; \alpha \in \mathbb{R}\}$, such that

$$\pi \hookrightarrow \theta_j \rtimes \pi_j.$$

C. Jantzen has proved in [Jn1] that representations $\pi_1, \ldots, \pi_k$ are uniquely determined by π, and that they are square integrable. Further $\pi \mapsto (\pi_1, \ldots, \pi_k)$ defines a bijection from $\mathcal{D}(\rho_1, \ldots, \rho_k; \sigma)$ onto $\prod_{i=1}^{k} \mathcal{D}(\rho_i; \sigma)$.

Each irreducible square integrable representation of a classical group belongs to some $\mathcal{D}(\rho_1, \ldots, \rho_k; \sigma)$. Further, if $\sigma \not\cong \sigma'$, then

$$\mathcal{D}(\rho_1, \ldots, \rho_k; \sigma) \cap \mathcal{D}(\rho'_1, \ldots, \rho'_{k'}; \sigma') = \emptyset,$$

and if

$$\{\rho_1, \ldots, \rho_k\} \cap \{\rho'_1, \ldots, \rho'_{k'}\} = \{\rho''_1, \ldots, \rho''_{k''}\}$$

then

$$\mathcal{D}(\rho_1, \ldots, \rho_k; \sigma) \cap \mathcal{D}(\rho'_1, \ldots, \rho'_{k'}; \sigma) = \mathcal{D}(\rho''_1, \ldots, \rho''_{k''}; \sigma)$$

In this way we obtain a reduction of the classification of irreducible square integrable representations of classical groups, to the problem of classification of sets

$$\mathcal{D}(\rho; \sigma)$$

($\rho \in \mathcal{C}$ is selfdual).

14.2. Consider the projection

$$\mathcal{O}_F \rightarrow \mathcal{O}_F / \mathfrak{p}_F.$$

Lift it to the level of groups. In this way one gets a natural homomorphism from the maximal compact subgroup K_0 in S_n to the group S_n over the field $\mathcal{O}_F / \mathfrak{p}_F$. The preimage in K_0 of the standard minimal parabolic subgroup in S_n over the field $\mathcal{O}_F / \mathfrak{p}_F$ is denoted by

$$I.$$

This open compact subgroup is called **Iwahori subgroup** of S_n.

The study of irreducible smooth representations with Iwahori fixed vectors has attracted lot of attention. In this case, for building the representation theory, one does not need non-trivial cuspidal representations (i.e. other than characters). Also, corresponding group algebras for this setting have a nice geometric realization. Using this, one can obtain construction of irreducible representations by geometric methods, what was done by D. Kazhdan and G. Lusztig.

For classical groups, to determine irreducible square integrable representations with Iwahori fixed vectors, it is equivalent to determining of

$$\mathcal{D}(1_{F^\times}; 1_{S_0}) \cup \mathcal{D}(\psi; 1_{S_0}),$$

where ψ is a (unique) character of order 2, which is unramified (i.e. which is trivial on $\mathcal{O}_F^\times$). Irreducible square integrable representations with Iwahori fixed vectors are parameterized by the Cartesian product $\mathcal{D}(1_{F^\times}; 1_{S_0}) \times \mathcal{D}(\psi; 1_{S_0})$.

15. Parameters of $\mathcal{D}(\rho; \sigma)$

15.1. For selfdual $\rho \in \mathcal{C}$ and an irreducible cuspidal representation σ of S_q, A. Silberger has proved there exists a unique

$$\alpha_{\rho,\sigma} \geq 0$$

such that

$$\nu^{\alpha_{\rho,\sigma}} \rho \rtimes \sigma$$

reduces.

Now we shall say what is the basic assumption (for ρ and σ), under which $\mathcal{D}(\rho, \sigma)$ is classified:

(BA) for ρ and σ $\qquad\qquad \alpha_{\rho,\sigma} - \alpha_{\rho,1_{S_0}} \in \mathbb{Z}.$

This assumption is needed (essentially only) in proofs.

F. Shahidi has proved that (BA) holds if σ is generic. It is also known that (BA) holds in some other cases. In general, (BA) would follow from some general Arthur's conjectures.

F. Shahidi has proved that

(15-1-1) $\qquad\qquad\qquad \alpha_{\rho,1_{S_0}} \in (1/2)\mathbb{Z}.$

15.2. We shall fix ρ and σ as above, and assume that (BA) holds for ρ and σ. Denote in the sequel

$$\alpha = \alpha_{\rho,\sigma}.$$

Now since we assume that (BA) holds for ρ and σ, (15-1-1) implies

$$\alpha = \alpha_{\rho,\sigma} \in (1/2)\mathbb{Z}_+.$$

Note that for $\pi \in \mathcal{D}(\rho, \sigma)$,

$$\pi_{cusp} = \sigma.$$

Therefore, since σ is fixed, for classification of $\mathcal{D}(\rho, \sigma)$ it is enough to consider instead of triples

$$(Jord(\pi), \pi_{cusp}, \epsilon_\pi)$$

pairs

$$(Jord(\pi), \epsilon_\pi)$$

(which form with σ admissible triples).

Further, for classification of $\mathcal{D}(\rho, \sigma)$ it is convenient (and enough) to work with

$$Jord_\rho(\pi) = \{a \in \mathbb{N}; (\rho, a) \in Jord(\pi)\}.$$

instead of $Jord(\pi)$. Now pairs

$$(Jord_\rho(\pi), \epsilon_\pi)$$

will be parameters of representation in $\mathcal{D}(\rho, \sigma)$. Note that $Jord_\rho(\pi)$ is a finite subset of either $2\mathbb{N}$ or $2\mathbb{N} - 1$ and ϵ_π is now regarded as a partially defined function on $Jord_\rho(\pi)$. Because of this, the parameters of $\mathcal{D}(\rho, \sigma)$ are now simpler than before.

There are two possibilities for α. The first is

$$\alpha \in \mathbb{Z}_+,$$

which will be called **integral case**, and the second is and

$$\alpha \in ((1/2)\mathbb{Z}_+ \setminus \mathbb{Z}_+),$$

which will be called **non-integral case**.

16. INTEGRAL CASE

We shall suppose in this section that

$$\alpha \in \mathbb{Z}_+,$$

and describe parameters $(Jord_\rho, \epsilon)$ of elements of $\mathcal{D}(\rho, \sigma)$ in this case.

16.1. In the integral case we have always

$$Jord_\rho \subseteq 2\mathbb{N} - 1.$$

Here partially defined function is defined on elements of $Jord_\rho$ if and only if $\alpha = 0$. If it is defined on $Jord_\rho$, then the values on $Jord_\rho$ completely determine the partially defined function.

If $\alpha \geq 1$, then ϵ is defined only on pairs from $Jord_\rho$ (and this partially defined function can be extended to a character of a free $\mathbb{Z}/2\mathbb{Z}$-module with basis $Jord_\rho$).

16.2. In the integral case, $Jord_\rho$ will be called of **alternated type** if

$$\mathrm{card}(Jord_\rho) = \alpha.$$

Here always exists a unique partially defined function ϵ such that $Jord_\rho$, ϵ and σ form an admissible triple. If $\alpha = 0$, then there is nothing to define. If $\alpha \geq 1$, then ϵ is not defined on elements on $Jord_\rho$, but it is defined on pairs. It is completely defined with the following property:

For each $a_-, a \in Jord_\rho, a_- < a$, such that

$$[a_-, a] \cap Jord_\rho = \{a_-, a\}$$

we have

$$\epsilon(a_-) \neq \epsilon(a)$$

(this is where the name alternated comes from).

Now we shall define the representation corresponding to alternated $Jord_\rho$ (and ϵ). Write $Jord_\rho = \{a_1, a_2, \ldots, a_\alpha\}$. After a renumeration we can assume

$$a_1 < a_2 < \cdots < a_\alpha.$$

Now the representation

$$\left(\prod_{i=1}^{\alpha} \delta([\nu^i \rho, \nu^{(a_i - 1)/2} \rho]) \right) \rtimes \sigma$$

has a unique irreducible subrepresentation, which will be denoted by

$$\pi_{(Jord_\rho, \sigma, \epsilon)}.$$

This representation is square integrable.

An example of such representations are Steinberg representations for symplectic groups.

16.3. We shall describe now general parameters of elements of $\mathcal{D}(\rho; \sigma)$. Take $Jord_\rho$ (and ϵ) of alternated type. Take any $a_-, a \in 2\mathbb{N} - 1$, $a_- < a$, such that

$$[a_-, a] \cap Jord_\rho = \emptyset.$$

Denote

$$Jord_\rho^{(1)} = Jord_\rho \cup \{a_-, a\}.$$

Extend ϵ in a way that

$$\epsilon(a_-) = \epsilon(a).$$

It is easy to see that there are exactly two such extensions. Denote them by ϵ_1, ϵ_2.

Now $(Jord_\rho^{(1)}, \sigma, \epsilon_i)$, $i = 1, 2$, are (new) admissible triples (in this setting). These triples are no more of alternated type. We can continue this construction, but now starting from $Jord_\rho^{(1)}, \sigma, \epsilon_i$.

Continuing this process, we construct $Jord_\rho^{(2)}$, $Jord_\rho^{(3)}$, ... (and corresponding partially defined functions). In this way, we shall get all the parameters of $\mathcal{D}(\rho; \sigma)$.

16.4. We shall describe now representations corresponding to these (new) parameters. Let $(Jord_\rho, \epsilon)$, a_-, a and $(Jord_\rho^{(1)}, \sigma, \epsilon_i)$ be as in 16.3. To alternated $(Jord_\rho, \epsilon)$ we have already attached in 16.2 a square integrable representation

$$\pi_{(Jord_\rho, \sigma, \epsilon)}.$$

Now the representation

$$\delta([\nu^{-(a_- - 1)/2} \rho, \nu^{(a-1)/2} \rho]) \rtimes \pi_{(Jord_\rho, \sigma, \epsilon)}$$

contains exactly two irreducible subrepresentations. One shows this using the strategy that we have used in Example 12.2. These irreducible subrepresentations are square integrable, and their parameters are $(Jord_\rho^{(1)}, \sigma, \epsilon_1)$ and $(Jord_\rho^{(1)}, \sigma, \epsilon_2)$.

If $Jord_\rho \neq \emptyset$, one determines from 13.8 which subrepresentation corresponds to which ϵ_i.

16.5. It remains to say which subrepresentation to attach to which ϵ_i if $Jord_\rho = \emptyset$. C. Mœglin has used normalized intertwining operators for this attaching.

One possibility would be to proceed in the following way. Suppose

$$Jord_\rho = \emptyset.$$

Then $\alpha = 0$. Write

$$\rho \rtimes \sigma = \tau_1 \oplus \tau_{-1}$$

as a sum of (two inequivalent) irreducible representations. The representations

$$\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \sigma \quad \text{and} \quad \delta([\nu\rho, \nu^{(a_--1)/2}\rho])^2 \rtimes \tau_i$$

have exactly one irreducible subquotient in common (for each $i = -1, 1$). Denote it by T_i. Now

$$\delta([\nu^{(a_-+1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes T_i$$

contains a unique irreducible subrepresentation. Denote it by π_i. Then π_1 and π_{-1} are two inequivalent irreducible subrepresentations of $\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$ (here $\pi_{(Jord_\rho,\sigma,\epsilon)} = \sigma$).

One natural possibility to distinguish irreducible subrepresentations of

$$\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$$

is to attach ϵ_i to $\pi_{\epsilon_i(a)}$ (for $i = -1, 1$). Let us note that we have not checked that this choice is the same as the choice that C. Mœglin made using normalized intertwining operators.

16.6. One proceeds further from $Jord_\rho^{(1)}$ to $Jord_\rho^{(2)}$, $Jord_\rho^{(2)}$ to $Jord_\rho^{(3)}$, ... recursively in the same way as we did in passing from $Jord_\rho$ to $Jord_\rho^{(1)}$ (it is even less complicated here, since always $Jord_\rho^{(1)} \neq \emptyset, Jord_\rho^{(2)} \neq \emptyset, \ldots$ and therefore we do not need to make choices as in 16.5).

17. Non-integral case

Now we shall assume that

$$\alpha \in ((1/2)\mathbb{Z}_+ \setminus \mathbb{Z}_+).$$

17.1. In this case.

$$Jord_\rho \subseteq 2\mathbb{N}.$$

Partially defined functions are defined on elements of $Jord_\rho$'s (and values on $Jord_\rho$'s completely determine partially defined functions).

17.2. In the non-integral case, $Jord_\rho$ will be called of **alternated type** if $\mathrm{card}(Jord_\rho) = \alpha \pm 1/2$, i.e. if

$$\mathrm{card}(Jord_\rho) = \alpha - 1/2 \quad \text{or} \quad \mathrm{card}(Jord_\rho) = \alpha + 1/2.$$

In this case, there exists a unique partially defined function ϵ on $Jord_\rho$ such that $Jord_\rho$, σ and ϵ form an admissible triple. This partially defined function ϵ is defined (and uniquely determined) by the following conditions:

For each $a_-, a \in Jord_\rho, a_- < a$, such that

$$[a_-, a] \cap Jord_\rho = \{a_-, a\}$$

we have

$$\epsilon(a_-) \neq \epsilon(a),$$

and

$$\epsilon(\min(Jord_\rho)) = \begin{cases} 1 & \text{if } \mathrm{card}(Jord_\rho) = \alpha + 1/2; \\ -1 & \text{if } \mathrm{card}(Jord_\rho) = \alpha - 1/2. \end{cases}$$

17.3. We shall now describe the representation corresponding to the above alternated $Jord_\rho$. Let $Jord_\rho = \{a_1, a_2, \ldots, a_{\alpha \pm 1/2}\}$. After a renumeration we can assume

$$a_1 < a_2 < \cdots < a_{\alpha \pm 1/2}.$$

Consider first the case

$$\mathrm{card}(Jord_\rho) = \alpha - 1/2.$$

Then the representation

$$\left(\prod_{i=1}^{\alpha - 1/2} \delta([\nu^{i+1/2}\rho, \nu^{(a_i-1)/2}\rho]) \right) \rtimes \sigma$$

has a unique irreducible subrepresentation. This subrepresentation will be denoted by

$$\pi_{(Jord_\rho, \sigma, \epsilon)}.$$

It is a square integrable representation which corresponds to $(Jord_\rho, \epsilon)$.

Now assume

$$\mathrm{card}(Jord_\rho) = \alpha + 1/2.$$

Then the representation

$$\left(\prod_{i=1}^{\alpha + 1/2} \delta([\nu^{i-1/2}\rho, \nu^{(a_i-1)/2}\rho]) \right) \rtimes \sigma$$

has a unique irreducible representation. We again denote this subrepresentation by

$$\pi_{(Jord_\rho, \sigma, \epsilon)}.$$

This is a square integrable representation which corresponds to $(Jord_\rho, \epsilon)$.

   We have described above how one attaches square integrable representations to alternated parameters.

17.3. Now we define general parameters of $\mathcal{D}(\rho; \sigma)$ in the same way as in the integral case. We also attach square integrable representations in the same way.

   The only difference which occurs between integral and non-integral case is in passing from $Jord_\rho$ to $Jord_\rho^{(1)}$ when $Jord_\rho = \emptyset$. In the non-integral case one uses (13-9-1) to determine which irreducible subrepresentation corresponds to which ϵ_i.

18. Local Langlands correspondences

18.1. Denote by W_F the Weil group of F. This is a dense subgroup of the Galois group of the separable algebraic closure of F over F (the topology is not the induced one from the Galois group, but a slightly modified one).

   Let G be a split connected reductive group over F (as $GL(n, F), Sp(2n, F)$ or $SO(2n + 1, F)$). By the Langlangs program, there should exist a natural partition of $\tilde{G}$ into finite subsets, called L-packets, which are indexed by (conjugacy classes of) admissible homomorphism of $W_F \times SL(2, \mathbb{C})$ into the complex dual group $^L G^0$ of G (admissible here means that the homomorphisms are continuous, that they carry W_F into semi simple elements and that they are algebraic on $SL(2, \mathbb{C})$).

   For the cases of the groups that we consider, complex dual groups are as follows:

$$^L GL(n, F)^0 = GL(n, \mathbb{C}),$$
$$^L Sp(2n, F)^0 = SO(2n + 1, \mathbb{C}),$$
$$^L SO(2n + 1, F)^0 = Sp(2n+, \mathbb{C})$$

(a property of the complex dual group $^L G^0$ is that it has the root system dual to the root system of G).

   We shall now concentrate our attention regarding the above aspect of the Langlands program to irreducible square integrable representations. In this case square integrable L-packets should be parameterized by admissible homomorphisms whose image is not contained in any proper Levi factor. Further, elements of a square integrable L-packet, which is indexed by an admissible homomorphism ψ, should be parameterized by irreducible representations of the component group of ψ (which is the quotient of the centralizer of the image of ψ by the connected component).

   The correspondence that one would get in this way is called local Langlands correspondence for G.

18.2. In the case of general linear groups, by the Langlands program there should be a bijection of irreducible square integrable representations of $GL(n, F)$ and n-dimensional

irreducible representations of $W_F \times SL(2, \mathbb{C})$ (which are admissible homomorphisms). Here component groups are trivial. In this bijection irreducible cuspidal representations of $GL(n, F)$ should correspond to irreducible representations of W_F

The work of Bernstein and Zelevinsky, which gave classification of irreducible square integrable representations of general linear groups modulo cuspidal representations, resulted with a reduction of establishing of local Langlands correspondence to the cuspidal case, i.e. to establishing a correspondence between (classes of) irreducible cuspidal representations of $GL(n, F)$ and (classes of) irreducible representations of W_F. More precisely, suppose that

$$\varphi$$

is such a correspondence for general linear groups between irreducible cuspidal representations of general linear groups and irreducible representations of W_F (we consider all general linear groups over F together). From the representation theory of $SL(2, \mathbb{C})$ one knows that for each $a \in \mathbb{N}$ there exists a unique irreducible algebraic representation

$$E_a$$

of $SL(2, \mathbb{C})$ on a-dimensional complex vector space (up to an equivalence). Then the formula for local Langlands correspondence on the set of (classes of) irreducible square integrable representations of general linear groups, which we shall denote also by φ, would be

$$\varphi(\delta(\rho, a)) = \varphi(\rho) \otimes E_a.$$

Local Langlands conjecture for $GL(n, F)$ has been recently proved in full generality (by M. Harris and R. Taylor in [HaTy], and by G. Henniart in [He]).

18.3. One may ask does the classification of irreducible square integrable representations of classical groups modulo cuspidal data give also a similar reduction.

The natural candidate for the Langlands correspondence Φ for classical groups is

$$\Phi(\pi) = \oplus_{(\rho, a) \in Jord(\pi)} \varphi(\rho) \otimes E_a$$

(φ is the local Langlands correspondence for general linear groups, which we have considered before). But it remains a number of facts to prove even to see that it is a good candidate (for the beginning, it is not clear at all that $\Phi(\pi)$ goes in the right group).

18.4. For classical groups, the centralizers of images of admissible homomorphism of $W_F \times SL(2, \mathbb{C})$ into the complex dual group, whose images are is not contained in any proper Levi factor, are finite groups which are $\mathbb{Z}/2\mathbb{Z}$-modules (these are component groups). Therefore, after choosing a basis, irreducible representations of the component group correspond to functions from the basis to $\{\pm 1\}$.

Now ϵ_π should give a part of the irreducible representation (i.e. character) of the component group corresponding to π. The rest should come from $\epsilon_{\pi_{cusp}}$ (once the local Langlands correspondence is established for cuspidal representations of classical groups).

Complete discussion regarding this reduction one can find in [Mg2].

19. Non-unitary duals of classical p-adic groups

19.1. The classification of irreducible square integrable representations of classical groups modulo cuspidal data implies also a classification of all the irreducible smooth representations of classical groups modulo cuspidal data (by cuspidal data we mean irreducible cuspidal representations of general linear and classical groups, and cuspidal reducibilities).

Suppose that a selfdual $\rho \in \mathcal{C}$ and irreducible square integrable representation π of a classical group are given. For understanding of tempered representations one needs to know the parity of reducibility.

If $Jord_\rho(\pi) \neq \emptyset$, then the parity which shows up in $Jord_\rho(\pi)$ is the parity of reducibility of ρ and π.

$Jord_\rho(\pi) = \emptyset$, then $Jord_\rho(\pi_{cusp}) = \emptyset$. Then the reducibility of ρ and π_{cusp} (and also π) is at 0 or 1/2. If the reducibility is at 0 (resp 1/2), then the parity of reducibility of ρ and π is odd (resp. even).

19.2. We can describe also the non-unitary dual by reduction to cuspidal lines.

Let σ be an irreducible cuspidal representation of a classical group S_q and let $\rho_1, \ldots, \rho_k \in \mathcal{C}$ be unitarizable such that for $i \neq j$, sets $\{\rho_i, \tilde\rho_i\}$ and $\{\rho_j, \tilde\rho_j\}$ have no equivalent representations (i.e. $\rho_i \not\cong \rho_j$ and $\rho_i \not\cong \tilde\rho_j$). Denote by

$$\mathcal{I}(\rho_1, \ldots, \rho_k; \sigma)$$

the set of all equivalence classes of irreducible subquotients of

$$\nu^{\alpha_1}\tau_1 \times \nu^{\alpha_2}\tau_2 \times \cdots \times \nu^{\alpha_\ell}\tau_\ell \rtimes \sigma,$$

where

$$\alpha_i \in \mathbb{R}, \tau_i \in \{\rho_1, \ldots, \rho_k, \tilde\rho_1, \ldots, \tilde\rho_k\}.$$

Then by [Jn1] there exists a bijection

$$\mathcal{I}(\rho_1, \ldots, \rho_k; \sigma) \longrightarrow \left(\prod_{i=1}^k \mathcal{I}(\rho_i; \sigma) \right),$$

$$\pi \longrightarrow (\pi_1, \ldots, \pi_k),$$

similarly as in 14.1 (for complete definition of the bijection one should consult [Jn1]). The classification of the non-unitary duals of classical groups reduces to the classification of the sets $\mathcal{I}(\rho_i; \sigma)$ in a similar way as the classification of the irreducible square integrable representations in 14.1 reduce to cuspidal lines (we shall not write details here, but the reduction is analogous).

19.3. Fix a unitarizable $\rho \in \mathcal{C}$ and fix an irreducible cuspidal representation σ of a classical group.

If ρ is not selfdual, then the tempered induction in $\mathcal{I}(\rho; \sigma)$ is always irreducible. Now irreducible tempered representations which show up as Langlands parameters of representations in $\mathcal{I}(\rho; \sigma)$ is easy to write (using Remark 13.3 to know equivalences among them).

Suppose now that ρ is selfdual. Let the reducibility of ρ and σ be $\alpha = \alpha_{\rho,\pi} \geq 0$. Now the parity of reducibility of ρ and σ (and also each square integrable representation in $\mathcal{I}(\rho; \sigma)$) is odd (resp. even) if $\alpha \in \mathbb{Z}$ (resp. $\alpha \notin \mathbb{Z}$). Further, one can describe easily irreducible tempered representations which show up as Langlands parameters of representations in $\mathcal{I}(\rho; \sigma)$, since we know the parity of reducibility of ρ and σ (one needs also to use Remark 13.3).

20. Unitary duals of general linear groups over local fields

20.1. Denote
$$D^u = \{\delta \in D; e(\delta) = 1\}.$$

The following theorem describes the unitary duals of general linear groups over any local field (archimedean or non-archimedean).

Theorem. *For a representation $\delta \in D^u$ and $m \geq 1$ denote*

$$u(\delta, m) = L(\nu^{(m-1)/2}\delta, \nu^{(m-3)/2}\delta, \ldots, \nu^{-(m-1)/2}\delta)$$

For $0 < \alpha < 1/2$ and δ and m as above, denote

$$\pi(u(\delta, m), \alpha) = \nu^{\alpha}u(\delta, m) \times \nu^{-\alpha}u(\delta, m).$$

Let B be the set of all possible $u(\delta, m)$ and $\pi(u(\delta, m), \alpha)$ with δ, m, α as above. Then
  (i) *If $\tau_1, \tau_2, \ldots, \tau_n \in B$, then the representation*

$$\tau_1 \times \tau_2 \times \cdots \times \tau_n$$

  is an irreducible unitarizable representation of a general linear group.
  (ii) *Let $\tau_1, \tau_2, \ldots, \tau_n, \tau_1', \tau_2', \ldots, \tau_{n'}' \in B$, Then*

$$\tau_1 \times \tau_2 \times \cdots \times \tau_n \cong \tau_1' \times \tau_2' \times \cdots \times \tau_{n'}'$$

  if and only if $n = n'$ and if one can obtain the sequence $(\tau_1, \tau_2, \ldots, \tau_n)$ from $(\tau_1', \tau_2', \ldots, \tau_n')$ by a permutation.
  (iii) *Each irreducible unitary representation of a general linear group is isomorphic to a representation*
$$\tau_1 \times \tau_2 \times \cdots \times \tau_n,$$

  for some $\tau_1, \tau_2, \ldots, \tau_n \in B$.

The above classification theorem is the same for all the local fields. The difference in the form of unitary duals comes from the difference of the sets D^u for different fields.

20.2. This theorem is proved in [T1] in the non-archimedean case. As we already mentioned, the theorem holds for archimedean fields in the same form (using the notion of $(\mathfrak{g}, K)$ modules), with the proof along the same strategy as in the non-archimedean case (see [T12]). D. Vogan in [Vo] has made quite different approach to the classification of unitary duals of general linear groups over archimedean fields.

Not to deal all the time with non-archimedean fields, we shall now describe the proof of the above theorem for $F = \mathbb{C}$. Since the proofs in the archimedean and non-archimedean case are along the same strategy, one will be able to get from this description quite good idea of the proof in the non-archimedean case.

20.3. In the sequel in this section, by a representation we shall mean corresponding $(\mathfrak{g}, K_0)$-module. Denote by

$$Irr^u = \cup_{n=0}^\infty GL(n, \mathbb{C})\hat{}.$$

Consider algebra R for complex general linear groups (constructed in 10.3). We shall consider

$$Irr^u \subseteq R.$$

Recall that R is a polynomial ring over D (Proposition 10.7). In particular, R is a factorial ring. Therefore, we can talk about prime elements in R.

In the complex case we have

$$D = GL(1, \mathbb{C})\tilde{}.$$

Note also that $|\ |_{\mathbb{C}}$ is the square of the usual absolute value on $\mathbb{C}$.

We shall now introduce several claims, whose proofs shall be discussed later:

(U0) $\sigma, \tau \in Irr^u \implies \sigma \times \tau \in Irr^u$.
(U1) $\delta \in D^u$ and $n \in \mathbb{N} \implies u(\delta, n) \in Irr^u$.
(U2) $\delta \in D^u$, $n \in \mathbb{N}$ and $0 < \alpha < 1/2 \implies \pi(u(\delta, n), \alpha) \in Irr^u$.
(U3) $\delta \in D$ and $n \in \mathbb{N} \implies u(\delta, n)$ is prime in R.
(U4) $a, b \in M(D) \implies L(a) \times L(b)$ contains $L(a + b)$ as a subquotient.

The addition of multisets in (which shows up in (U4)) is defined in obvious way:

$$(x_1, \ldots, x_u) + (y_1, \ldots, y_v) = (x_1, \ldots, x_u, y_1, \ldots, y_v).$$

Proposition. *Claims (U0) - (U4) imply Theorem 9.1.*

Proof. First observe that (U1), (U2) and (U0) imply (i) of the theorem.

Further, commutativity of R give implication $\impliedby$ in (ii). The implication $\implies$ follows from (U3).

It remains to prove (iii) (i.e. exhaustion). Suppose $\pi \in Irr^u$. Then

$$\pi = L(\gamma_1, \gamma_2, \ldots, \gamma_\ell)$$

for some $\gamma_1, \gamma_2, \ldots, \gamma_\ell \in D$ (see Remark 9.5). Note that π is hermitian (since it is unitary). This, together with (9-5-1) and (9-5-2), imply that

$$\pi = L(\nu^{\alpha_1}\delta_1, \nu^{-\alpha_1}\delta_1, \ldots, \nu^{\alpha_k}\delta_k, \nu^{-\alpha_k}\delta_k, \delta_{k+1}, \ldots, \delta_s),$$

for some $\alpha_i > 0$, $1 \le i \le k$ and $\delta_j \in D^u$, $1 \le j \le s$.

We shall use in the sequel the following simple fact

(20-3-1)   If $a_1, a_2, \ldots, a_m \in M(D)$ and $L(a_1), L(a_2), \ldots, L(a_m) \in Irr^u$,
$$\text{then } L(a_1) \times L(a_2) \times \cdots \times L(a_m) = L(a_1 + a_2 + \cdots + a_m).$$

This follows directly from (U4) and (U0) (by induction).

To get an idea of proof of (iii), we shall now give a proof of it in the rigid case, i.e. when all $\alpha_i \in (1/2)\mathbb{Z}$.

Denote
$$a(\delta, m) = (\nu^{(m-1)/2}\delta, \nu^{(m-3)/2}\delta, \ldots, \nu^{-(m-1)/2}\delta).$$

Then obviously
$$u(\delta, m) = L(a(\delta, m)).$$

Using the fact that

$$(\nu^{\alpha_i}\delta_i, \nu^{-\alpha_i}\delta_i) + a(\delta_i, 2\alpha_i - 1) = a(\delta_i, 2\alpha_i + 1)$$

and (20-3-1) (several times), from (U1) we get that

$$\pi \times u(\delta_i, 2\alpha_1 - 1) \times \cdots \times u(\delta_k, 2\alpha_k - 1)$$
$$\cong u(\delta_i, 2\alpha_1 + 1) \times \cdots \times u(\delta_k, 2\alpha_k + 1) \times \delta_{k+1} \times \cdots \times \delta_s.$$

By (U3), on the right hand side we have prime elements (from B). Since R is factorial, π must be a subproduct of the right hand side (up to a sign). So, π must be a product of elements from B. This proves (iii) in the rigid case.

The proof of (iii) in the non-rigid case proceeds along a similar idea, but it is slightly technically more complicated in this case. $\square$

By the above proposition, to prove the theorem, it is enough to prove (U0) - (U4). Now we shall explain how the proofs of each of these claims go.

20.4. (U1): Considering the modular function of the standard minimal parabolic subgroup in $GL(m, \mathbb{C})$, we get easily that for $\delta \in D = GL(1, \mathbb{C})\hat{\ }$ (i.e. a character of $\mathbb{C}^\times$) we have
$$u(\delta, m) = \delta \circ \det : GL(m, \mathbb{C}) \to \mathbb{C}^\times.$$

Thus, $u(\delta, m)$ is unitarizable. Therefore, (U1) holds.

20.5. (U2): The restriction of the representation $\pi(u(\delta, m), \alpha)$ (which we consider in (U2)) to $SL(2m, \mathbb{C})$, is a Stein's complementary series representation from [Stein] (if $m > 1$;

if $m = 1$, then this is a well known complementary series representation of $SL(2, \mathbb{C})$. Therefore, it is unitarizable. From this one gets directly that $\pi(u(\delta, m), \alpha)$ is unitarizable as a representations of $GL(2m, \mathbb{C})$ (since it has unitarizable central character).

One can get the unitarizability of representations $\pi(u(\delta, m), \alpha)$ by standard construction of complementary series representations (which are unitarizable). For this, see 20.11 bellow.

20.6. (U3): We shall illustrate the proof of (U3) on the example of $u(\delta, 2)$.

Note first that R is a graded ring (by definition). The degree of $u(\delta, 2)$ is two.

The representation theory of $SL(2, \mathbb{C})$ implies that

$$(20\text{-}6\text{-}1) \qquad\qquad u(\delta, 2) = X_1 \times X_2 - X_3 \times X_4$$

for some $X_i \in D$, $i = 1, 2, 3, 4$, where all X_i are different. Suppose that $u(\delta, 2)$ is not prime. Since it is primitive (the greatest common divisor of coefficients is 1), it must be a product of homogeneous elements f_1 and f_2 of degree one. Write

$$f_i = c_1^{(i)} X_1 + c_2^{(i)} X_2 + c_3^{(i)} X_3 + c_4^{(i)} X_4, \quad i = 1, 2.$$

Since $X_1 \times X_2$ shows up in $u(\delta, 2)$ (see (20-6-1)), it follows that $c_1^{(1)} \neq 0$ and $c_2^{(2)} \neq 0$ (after possible changing indexes of f_1 and f_2). Since $X_3 \times X_4$ shows up in $u(\delta, 2)$, it follows that $c_3^{(1)} \neq 0$ and $c_4^{(2)} \neq 0$ (after possible changing indexes of X_3 and X_4). These observations imply that the total degree of $u(\delta, 2)$ in variables X_1 and X_4 is 2. This obviously contradicts to the expression (20-6-1). This contradiction completes the proof that $u(\delta, 2)$ is prime (in R).

The proof of (U3) in the general case follows the same strategy and uses only a very basic facts about composition series of standard modules, i.e. principal series (which are standard facts of Langlands classification).

20.7. (U4): This claim follows from basic properties of composition series of principal series (which are standard facts of Langlands classification). We shall explain now how it follows.

We shall consider s.s.$(\lambda(a)) \in R$ for $a \in M(D)$. For simplicity, we shall write s.s.$(\lambda(a))$ as an element of R simply as $\lambda(a) \in R$.

Let $a_1, a_2 \in M(D)$. There exists a partial order $\leq$ on $M(D)$ (which is quite explicit and which is simple to describe), such that we have

$$L(a_i) = \lambda(a_i) + \sum_{b^{(i)} < a_i} m_{b^{(i)}}^{(i)} \lambda(b^{(i)}), \quad i = 1, 2,$$

in R (here $m_{b^{(i)}}^{(i)} \in \mathbb{Z}$). Now

$$L(a_1) \times L(a_2)$$
$$= \lambda(a_1) \times \lambda(a_2) + \sum_{b^{(1)} < a_1} m_{b^{(1)}}^{(1)} \lambda(b^{(1)}) \times \lambda(a_2) + \sum_{b^{(2)} < a_2} m_{b^{(2)}}^{(2)} \lambda(b^{(2)}) \times \lambda(a_1)$$
$$+ \sum_{b^{(1)} < a_1} m_{b^{(1)}}^{(1)} \lambda(b^{(1)}) \times \sum_{b^{(2)} < a_2} m_{b^{(2)}}^{(2)} \lambda(b^{(2)}).$$

We know that $L(a_1 + a_2)$ is a subquotient of $\lambda(a_1) \times \lambda(a_2)$. Standard properties of the Langlands classification imply that $L(a_1 + a_2)$ is not a subquotient of any of three sums on the right hand side of the above equality. This proves (U4).

20.8. (U0): Let P_n be the subgroup of all the matrices in $GL(n, \mathbb{C})$ which have bottom row equal to $(0, 0, \ldots, 0, 1) \in \mathbb{C}^n$. Suppose that we know that

(K) $\pi \in GL(n, \mathbb{C})\hat{} \implies \pi | P_n$ is irreducible

(in the above claim, we consider π as an irreducible unitary representation on a Hilbert space, not as a $(\mathfrak{g}, K_0)$-module).

It has been known for a long time that (K) implies (U0). The implication follows using small Mackey theory. For the implication, for irreducible unitary representations π_1 and π_2 of $GL(n_1, \mathbb{C})$ and $GL(n_2, \mathbb{C})$, using (K) one shows that $(\pi_1 \times \pi_2)|P_{n_1+n_2}$ is an induced unitary representation, which is irreducible by small Mackey theory. A complete proof of this implication can be found in [Sah] (but the proof is implicit already in [GlN]).

M. Baruch proved (K) in [Bu].

Additional comments

20.9. We shall give here a little bit more explanations regarding Baruch's proof of (K) and the history of proving of (K).

A.A. Kirillov observed in [Ki] that on a dense open subset of $GL(n, \mathbb{C})$, in each $GL(n, \mathbb{C})$-conjugacy classes there exists an open dense P_n-conjugacy class. This clearly implies that each continuous function on $GL(n, \mathbb{C})$, which is constant on P_n-conjugacy classes, is constant on $GL(n, \mathbb{C})$-conjugacy classes. Further, the last observation implies that if a P_n-invariant distribution on $GL(n, \mathbb{C})$ is represented by a continuous function (with respect to the Haar measure), then it is an invariant distribution (i.e. invariant for conjugation by elements of $GL(n, \mathbb{C})$).

Kirillov expected that this property holds for any P_n-invariant distribution on $GL(n, \mathbb{C})$ (not only for those one which are give by integration against continuous function). He observed that this property would imply (K) in the following way. Take any T in the commutator of the representation $\pi | P_n$. For proof of (K), by Schur lemma it is enough to show that T is scalar.

Kirillov considered the distribution

(20-9-1) $\Lambda_T : \varphi \mapsto \mathrm{Trace}(T\pi(\varphi)),$

which is P_n-invariant (T is in the commutator of $\pi | P_n$). Now the property that Kirillov expected for general P_n-invariant distributions, would imply that the above distribution is invariant for the whole group. Using irreducibility of π, it is easy to show that this would imply that T must be a scalar operator.

Note that for proving (K), it is enough to prove Kirillov expectation only for P_n-invariant eigen-distributions (since Λ_T is an eigen-distribution).

At this point, let we recall of the Harish-Chandra regularity theorem for invariant eigen-distributions. He showed that such a distribution is represented by a locally integrable

function, which is analytic on regular semi simple elements. If one could prove such a type of result for P_n-invariant eigen-distributions, then the Kirillov's argument for P_n-invariant distributions represented by continuous functions could be used to see the invariance for the whole group (and we would prove (K) in this way). Since the geometry of P_n and $GL(n, \mathbb{C})$-conjugacy classes is the same on a big open set in $GL(n, \mathbb{C})$, it make sense to try to follow Harish-Chandra's strategy of proof of the regularity theorem, to try to prove such a type of result for P_n-invariant eigen-distributions.

As we already have mentioned, M. Baruch proved (K) in [Bu]. The strategy of his proof may be considered as a further development of the ideas that we discussed above (there is a plenty of new moments).

20.10. J. Bernstein proposed the following strategy for proving (K) (and also (U0)). Consider $GL(n-1, \mathbb{C}) \subseteq GL(n, \mathbb{C})$ in obvious way. Bernstein asked if each $GL(n-1, \mathbb{C})$-invariant distribution on $GL(n, \mathbb{C})$ is invariant for transposition (for our purpose, it is enough to consider only $GL(n-1, \mathbb{C})$-invariant eigen-distributions). Positive answer to this question would imply (K).

One can consider also other local fields regarding the above question. As far as we remember, in a discussion with D. Miličić we saw that it is an easy exercise to show that the answer to this question is positive for $GL(2, \mathbb{R})$. Unfortunately, this argument cannot be extended to higher $GL(n, \mathbb{R})$.

20.11. Now we shall explain how one can prove (U2), using the standard construction of complementary series representations. For this one needs a simple

Lemma. *Let* $\gamma_1, \ldots, \gamma_u, \delta_1, \ldots, \delta_v \in D$. *Suppose that* $\delta_i \times \gamma_j$ *is irreducible for all indexes* $1 \leq i \leq u$, $1 \leq j \leq v$. *Then*

$$L(\gamma_1, \ldots, \gamma_u) \times L(\delta_1, \ldots, \delta_v)$$

is irreducible.

Proof. Using the fact that all $\delta_i \times \gamma_j$ are irreducible (which implies $\delta_i \times \gamma_j \cong \gamma_j \times \delta_i$), and associativity of operation $\times$ among representations representations (see Remark 9.4), we get
$$\lambda(\gamma_1, \ldots, \gamma_u, \delta_1, \ldots, \delta_v) \cong \lambda(\gamma_1, \ldots, \gamma_u) \times \lambda(\delta_1, \ldots, \delta_v)$$

(for the definition of $\lambda(a)$, see 9.5). From the above isomorphism, one concludes that $L(\gamma_1, \ldots, \gamma_u) \times L(\delta_1, \ldots, \delta_v)$ has a unique irreducible quotient, and that this quotient is isomorphic to $L(\gamma_1, \ldots, \gamma_u, \delta_1, \ldots, \delta_v)$.

Since $\tilde{\delta}_1 \times \tilde{\gamma}_j$ are also all irreducible, the above conclusion hold also for them: the representation $L(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_u) \times L(\tilde{\delta}_1, \ldots, \tilde{\delta}_v)$ has a unique irreducible quotient, which is $L(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_u, \tilde{\delta}_1, \ldots, \tilde{\delta}_v)$. Now passing to contragredients (and using (9-5-1)), we shall get that $L(\gamma_1, \ldots, \gamma_u) \times L(\delta_1, \ldots, \delta_v)$ has a unique irreducible subrepresentation, which is isomorphic to the representation $L(\gamma_1, \ldots, \gamma_u, \delta_1, \ldots, \delta_v)$.

A basic property of Langlands classification is that the multiplicity of a Langlands quotient in corresponding standard module is one. This implies irreducibility of $L(\gamma_1, \ldots, \gamma_u) \times L(\delta_1, \ldots, \delta_v)$. $\square$

The above lemma and the representations theory of $SL(2,\mathbb{C})$ imply that the continuous family of representations

$$\pi(u(\delta,m),\alpha), \quad 0 \leq \alpha < 1/2,$$

is irreducible (irreducibility at 0 follows from (U0)). From this, using standard integral intertwining operators we can get on these representations (non-degenerate) hermitian forms, which depend continuously on α, and which make these representations hermitian. Since we have positive definiteness in 0 and we have continuous family of hermitian representations, this imply positive definiteness for all $0 \leq \alpha < 1/2$ (this easily follows from a finite dimensional argument; see [T6]). Therefore, representations $\pi(u(\delta,m),\alpha)$ from (U2) are unitarizable.

21. On the unitarizability problem for classical p-adic groups

21.1. Constructing new irreducible unitarizable representations is a very interesting and puzzling problem. It may be related to a number of other problems.

Most interesting (and hardest) question is constructing of irreducible unitarizable representations which show up from "nowhere", i.e., of isolated irreducible unitarizable representations. Namely, there is a natural topology on unitary duals (defined by approximation of matrix coefficients) and isolated representations are those those ones for which $\{\pi\}$ is an open set (if the center is not compact, one defines isolated representations modulo center; these representations play the role of isolated representations in this case).

As we mentioned above, the construction of isolated irreducible unitarizable representations can be related to a number of other questions. Let us mention some of them: representations in residual spectrum of the group over adels, θ-correspondences, (conjecturaly) involution (which we mention bellow in 21.2) of square integrable representations. A natural question is: how big portion of isolated representations is in a range of each of these methods, combined with some standard constructions of irreducible unitarizable representations (see section 3. of [T6] for standard constructions). We plan to address these questions in the future.

Now we shall formulate some other (precise) questions regarding the unitarizability problem (the first question is around for a long time and we do not know who posed it for the first time in full generality). These questions may provide a strategy (or may be considered as a part of a general strategy) for attacking unitarizability problem for classical p-adic groups.

It may happen that answers to (at least some of these) questions will be obtained in the same time as we will get the solution of the unitarizability problem for classical p-adic groups. Nevertheless, as we already mentioned above, some of these questions may be useful guidance on the way to the solution of the unitarizability problem for classical p-adic groups. This is the reason that we collect them here.

21.2. A.-M. Aubert and also P. Schneider and U. Stuhler ([Au], [SchStu]) defined an involution on irreducible representations of connected reductive p-adic groups. This involution carries irreducible unitarizable representations of general linear groups to the unitarizable ones. This was conjectured by J. Bernstein in [Be2] (and shown by this author; see for

example [T1]). Further, this involution in the case of irreducible representations with Iwahori fixed vectors carries unitarizable representations to the (irreducible) unitarizable ones. This was proved by D. Barbasch and A. Moy in [BbMy].

It is natural to ask if this is the case in general. It would be very important to show this (if this is the case).

21.3. Fix an irreducible cuspidal representation σ of a classical group S_q and fix unitarizable $\rho_1, \ldots, \rho_k \in \mathcal{C}$ such that for $i \neq j$, sets $\{\rho_i, \tilde{\rho}_i\}$ and $\{\rho_j, \tilde{\rho}_j\}$ have no equivalent representations (i.e. $\rho_i \not\cong \rho_j$ and $\rho_i \not\cong \tilde{\rho}_j$). We have already observed in 19.2 that there exists a bijection

$$\mathcal{I}(\rho_1, \ldots, \rho_k; \sigma) \longrightarrow \left(\prod_{i=1}^{k} \mathcal{I}(\rho_i; \sigma) \right),$$

$$\pi \longrightarrow (\pi_1, \ldots, \pi_k)$$

(for details see [Jn1])

The question is:

Is π unitarizable if and only if all $\pi_1, \pi_2, \ldots, \pi_k$ are unitarizable?

21.4. Suppose

$$\rho \not\cong \tilde{\rho},$$

where $\rho \in \mathcal{C}$ is unitarizable. Take an irreducible unitarizable representation π of a general linear group which is an irreducible subquotients of

$$\nu^{\alpha_1} \tau_1 \times \nu^{\alpha_2} \tau_2 \times \cdots \times \nu^{\alpha_\ell} \tau_\ell,$$

where $\alpha_i \in \mathbb{R}, \tau_i \in \{\rho, \tilde{\rho}\}$. We expect that $\pi \rtimes \sigma$ is irreducible (this should not be hard to prove). Then $\pi \rtimes \sigma$ is an irreducible unitarizable representation.

The question is:

Does every unitarizable representation in $\mathcal{I}(\rho, \sigma)$ comes in this way?[1]

Possible positive answer to this question would give reduction of unitarizability problem in $\mathcal{I}(\rho, \sigma)$ to the case of general linear groups (where the unitarizability problem has been solved).

21.5. Suppose that $\delta_1, \ldots, \delta_k \in D$ are unitarizable, and π is an irreducible square integrable representations of a classical group. Let τ_1 and τ_2 be irreducible subrepresentations of

$$\delta_1 \times \cdots \times \delta_k \rtimes \sigma.$$

Let $\delta'_1, \ldots, \delta'_\ell \in D_+$. One may ask the following question:

Is $L(\delta'_1, \ldots, \delta'_\ell, \tau_1)$ unitarizable if and only if $L(\delta'_1, \ldots, \delta'_\ell, \tau_2)$ is unitarizable?

---

[1]After this paper has been written, we have noticed that it is pretty easy to show (as we have expected), that the answer to this question is postive.

The answer to this question is negative (the first example that we know, which shows that the answer is negative, is for $SO(7, F)$).

21.6. Consider $\mathcal{I}(\rho; \sigma)$, with ρ selfdual. The question is

Can the description of unitarizable representations in $\mathcal{I}(\rho; \sigma)$ be expressed only in terms of the reducibility point $\alpha_{\rho,\sigma}$ (similarly as for $\mathcal{D}(\rho; \sigma)$)?

21.7. In the case of general linear groups, solution of the unitarizability problem may be expressed independently of the nature of the (local) field F (see 20.1). This may be viewed as a (very strong) example of Lefschetz principle. The question on the same line is:

Can one get also a descriptions of unitary duals for each of series of classical groups independent of the nature of the field?

21.8. W. Casselman has proved that in the induced representation from minimal parabolic subgroup, where the Steinberg representation shows up, only two irreducible subquotients are unitarizable, the Steinberg and the trivial one (supposing that the group is not compact modulo center). This fact is not important only regarding unitarizability problem (see [BlWh]). Let us recall that this fact can be also derived from Howe-Moore theorem on asymptotic behavior of infinite dimensional irreducible unitary representations of reductive groups over local fields.

This fact concerns only finitely many irreducible representations regarding unitarizability (for a fixed group). Nevertheless, its importance much overcomes this finite set. Namely, it implies that Steinberg and trivial representation are isolated in unitary dual if the the group has compact center and if its split rank is greater than 1. It also implies that the induced representations around the induced representation where the Steinberg and the trivial representation show up, does not have unitarizable subquotients (these regions are determined by certain irreducibility conditions). Using this fact and some simple standard results, one can solve the unitarizability problem for rank two groups (let us note that to apply this result, one needs to understand reducibility of parabolically induced representations).

Such a fact about exactly two unitarizable subquotients in the whole induced representation where the arbitrary irreducible square integrable representation shows up, holds in the case of general linear groups (the induced representation is always multiplicity one, and further, there is exactly one irreducible square integrable subquotient there).

One may ask if this holds for other groups.

Already the first example other than general linear group, the example $Sp(4, F)$, tells that neither one of the nice properties discussed above regarding the whole induced representation, where the arbitrary irreducible square integrable representation shows up hold in this case. Namely, already for $Sp(4, F)$ there is an example of such induced representation which is not multiplicity one, which contains two inequivalent irreducible square integrable subquotients, which has total length 6, and where all the irreducible subquotients are unitarizable.

In general, for classical groups we have quite often plenty of unitarizable subquotients in whole induced representations where the irreducible square integrable representations show up (for the beginning, we can have as many square integrable subquotients as we want). In the moment, we shall ask only the following question:

Suppose that $\rho \in \mathcal{C}$ is selfdual. Let σ be an irreducible cuspidal representation of a classical group. Suppose that $\nu^{\alpha}\rho \rtimes \sigma$ reduces for some $\alpha > 0$. Then

$$\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$$

is a multiplicity one representation. It contains exactly one irreducible square integrable subquotient (actually, it is a unique subrepresentation in the above representation; such representations we call square integrable representations of Steinberg type). The question is:

Does the above induced representation have exactly two irreducible subquotients which are unitarizable (a weaker question is, does it have at most two)?

References

[Ar1]   Arthur, J., *On some problems suggested by the trace formula*, Lie Group Representations II, Proceedings, University of Maryland 1982-83, Lecture Notes in Math. 1041, Springer-Verlag, Berlin, 1984, pp. 1-49.

[Ar2]   Arthur, J., *Unipotent automorphic representations: conjectures*, Astérisque **171-172** (1989), 13-71.

[Au]   Aubert, A.-M., *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p-adique*, Trans. Amer. Math. Soc. **347** (1995), 2179-2189 (and *Erratum*, Trans. Amer. Math. Soc **348** (1996), 4687-4690).

[Bb]   Barbasch, D., *Unitary spherical spectrum for split classical groups*, preprint (2002).

[BbMy]   Barbasch, D. and Moy, A., *A unitarity criterion for p-adic groups*, Invent. Math. **98** (1989), 19-37.

[Bu]   Baruch, E.M., *Proof of Kirillov's conjecture*, Ann. of Math. (to appear).

[Be1]   Bernstein, J.N., *All reductive p-adic groups are tame*, Functional Anal. Appl. **8** (1974), 91-93.

[Be2]   Bernstein, J., *P-invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-archimedean case)*, Lie Group Representations II, Proceedings, University of Maryland 1982-83, Lecture Notes in Math. 1041, Springer-Verlag, Berlin, 1984, pp. 50-102.

[Be3]   Bernstein, J. (written by K. Rumelhart), *Draft of: Representations of p-adic groups*, preprint.

[BeZe1] Bernstein, I. N. and Zelevinsky, A.V., *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspekhi Mat. Nauk. **31** (1976), 5-70.

[BeZe2] Bernstein, J. and Zelevinsky, A.V., *Induced representations of reductive p-adic groups I*, Ann. Sci. École Norm Sup. **10** (1977), 441-472.

[BlWh]   Borel, A. and Wallach, N., *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Princeton University Press, Princeton, 1980.

[BuKu]   Bushnell, C.J. and Kutzko, P., *The admissible dual of $GL(N)$ via compact open subgroups* (1993), Princeton University Press, Princeton.

[Ct]   Cartier, P., *Representations of p-adic groups; a survey*, Symp. Pure Math. 33, part 1, Amer. Math. Soc., Providence, Rhone Island, 1979, pp. 111-155.

[Cs]   Casselman, W., *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint.

[CsMi]   Casselman W. and Miličić D., *Asymptotic behavior of matrix coefficients of admissible representations*, Duke Math. J. **49, No. 4** (1982), 869-930.

[Cl]   Clozel L., *Progrès récents vers la classification du dual unitaire des groupes réductifs réels*, Séminaire Bourbaki no 681 (1986-87), Astérisque.

[Gb]   Gelbart, S.S., *An elementary introduction to the Langlands program*, Bulletin Amer. Math. Soc. **10** (1984), 177-219.

[GlGrPS] Gelfand, I.M., Graev, M. and Piatetski-Shapiro, *Representation theory and automorphic functions*, Saunders, Philadelphia, 1969.

[GlKz]   Gelfand, I.M. and Kazhdan, D.A., *Representations of $GL(n, k)$*, Lie groups and their Representations, Halstead Press, Budapest, 1974, pp. 95-118.

[GlN]   Gelfand, I.M. and Naimark, M.A. Unitäre Darstellungen der Klassischen Gruppen (1957), Berlin; German translationAkademie Verlag (ed.).

[Go]   Goldberg, D., *Reducibility of induced representations for $Sp(2n)$ and $SO(n)$*, Amer. J. Math. **116** (1994), no. 5, 1101-1151.

[HC]   Harish-Chandra, *Collected papers*, Springer-Verlag, Berlin, 1983.

[HaTy]   Harris, M. and Taylor, R., *On the geometry and cohomology of some simple Shimura varieties* (2001), Princeton University Press, Annals of Math. Studies 151.

[He]   Henniart, G., *Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439-455.

[Ho]   Howe, R., *θ-series and automorphic forms*, Symp. Pure Math. 33, part 1, Amer. Math. Soc., Providence, Rhone Island, 1979, pp. 275-286.

[Hu]   Humpreys, J.E., *Linear algebraic groups*, Springer-Verlag, New York, 1975.

[Jc1]   Jacquet, H, *Generic representations*, Non-Commutative Harmonic Analysis, Lecture Notes in Math. 587, Spring Verlag, Berlin, 1977, pp. 91-101.

[Jc2]   Jacquet, H., *Principal L-functions of the linear group*, Symp. Pure Math. 33, part 2, Amer. Math. Soc., Providence, Rhone Island, 1979, pp. 63-86.

[Jc3]   Jacquet, H., *On the residual spectrum of GL(n)*, Lie Group Representations II, Proceedings, University of Maryland 1982-83, Lecture Notes in Math. 1041, Springer-Verlag, Berlin, 1984, pp. 185-208.

[Jn1]   Jantzen, C., *On supports of induced representations for symplectic and odd-orthogonal groups*, Amer J. Math. **119** (1997), 1213-1262.

[Jn2]   Jantzen, C., *On square integrable representations of classical p-adic groups II*, Representation Theory **4** (2000), 127-180.

[KzLu]   Kazhdan, D. and Lusztig, G., *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), 153-215.

[Ki]   Kirillov, A. A,, *Infinite dimensional representations of the general linear group*, Dokl. Akad. Nauk SSSR **114** (1962), 37–39; Soviet Math. Dokl. **3** (1962), 652–655.

[Kn]   Knapp, A. W., *Introduction to the Langlands program*, Representation Theory and Automorphic Forms; An Instructional Conference at the International Centre for Mathematical Sciences, Edinburgh, 1996, American Mathematical Society, Proceedings of Symposia in Pure Mathematics 61, Providence, Rhode Island, 1997, pp. 245-302.

[Ls]   Langlands, R.P., *On the classification of irreducible representations of real algebraic groups*, Representation Theory and Harmonic Analysis on Semisimple Lie Groups, P.J. Sally, Jr. and D. A. Vogan, Jr. editors, Amer. Math. Soc., Providence, 1989.

[Mi]   Miličić, D., *On C^*-algebras with bounded trace*, Glasnik Mat. **8(28)** (1973), 7-21.

[Mg1]   Mœglin C., *Normalisation des opérateurs d'entrelacement et réductibilité des induites de cuspidales; le cas des groupes classiques p-adiques*, Annals of Math. **151** (2000), no. 2, 817-847.

[Mg2]   Mœglin, C., *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité.*, in Journal of the European Mathematical Society (to appear).

[Mg3]   Mœglin, C., *Représentations quadratiques unipotentes des groupes classiques p-adiques.*, Duke Math. J. **84** (1996), 267-332.

[Mg4]   Moeglin, C., *Points de réducibilité pour les induites de cuspidales*, preprint, Institute de mathématiques de Jussieu (2001).

[MgT]   Mœglin, C. and Tadić, M., *Construction of discrete series for classical p-adic groups*, J. Amer. Math. Soc. **15** (2002), 715-786.

[MgVgWd]   Mœglin, C., Vignéras, M.-F. and Waldspurger J.-L., *Correspondance de Howe sur un corps p-adique*, Lecture Notes in Math 1291, Springer-Verlag, Berlin, 1987.

[My]   Moy, A., *Representations of GSp(4) over a p-adic field: parts 1 and 2*, Compositio Math. **66** (1988), 237-328.

[Mu]   Muić, G., *On generic irreducible representations of $Sp(n, F)$ and $SO(2n+1, F)$*, Glasnik Mat. **33(53)** (1998), 19-31.

[Ro]   Rodier, F., *Représentations de $GL(n, k)$ où k est un corps p-adique, Séminaire Bourbaki no 587 (1982)*, Astérisque **92-93** (1982), 201-218.

[Sah]   Sahi, S., *On Kirillov's conjecture for archimedean fields*, Compos. Math. **72** (1989), 67-86.

[Sav]   Savin, G., *Lectures on representations of p-adic groups*, preprint (2002).

[SalT]   Sally, P.J. and Tadić, M., *Induced representations and classifications for $GSp(2, F)$ and $Sp(2, F)$*, Mémoires Soc. Math. France **52** (1993), 75-133.

[SchStu]   Schneider, P. and Stuhler, U., *Representation theory and sheaves on the Bruhat-Tits building*, Publ. Math. IHES (1997), 97-191.

[Sha1]   Shahidi, F., *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. **132** (1990), 273-330.

[Sha2]   Shahidi, F., *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J. **66** (1992), 1-41.

[Sil1]   Silberger, A., *The Langlands quotient theorem for p-adic groups*, Math. Ann. **236** (1978), 95-104.

[Sil2]   Silberger, A., *Special representations of reductive p-adic groups are not integrable*, Ann. of Math. **111** (1980), 571-587.

[Spe]   Speh B., *Unitary representations of $GL(n, \mathbb{R})$ with non-trivial $(\mathfrak{g}, K)$-cohomology*, Invent. Math. **71** (1983), 443-465.

[Stein] Stein, E.M., *Analysis in matrix spaces and some new representations of $SL(N, \mathbb{C})$*, Ann. of Math. **86** (1967), 461–490.

[Stb]   Steinberg, R., *Lectures on Chevalley groups*, Yale University, 1968.

[T1]   Tadić, M., *Classification of unitary representations in irreducible representations of general linear group (non-archimedean case)*, Ann. Sci. École Norm. Sup **19** (1986), 335-382.

[T2]   Tadić, M., *Topology of unitary dual of non-archimedean $GL(n)$*, Duke Math. J. **55** (1987), 385-422.

[T3]   Tadić, M., *Representations of p-adic symplectic groups*, Compositio Math. **90** (1994), 123-181.

[T4]   Tadić, M., *Structure arising from induction and Jacquet modules of representations of classical p-adic groups*, Journal of Algebra **177** (1995), no. 1, 1-33.

[T5]   Tadić, M., *Correspondence on characters of irreducible unitary representations of $GL(n, \mathbb{C})$*, Mathematischen Annalen **305** (1996), 419-438.

[T6]   Tadić, M., *An external approach to unitary representations*, Bulletin Amer. Math. Soc. **28** (1993), no. 2, 215-252.

[T7]   Tadić, M., *Representations of classical p-adic groups*, Representations of Lie groups and quantum groups, Pitman Research Notes in Mathematics Series 311, Longman, Essex, 1994, pp. 129-204.

[T8]   Tadić, M., *On regular square integrable representations of p-adic groups*, Amer. J. Math. **120** (1998), no. 1, 159-210.

[T9]   Tadić, M., *On reducibility of parabolic induction*, Israel J. Math. **107** (1998), 29-91.

[T10]   Tadić, M., *Square integrable representations of classical p-adic groups corresponding to segments*, Representation Theory **3** (1999), 58-89.

[T11]   Tadić, M., *A family of square integrable representations of classical p-adic groups*, Glasnik Mat. **37(57)** (2002), 21-57.

[T12]   Tadić, M., *$GL(n, \mathbb{C})\hat{}$ and $GL(n, \mathbb{R})\hat{}$*, preprint (2002).

[Vo]   Vogan, D.A., *The unitary dual of $GL(n)$ over an archimedean field*, Invent. Math. **82** (1986), 449-505.

[Wd]   Waldspurger, J.-L., *La formule de Plancherel pour les groupes p-adiques, d'après Harish-Chandra*, Journal de l'Institut de Math. de Jussieu (to appear).

[Wr]   Warner, G., *Harmonic analysis on Semi-Simple Lie Groups I, II*, Springer-Verlag, Berlin, 1972.

[We]   Weil, A., *Basic number theory*, Springer-Verlag, New York, 1974.

[Ze1]   Zelevinsky, A.V., *Induced representations of reductive p-adic groups II, On irreducible representations of $GL(n)$*, Ann. Sci École Norm Sup. **13** (1980), 165-210.

[Ze2]   Zelevinsky, A. V., *Representations of Finite Classical Groups, A Hopf Algebra Approach*, Lecture Notes in Math 869, Springer-Verlag, Berlin, 1981.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA 30, 10000 ZAGREB, CROATIA