SQUARE INTEGRABLE REPRESENTATIONS OF SEGMENT TYPE (GENERIC REDUCIBILITIES)

Marko Tadić

Introduction

The best known irreducible square integrable representation of a reductive p-adic group is the Steinberg representation. This representation has a simple and natural construction. In some cases, one can construct a family of relatively simple irreducible square integrable representations in a way which is a natural generalizations of the construction of the Steinberg representation, starting from cuspidal representations of Levi subgroups (let us note that each square integrable representation of a general linear group can be constructed in this way). Properties of such square integrable representations are very similar to the properties of irreducible square integrable representations of general linear groups. If we consider p-adic symplectic or split odd-orthogonal group, what we shall do in this paper, the first class of substantially more complicated irreducible square integrable representations are so called irreducible square integrable representations of segment type. These representations show a number of new properties and sharp differences with the case of general linear groups. These square integrable representations show up as square integrable subquotients of the reducible representations, parabolically induced from

$$\delta \otimes \sigma,$$

where δ is an irreducible essentially square integrable representations of a general linear group and σ is an irreducible cuspidal representation of a classical group.

Nowadays we have several ways to construct the square integrable representations of segment type. In this paper we present the old original (initial) construction of the square integrable representations of segment type (which goes back to 1992. and 1993.[1]).

A very direct (and much newer) construction of irreducible square integrable representations of segment type is realized in [T10] (the construction in [T10] is

[1]This approach was undertaken in the first part of [T9] (let us note that the first part of preprint [T9] is just the reorganized "union" of the first part of the preprint [T6], and of the preprint [T8]).

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

realized for more general reducibilities than our initial construction; [T10] treats general half-integral reducibilities).

We shall explain now the reasons for publishing the initial construction. The initial construction introduced a number of new ideas, which then enabled introduction of new ideas and methods in the study of reducibility questions of parabolically induced representations (such a study of reducibility was initiated in [T7], and continued very successfully by C. Jantzen in [J] and G. Muić in [Mu2], [Mu3]). Several important ideas and methods are not present in [T10]. This construction, which is very natural, gives an additional insight to the square integrable representations of segment type, which is also not present in [T10]. This insight may be useful in the further study of the square integrable representations of segment type.

All the irreducible square integrable representations of the classical p-adic groups are constructed modulo cuspidal data in [MœT] (under a natural assumption which is proved in some cases and which is expected to hold in general). The methods and ideas of the first construction of square integrable representations of segment type played an important role in development of the ideas which lead to the general construction (the ideas of [T11] also played a role in the general construction[2]).

Further, after the construction in [MœT], a number of important and difficult questions arises about the constructed square integrable representations (characters, formal degrees, Plancherel measures, ...). About most of these questions we know very little, or almost nothing, even for the square integrable representations of the segment type (this lack of knowledge starts already with $Sp(4)$). About square integrable representations of segment type exists a much more explicit understanding then about the general ones. Therefore, it is natural to try to understand first these questions for for the square integrable representations of the segment type. Because of this, we shall probably deal a lot in the future with them (at least for some time).

The first approach to the segment representations may be useful for the study of generic irreducible square integrable representations.[3]

These observations suggest us that it may be of interest to have available the original approach to the square integrable representation of segment type. Further, after reading this original approach to the square integrable representations of the segment type, it might be easier to understand some much more complicated ideas, which show up in the later papers, in particular in [MœT]. These are some of the reasons that convinced us to publish the initial original approach to the square integrable representations of the segment type

We shall now describe the content of the paper. The basic notation is introduced in the first section. The second section describes the square integrable representations of Steinberg type and introduces generic reducibilities. In the third and

[2]Let us note that preprint [T11] is a slight generalization of the second part of preprint [T9].

[3]Square integrable representations of segment type whose construction we present in this paper, can be used to construct in a pretty direct way a significant family of much more general irreducible square integrable representations, as it was done in [T11]. The importance of this more general family follows from the fact that it includes, for example, all the generic irreducible square integrable representations ([Mu1]).

fourth sections we give a construction of the square integrable representations of the segment type, in the case of cuspidal reducibility at 1/2. The fifth section is giving such construction for the unitary reducibility, while the sections six and seven present the construction for the reducibility at 1.

1. Preliminaries

In this section we shall recall of the notation used in this paper. Since the same notation was introduced and used in [T10], [T7], [T5] and some other papers, we shall very briefly recall of the notation (for complete explanations one needs to consult these papers). Part of this notation is a standard notation introduced by J. Bernstein and A.V. Zelevinsky in the case of p-adic general linear groups.

We shall denote by F a local non-archimedean field of characteristic different from two. The modulus character of F will be denoted by $|\ \ |_F$. The character $|\det|_F$ of $GL(n,F)$ will be denoted by ν. For admissible representations π_i of $GL(n_i,F)$, for $i=1,2$, J. Bernstein and A.V. Zelevinsky denote by $\pi_1\times\pi_2$, the representation of $GL(n_1+n_2,F)$ parabolically induced from $\pi_1\otimes\pi_2$ (then $\pi_1\times(\pi_2\times\pi_3)\cong(\pi_1\times\pi_2)\times\pi_3)$.

There is a natural ordering on the Grothendieck group $\mathfrak{R}(G)$ of the category of all admissible representations which have finite length, of a reductive group G over F. The canonical mapping from the objects of the category to the Grothendieck group will be denoted by s.s. (the image is just a cone of positive elements). The set of all equivalence classes of irreducible admissible representations of G is denoted by $\tilde G$, while the set of unitarizable classes in $\tilde G$ is denoted by $\hat G$.

Let $R_n=\mathfrak{R}(GL(n,F))$ and $R=\oplus_{n\geq 0}R_n$. The multiplication $\times$ between representations lifts to a multiplication on R, which we denote again by $\times$. The mapping $R\otimes R\to R$ which we get by factorization of $\times : R\times R\longrightarrow R$ is denoted by m.

Let π be an admissible representation π of $GL(n,F)$ of finite length and, let $\alpha=(n_1,\dots,n_k)$ be an ordered partition of n. Denote by P_α^{GL} the standard parabolic subgroup of $GL(n,F)$ whose Levi factor M_α^{GL} is naturally isomorphic to $GL(n_1,F)\times\ldots\times GL(n_k,F)$, and denote by $r_\alpha(\pi)$ the Jacquet module of π with respect to P_α^{GL}. We can, and we shall consider s.s. $(r_\alpha(\pi))$ as an element of $R_{n_1}\otimes\ldots\otimes R_{n_k}$. Let $m^*(\pi)=\sum_{k=0}^n \text{s.s.}\big(r_{(k,n-k)}(\pi)\big)\in R\otimes R$. We lift m^* to an additive homomorphism from R to $R\times R$.

We denote by ${}^tg$ (resp. ${}^\tau g$) the transposed matrix of a matrix g (resp. the transposed matrix of g with respect to the second diagonal). By ${}^\tau\pi^{-1}$ we denote the representation $g\mapsto\pi({}^\tau g^{-1})$ (π is a representation of $GL(n,F)$). Further, $\tilde\pi$ denotes the contragredient representation of π. For irreducible π we have ${}^\tau\pi^{-1}\cong\tilde\pi$.

If an irreducible admissible representation π of $GL(n,F)$ is a subquotient of $\rho_1\times\cdots\times\rho_k$, where ρ_i are irreducible cuspidal representations of $GL(n_i,F)$, then the multiset $(\rho_1,\dots,\rho_k)$ is called the support of π, and denoted by $\text{supp}(\pi)=(\rho_1,\dots,\rho_k)$. Suppose that π has finite length and that any irreducible subquotient π' of π has $\text{supp}(\pi')=(\rho_1,\dots,\rho_k)$. Then we say that π has a support and we shall write $\text{supp}(\pi)=(\rho_1,\dots,\rho_k)$. Similarly we define if $\pi\in R_n,\ \pi>0$ has support.

The $n\times n$ matrix having 1's on the diagonal (resp. on the second diagonal) and all other entries 0, will be denoted by I_n (resp. J_n). Then

$$Sp(n,F) = \left\{ S \in GL(2n,F); \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^tS \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} S = I_{2n} \right\}$$

and $SO(2n+1,F) = \{S \in GL(2n+1,F); {}^\tau SS = I_{2n+1}\}$ (we take $Sp(0,F)$ to be the trivial group). In the rest of the paper we shall fix one of the above two series of groups, and denote by S_n the group of the rank n from the fixed series. Fix the minimal parabolic subgroup $P_{\min}$ in S_n consisting of all upper triangular matrices in the group (parabolic subgroups containing $P_{\min}$ are called standard).

If X_i are square matrices, then we denote by q-diag $(X_1, \cdots, X_k)$ the quasi diagonal matrix which has on the quasi diagonal matrices $X_1, \cdots, X_k$. For an ordered partition $\alpha = (n_1, \dots, n_k)$ of some non-negative integer $m \leq n$ into positive integers, let

$$M_\alpha = \left\{ \text{q-diag}\,(g_1, \cdots, g_k, h, \ {}^\tau g_k^{-1}, \cdots, \ {}^\tau g_1^{-1}); g_i \in GL(n_i,F), h \in S_{n-m} \right\}$$

(for $m=0$, the only such partition will be denoted by (0)). Now $P_\alpha = M_\alpha P_{\min}$ is a standard parabolic subgroup of S_n and M_α is its Levi factor (we denote by N_α the unipotent radical of P_α). Obviously, M_α is naturally isomorphic to $GL(n_1,F) \times \ldots \times GL(n_k,F) \times S_{n-m}$. This fact enables us to identify $\tilde{M}_\alpha$ with $GL(n_1,F)\tilde{} \times \cdots \times GL(n_k,F)\tilde{} \times \tilde{S}_{n-m}$.

For admissible representations π and σ of $GL(n,F)$ and S_m respectively, we denote by $\pi \rtimes \sigma$ the representation of S_{n+m} parabolically induced from the representation $\pi \otimes \sigma$ of $P_{(n)}$. Now

$$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma, \tag{1-1}$$

$$(\pi \rtimes \sigma)^{\sim} \cong \tilde{\pi} \rtimes \tilde{\sigma}. \tag{1-2}$$

Let $R_n(S) = \mathfrak{R}(S_n)$ and $R(S) = \oplus_{n \geq 0} R_n(S)$ (note that we have natural orderings on R and $R(S)$, and therefore also on $R \otimes R(S)$). Lift the multiplication $\rtimes$ among representations, to a multiplication $\rtimes : R \times R(S) \to R(S)$. With respect to this multiplication, $R(S)$ is an R-module. For $\pi \in R$ and $\sigma \in R(S)$ we have

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma, \tag{1-3}$$

where $\sim$ denotes the contragredient involution on R. Denote by $\mu : R \otimes R(S) \to R(S)$ the factorization of $\rtimes : R \times R(S) \to R(S)$.

For a smooth representation σ of S_n of finite-length and an ordered partition $\alpha = (n_1, \dots, n_k)$ of $0 \leq m \leq n$, the Jacquet module of σ with respect to P_α will be denoted by $s_\alpha(\sigma)$ (we shall consider s.s. $(s_\alpha(\sigma)) \in R_{n_1} \otimes \cdots \otimes R_{n_k} \otimes R_{n-m}(S)$). For irreducible σ let

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}\left(s_{(k)}(\sigma)\right),$$

and extend μ^* to an additive mapping $R(S) \to R \otimes R(S)$. Define $s : R \otimes R \to R \otimes R$ by $s(\sum_i x_i \otimes y_i) = \sum_i y_i \otimes x_i$. Since R is R-module, $RR \otimes (S)$ is $R \otimes R$-module in a natural way. If we denote

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*,$$

then

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma). \tag{1-4}$$

For an admissible representation $\pi \otimes \sigma$ of $GL(n, F) \times S_m$, we shall say that $\pi \otimes \sigma$ has GL-support if σ is an irreducible cuspidal representation and if π has support. In that case we define

$$\mathrm{supp}_{GL}(\pi \otimes \sigma) = \mathrm{supp}(\pi).$$

Further, for $\pi \otimes \sigma \in R_n \otimes R_m$ such that $\pi > 0$ and that σ is an irreducible cuspidal representation, we define in a similar way if $\pi \otimes \sigma$ has a GL-support. If τ is an irreducible admissible representation of S_m, then there exist an irreducible representations π of $GL(n, F)$, and an irreducible cuspidal representation σ of S_{m-n} such that τ is a subquotient of $\pi \rtimes \sigma$.Denote

$$\mathrm{depth}_{GL}(\tau) = n.$$

For an admissible representation τ of S_m of finite length, such that $\mathrm{depth}_{GL}(\tau') = d$ for any irreducible subquotient τ' of τ, we say that it has a depth (and write $\mathrm{depth}_{GL}(\tau) = d$). Similarly we define depth of $\tau \in R_n(S)$, $\tau > 0$. For an admissible representation τ of finite length which has a depth, define

$$s_{GL}(\tau) = s_{(\mathrm{depth}_{GL}(\tau))}(\tau).$$

Similarly we define $s_{GL}(\tau)$ for $\tau \in R_n(S)$, $\tau > 0$, if τ has a depth.

2. Square integrable representations of Steinberg type

Denote by $\mathcal{C}$ all equivalence classes of irreducible cuspidal representations of $GL(p, F)$ for all $p \geq 1$. The set of all segments $[\rho, \nu^n\rho] = \{\rho, \nu\rho, \nu^2\rho, \dots, \nu^n\rho\}$ in irreducible cuspidal representations of general linear groups (i.e. in $\mathcal{C}$) is denoted by $\mathcal{S}(\mathcal{C})$. The irreducible essentially square integrable representation attached to $[\rho, \nu^n\rho] \in \mathcal{S}(\mathcal{C})$ will be denoted by $\delta([\rho, \nu^n\rho])$ (it is the unique irreducible subrepresentation of $\nu^n\rho \times \nu^{n-1}\rho \times \dots \times \nu\rho \times \rho$). For $n < 0$ we take $[\rho, \nu^n\rho] = \emptyset$, and define $\delta(\emptyset)$ to be $1 \in R$. Then

$$m^* \left(\delta([\rho, \nu^n\rho])\right) = \sum_{k=-1}^{n} \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \delta([\rho, \nu^k\rho]). \tag{2-1}$$

We shall often use a simple modification of this formula:

$$s \left(m^* \left(\delta([\rho, \nu^n\rho])\right)\right) = \sum_{k=-1}^{n} \delta([\rho, \nu^k\rho]) \otimes \delta([\nu^{k+1}\rho, \nu^n\rho]).$$

Further, $r_{(m)^{n+1}}(\delta([\rho,\nu^n\rho])) = \nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\rho$, where $(m)^{n+1}$ denotes $(m,m,\dots,m)\in\mathbb{Z}^{n+1}$.

Let D be the set of all equivalence classes of the irreducible essentially square integrable representations of $GL(n,F)$'s for all $n\geq 1$. For $\delta\in D$ there exists a unique $e(\delta)\in\mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Denote $\nu^{-e(\delta)}\delta$ by δ^u. Then $\delta=\nu^{e(\delta)}\delta^u$, where $e(\delta)\in\mathbb{R}$ and δ^u is unitarizable. Denote by $M(D)$ the set of all finite multisets in D. For $d=(\delta_1,\dots,\delta_k)\in M(D)$ take a permutation p of the set $\{1,\dots,k\}$ such that $e(\delta_{p(1)})>e(\delta_{p(2)})\cdots>e(\delta_{p(k)})$. Then the representation $\delta_{p(1)}\times\cdots\times\delta_{p(k)}$ has a unique irreducible quotient, which is denoted by $L(d)$. Now $d\mapsto L(d)$ is Langlands' classification for general linear groups. We shall usually write $L(\delta_1,\dots,\delta_k)$ instead of $L((\delta_1,\dots,\delta_k))$.

Denote $D_+=\{\delta\in D; e(\delta)>0\}$ and let $M(D_+)$ denote the set of all finite multisets in D_+. Denote by $T(S)$ be the set of all equivalence classes of the irreducible tempered admissible representations of S_n's for all $n\geq 0$. For $t=((\delta_1,\dots,\delta_n),\tau)\in M(D_+)\times T(S)$ take a permutation p of the set $\{1,2,\dots,n\}$ such that $e(\delta_{p(1)})\geq e(\delta_{p(2)})\geq\dots\geq e(\delta_{p(n)})$. Then representation $\delta_{p(1)}\times\delta_{p(2)}\times\dots\times\delta_{p(n)}\rtimes\tau$ has a unique irreducible quotient, which is denoted by $L(t)$. This is Langlands' classification for groups S_m and $t\mapsto L(t)$ is a one-to-one parameterization. Similarly as before, we write usually $L(t)=L(((\delta_1,\dots,\delta_n),\tau))$ simply as $L((\delta_1,\dots,\delta_n),\tau)$ or $L(\delta_1,\dots,\delta_n,\tau)$.

Now we shall recall of the square integrable representations of Steinberg type ([T5]).

2.1. Theorem. *Fix an irreducible unitarizable cuspidal representation ρ of $GL(\ell,F)$ and fix a similar representation σ of S_m. Suppose that $\nu^\alpha\rho\rtimes\sigma$ reduces for some $\alpha>0$. Then $\rho\cong\tilde\rho$. The representation $\nu^{\alpha+n}\rho\times\nu^{\alpha+n-1}\rho\times\cdots\nu^{\alpha+1}\rho\times\nu^\alpha\rho\rtimes\sigma$ has a unique irreducible subrepresentation which we denote by $\delta([\nu^\alpha\rho,\nu^{\alpha+n}\rho],\sigma)$ $(n\geq 0)$. We have $s_{(\ell)^{n+1}}(\delta([\nu^\alpha\rho,\nu^{\alpha+n}\rho],\sigma))=\nu^{\alpha+n}\rho\otimes\nu^{\alpha+n-1}\rho\otimes\cdots\otimes\nu^{\alpha+1}\rho\otimes\nu^\alpha\rho\otimes\sigma$ (here $(\ell)^{n+1}=(\ell,\ell,\dots,\ell)\in\mathbb{Z}^{n+1}$) and*

$$\mu^*\left(\delta([\nu^\alpha\rho,\nu^{\alpha+n}\rho],\sigma)\right)=\sum_{k=-1}^{n}\delta([\nu^{\alpha+k+1}\rho,\nu^{\alpha+n}\rho])\otimes\delta([\nu^\alpha\rho,\nu^{\alpha+k}\rho],\sigma)$$

The representation $\delta([\nu^\alpha\rho,\nu^{\alpha+n}\rho],\sigma)$ is square integrable and we have $\delta([\nu^\alpha\rho,\nu^{\alpha+n}\rho],\sigma)\tilde{}$ $\cong\delta([\nu^\alpha\rho,\nu^{\alpha+n}\rho],\tilde\sigma)$. (We take $\delta(\emptyset,\sigma)$ in the above formula to be just σ.)

Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ and let σ be an irreducible cuspidal representation of S_q. It is well-known that if $\nu^\alpha\rho\rtimes\sigma$ reduces for same $\alpha\in\mathbb{R}$, then $\rho\cong\tilde\rho$. One proves this in a similar way as in the GSp-case in [T2] (here the proof is even much simpler then there). The converse of this fact holds: if $\rho\cong\tilde\rho$, then $\nu^\alpha\rho\rtimes\sigma$ reduces for some $\alpha\in\mathbb{R}$. The argument is following. Suppose that $\rho\cong\tilde\rho$ and that $\rho\rtimes\sigma$ does not reduce. Then one can chose $\alpha_0>0$ such that $\nu^\alpha\rho\rtimes\sigma$ is irreducible for $0\leq\alpha<\alpha_0$,. These representations are unitarizable (they form a complementary series). Since matrix coefficients of unitarizable representations are bounded and the Jacquet module is s.s.$(s_{(p)}(\nu^\alpha\rho\rtimes\sigma))=\nu^\alpha\rho\otimes\sigma+\nu^{-\alpha}\rho\otimes\sigma$, the connection of asymptotic of matrix coefficients and

Jacquet modules in [C] implies that there must exist $\alpha_0 > 0$ such that $\nu^{\alpha_0}\rho \rtimes \sigma$ reduces (one can even get an explicit upper bound for such α_0).

An admissible representation ρ shall be called selfdual if $\rho \cong \tilde{\rho}$. If representation is selfdual, then it is unitarizable. Let $\rho \in \mathcal{C}$ be selfdual, and let σ be an irreducible cuspidal representation of S_q. In this paper we shall deal with pairs (ρ, σ) which satisfy the following condition.

(C) There exists $\alpha_0 \in \{0, 1/2, 1\}$ such that $\nu^{\alpha_0}\rho \rtimes \sigma$ reduces, and $\nu^{\beta}\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}$, $|\beta| \neq \alpha_0$.

The condition (C) holds for any ρ, if $q = 0$ ([Sh2]).

If (ρ, σ) as above satisfies (C), then it satisfies exactly one of the following three conditions:

(C0) $\rho \rtimes \sigma$ reduces and $\nu^{\beta}\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}^{\times}$;

(C1/2) $\nu^{1/2}\rho \rtimes \sigma$ reduces and $\nu^{\beta}\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}\backslash\{\pm 1/2\}$;

(C1) $\nu\rho \rtimes \sigma$ reduces and $\nu^{\beta}\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}\backslash\{\pm 1\}$

(we follow the notation of the Jantzen's paper [J]).

The following fact proved in [T5] explains why only selfdual irreducible cuspidal representations of general linear groups are interesting for the construction of irreducible square integrable representations of groups S_m.

2.2. Proposition. *Let $\rho_1, \rho_2, \dots, \rho_n \in \mathcal{C}$, and let σ be an irreducible cuspidal representation of S_q. Suppose that $\rho_1 \times \rho_2 \times \cdots \times \rho_n \rtimes \sigma$ contains a square integrable subquotient. Then all ρ_i^u are selfdual representations.*

In [T5] we have got a number of other conditions which must be satisfied by $\rho_1, \rho_2, \dots, \rho_n$ and σ as above.

3. Reducibility at 1/2, I

We fix an irreducible unitarizable cuspidal representation ρ of $GL(p, F)$ and an irreducible cuspidal representation σ of S_q. We shall assume in this section that $\nu^{1/2}\rho \rtimes \sigma$ reduces (thus $\rho \cong \tilde{\rho}$), and that $\nu^{\alpha}\rho \cong \tilde{\rho}$ is irreducible for $\alpha \in \mathbb{R}\backslash\{\pm 1/2\}$. In other words, we assume that (ρ, σ) satisfies (C1/2).

3.1. Lemma. *Suppose that $m_1, m_2, \dots, m_k$ are integers which satisfy $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_{k-1} \geq m_k \geq 0$. Let $\Delta_i = [\nu^{1/2}\rho, \nu^{m_i+1/2}\rho]$ and*

$$\tau = \nu^{1/2}\rho \times \nu^{3/2}\rho \times \cdots \times \nu^{m_1+1/2}\rho \times \nu^{1/2}\rho \times \nu^{3/2}\rho \times \cdots \times \nu^{m_2+1/2}\rho \times \cdots \times \nu^{1/2}\rho \times \nu^{3/2}\rho \times \cdots \times \nu^{m_k+1/2}\rho \rtimes \sigma.$$

Then

(i) $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \otimes \sigma$ *is a subquotient of* $s_{GL}(\tau)$*. The multiplicity in* $s_{GL}(\tau)$ *is one.*
(ii) There exists a unique irreducible subquotient π *of* τ *such that* $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \otimes \sigma$ *is a subquotient of* $s_{GL}(\pi)$*. The multiplicity of* π *in* τ *is one and* π *is the unique irreducible subrepresentation of* $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$*.*

Proof. From (1-4) we get by induction

$$\begin{aligned} s_{GL}(\tau) = \sum \nu^{e_{(1,1/2)}1/2}\rho \times \nu^{e_{(1,3/2)}3/2}\rho \times \cdots \times \nu^{e_{(1,m_1+1/2)}(m_1+1/2)}\rho \\ \times \nu^{e_{(2,1/2)}1/2}\rho \times \nu^{e_{(2,3/2)}3/2}\rho \times \cdots \times \nu^{e_{(2,m_2+1/2)}(m_2+1/2)}\rho \times \dots \\ \dots \times \nu^{e_{(k,1/2)}1/2}\rho \times \nu^{e_{(k,3/2)}3/2}\rho \times \cdots \times \nu^{e_{(k,m_k+1/2)}(m_k+1/2)}\rho \otimes \sigma \end{aligned}$$

where the sum runs over all possible $e_{(i,j+1/2)} \in \{\pm 1\}, 1 \leq i \leq k, 0 \leq j \leq m_i$. Now (i) follows directly. Further, (i) implies that the multiplicity of π in τ is one. The Frobenius reciprocity implies that every irreducible subrepresentation of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$ has $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \otimes \sigma$ for a quotient of suitable Jacquet module. Therefore, there exists a unique irreducible subrepresentation, and it is π. $\square$

Note that above definition in the case of $k = 1$ agrees with the definition of square integrable representation of Steinberg type (see Theorem 2.1), which was denoted by $\delta(\Delta_1) = \delta([\nu^{1/2}\rho, \nu^{m_1+1/2}\rho], \sigma)$. If $k = 2$, then we shall denote the representation defined in the lemma by

$$\delta([\nu^{-1/2-m_2}\rho, \nu^{m_1+1/2}\rho], \sigma).$$

The tempered representations which we consider in the following theorem play an important role in the construction of irreducible square integrable representations.

3.2. Theorem. *Let* $n \in \mathbb{Z}$*,* $n \geq 0$*, and suppose that* $\nu^{1/2}\rho \rtimes \sigma$ *reduces. Then:*
(i) $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma$ *and* $\delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho], \sigma)$ *contain a unique common irreducible subquotient. That subquotient is* $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$*.*

(ii)
$$\begin{aligned} &s.s.\left(s_{((2n+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)\right)\right) \\ &= \sum_{k=0}^{n+1} \delta([\nu^{-k+1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{k+1/2}\rho, \nu^{n+1/2}\rho]) \otimes \sigma. \end{aligned}$$

(iii) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma$ *is a direct sum of two irreducible nonequivalent subrepresentations. One of them is* $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$*. Denote the other one by* $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$*. We have* $s.s.\left(s_{((2n+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)\right)\right) + \delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho])^2 \otimes \sigma = s.s.\left(s_{((2n+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)\right)\right)$*.*
(iv) Representations $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$ *and* $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$ *are tempered. They are not square integrable.*

(v)
$$\begin{aligned} \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)\tilde{} \cong \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \tilde{\sigma}), \\ \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)\tilde{} \cong \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \tilde{\sigma}). \end{aligned}$$

Proof. From (2-1) and (1-4) we obtain

$$\text{(3-1)}\qquad \text{s.s.}\left(s_{((2n+2)p)}\left(\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\right)\right)$$
$$=\sum_{k=-n-1}^{n+1}\delta([\nu^{-k+1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma$$
$$=\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\otimes\sigma+2\sum_{k=1}^{n+1}\delta([\nu^{-k+1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma$$

From this we can conclude that $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$ is a multiplicity one representation of length ≤ 2 (use the Frobenius reciprocity and the fact that $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$ is completely reducible, because this representation is unitarizable). We look further at

$$\text{(3-2)}\qquad \text{s.s.}\left(s_{((2n+2)p)}\left(\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma)\right)\right)$$
$$=\left[\sum_{k=0}^{n+1}\delta([\nu^{-k+1/2}\rho,\nu^{-1/2}\rho])\times\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\right]\times\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma$$

We shall now write all common irreducible subquotients of (3-1) and (3-2). They are

$$\text{(3-3)}\qquad \delta([\nu^{-k+1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma,\ \ k=0,1,\dots,n+1.$$

Multiplicities in (3-1) of above representation are all two, except of the first one (for $k=0$), which is one. The multiplicities of above representation in (3-2) are all 1. Write now

(3-4)
$$\text{s.s.}\left(s_{((2n+2)p)}\left(\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\rtimes\sigma\right)\right)$$
$$=\left[\sum_{k=0}^{n+1}\delta([\nu^{-k+1/2}\rho,\nu^{-1/2}\rho])\times\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\right]^2\otimes\sigma.$$

We shall determine multiplicities of representations from (3-3) in (3-4). Note that if we look at a fixed representation from (3-3), then the cuspidal representations which appear in the support form a segment which ends with $\nu^{n+1/2}\rho$. In general, the support is of the form $(\nu^{1/2-k}\rho,\nu^{1/2-k+1}\rho,\dots,\nu^{1/2+n}\rho)+(\nu^{k+1/2}\rho,\nu^{k+3/2}\rho,\dots,\nu^{1/2+n}\rho)$ where $k=0,1,\dots,n,n+1$. It is now easy to conclude that the multiplicities of representation from (3-3) in (3-4) are the same as the multiplicities in (3-1).

Note that $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\leq\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\rtimes\sigma$ and

$$\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma)\leq\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\rtimes\sigma.$$

We always consider inequalities as above, as inequalities between semi simplifications in the Grothendieck group of the corresponding category of smooth representations of finite length. We obtain easily from (3-1) and (3-2)

$$\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\not\leq\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma),$$
$$\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma)\not\leq\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$$

(if we would have somewhere above inequality, then the inequality would hold between all Jacquet modules, what can not be by (3-1) and (3-2)). This, and the multiplicities of representations of (3-3) in (3-1), (3-2) and (3-4) imply that there exists a unique common irreducible subquotient of $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$ and $\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma)$. Since this common irreducible subquotient must have in the Jacquet module $\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\otimes\sigma$ (as a subquotient), it must be $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)$. Therefore, (i) holds. The calculation of multiplicities implies (ii). Now (iii) follows from (ii) and (3-1). Further, (iv) is a consequence of the square integrability criterion. Finally, we get (v) using the characterization in (i). $\square$

3.3. Theorem. *Let $n,m\in\mathbb{Z}$, $m>n\geq 0$. Suppose that (ρ,σ) satisfies (C1/2). Then:*

(i)
$$s.s.\left(s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)\right)\right)=\sum_{k=0}^{n+1}\delta([\nu^{1/2-k}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{1/2+k}\rho,\nu^{n+1/2}\rho])\otimes\sigma.$$

(ii) The representation $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ is square integrable.
(iii) The representation $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ is a unique common irreducible subquotient of $\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m-1/2}\rho],\sigma)$ and $\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho],\sigma)$.
(iv) $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)\tilde{}\cong\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\tilde{\sigma})$.

Proof. Write

$$\text{(3-5)}\qquad \text{s.s.}\left(s_{((n+m+2)p)}\left(\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho],\sigma)\right)\right)$$
$$=(\nu^{n+1/2}\rho+\nu^{-n-1/2}\rho)\times\left[\sum_{k=0}^{n}\delta([\nu^{1/2-k}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{1/2+k}\rho,\nu^{n-1/2}\rho])\right]\otimes\sigma,$$

$$\text{(3-6)}\qquad \text{s.s.}\left(s_{((n+m+2)p)}\left(\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m-1/2}\rho],\sigma)\right)\right)$$
$$=(\nu^{m+1/2}\rho+\nu^{-m-1/2}\rho)\times\left[\sum_{k=0}^{n+1}\delta([\nu^{1/2-k}\rho,\nu^{m-1/2}\rho])\times\delta([\nu^{1/2+k}\rho,\nu^{n+1/2}\rho])\right]\otimes\sigma.$$

Common irreducible subquotients of (3-5) and (3-6) can not contain in the GL-supports $\nu^{-m-1/2}\rho$ (see (3-5)). Also, the representations in the GL-supports of each common irreducible subquotient will form a segment which ends with $\nu^{m+1/2}\rho$ (see (3-6) and use the above remark about $\nu^{-m-1/2}\rho$). We shall now write all pairs from (3-5) and (3-6) which can have common irreducible subquotients. They are

$$\begin{gathered}\nu^{m+1/2}\rho \times \delta([\nu^{-n-1/2}\rho, \nu^{m-1/2}\rho]) \otimes \sigma \quad \text{and} \\ \nu^{-n-1/2}\rho \times \delta([\nu^{-n+1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma; \\ \nu^{m+1/2}\rho \times \delta([\nu^{1/2-k}\rho, \nu^{m-1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n+1/2}\rho]) \otimes \sigma \quad \text{and} \\ \nu^{n+1/2}\rho \times \delta([\nu^{1/2-k}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n-1/2}\rho]) \otimes \sigma, \ \text{for } k = 0, 1, \ldots, n.\end{gathered} \tag{3-7}$$

We can now write easily the common irreducible factors of (3-5) and (3-6) from (3-7). They are

$$\delta([\nu^{1/2+k}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{1/2-k}\rho, \nu^{m+1/2}\rho]) \otimes \sigma, \ k = 0, 1, \ldots, n+1. \tag{3-8}$$

Multiplicities of the representations from (3-8) in (3-5) and (3-6) are all equal to one.

We further consider

$$\begin{aligned} \text{s.s.} &\left(s_{((n+m+2)p)}\left(\nu^{n+1/2}\rho \times \nu^{m+1/2}\rho \rtimes \delta([\nu^{-n+1/2}\rho, \nu^{m-1/2}\rho], \sigma\right)\right) \\ &= (\nu^{n+1/2}\rho + \nu^{-n-1/2}\rho) \times (\nu^{m+1/2}\rho, \nu^{-m-1/2}\rho) \\ &\times \left[\sum_{k=0}^{n} \delta([\nu^{1/2-k}\rho, \nu^{m-1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n-1/2}\rho])\right] \otimes \sigma. \end{aligned} \tag{3-9}$$

We want to see multiplicities of representations from (3-8) in (3-9). We need to consider only the following terms in the sum

$$\begin{gathered}\nu^{-n-1/2}\rho \times \nu^{m+1/2}\rho \times \delta([\nu^{-n+1/2}\rho, \nu^{m-1/2}\rho]) \otimes \sigma, \\ \nu^{n+1/2}\rho \times \nu^{m+1/2}\rho \times \delta([\nu^{1/2-k}\rho, \nu^{m-1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n-1/2}\rho]) \otimes \sigma, \ k = 0, 1, \ldots, n.\end{gathered}$$

It is easy to get now that all multiplicities are 1.

From the definition of representations $\delta([\nu^{-n'-1/2}\rho, \nu^{m'+1/2}\rho], \sigma)$ we get

$$\begin{aligned}\nu^{n+1/2}\rho \rtimes \delta([\nu^{-n+1/2}\rho, \nu^{m+1/2}\rho], \sigma) &\leq \nu^{n+1/2}\rho \times \nu^{m+1/2}\rho \rtimes \delta([\nu^{-n+1/2}\rho, \nu^{m-1/2}\rho], \sigma), \\ \nu^{m+1/2}\rho \rtimes \delta([\nu^{-n-1/2}\rho, \nu^{m-1/2}\rho], \sigma) &\leq \nu^{n+1/2}\rho \times \nu^{m+1/2}\rho \rtimes \delta([\nu^{-n+1/2}\rho, \nu^{m-1/2}\rho], \sigma).\end{aligned}$$

This, together with the multiplicities that we have computed, implies that if we write $\leq$ instead of $=$ in (i), then such inequality holds. For the opposite inequality we shall first prove

$$\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{((n+m+2)p)}(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma). \tag{3-10}$$

To prove this, observe that

$$\delta([\nu^{-m-1/2}\rho,\nu^{m+1/2}\rho],\sigma)\le\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$$

(one checks that the subquotient of the Jacquet module of $\delta([\nu^{-m-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ which characterizes this representation must be in the Jacquet module of the right hand side). Thus

$$\delta([\nu^{-m-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$$
$$\le s_{((2m+2)p)}(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)).$$

The formula for the above Jacquet module and the inequality (i) that we have already proved, imply (3-10) now.

We shall use now (3-10). The representation on the left hand side of (3-10) must be a direct summand of the Jacquet module on the right hand side of (3-10) (see the central characters and use the inequality $\le$ from (i) which we have proved). Thus for $n>0$

$$\begin{aligned}\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)\hookrightarrow&\nu^{m+1/2}\rho\times\cdots\times\nu^{-n+1/2}\rho\times\nu^{-n-1/2}\rho\rtimes\sigma\\ \cong&\nu^{m+1/2}\rho\times\cdots\times\nu^{-n+1/2}\rho\times\nu^{n+1/2}\rho\rtimes\sigma.\end{aligned}$$

Using the Frobenius reciprocity and comparing with GL-supports of representations in (3-7), we can conclude that $\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho])\times\nu^{n+1/2}\rho\otimes\sigma$ is in the Jacquet module. Proceeding in the same way we shall get all other members except $\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma$. The last representation is by definition in the Jacquet module of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$. This finishes the proof of (i). The square integrability criterion and (i) imply (ii) (use [Z1]). Now it is easy to get (iii) from (i) and our previous considerations. One gets (iv) by induction using the characterization in (iii), and Theorems 2.1 and 3.2. $\square$

3.4. Remark. It seems that it would be equally convenient to use the representations $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)$ and $\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ for the upper estimate of the Jacquet module of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ in the last proof.

4. Reducibility at 1/2, II

As in the previous section, we fix an irreducible unitarizable cuspidal representation ρ of $GL(p,F)$ and an irreducible cuspidal representation σ of S_q such that (ρ,σ) satisfies (C1/2).

4.1. Lemma. *Let $n,m\in\mathbb{Z}$, $m\ge n\ge 0$. Then:*
(i) For $k=1,2,\dots,n,n{+}1$, multiplicity of $\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-k+1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma\right)$ is 2. In particular, the multiplicities of

$$\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\quad\textit{and}\quad\delta([\nu^{3/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$$

in $s_{GL}\left(\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma\right)$ are both 2.

(ii) Multiplicity of $\delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ *in*

$$s_{((n+m+2)p)}(\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \nu^{-n+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma)$$

is 2.
(iii) If π *is an irreducible subquotient of* $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ *such that*

$$\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{((n+m+2)p)}(\pi),$$

then $2\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \not\leq s_{((n+m+2)p)}(\pi)$.

Proof. The claim (i) follows from the following formula

$$\text{s.s.}\left(s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma\right)\right)$$
$$= \sum_{i=-m-1}^{n+1} \delta([\nu^{i+1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-i+1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$$

(use (1-4) and (2-1) to get the formula). The claim (ii) follows from the first formula in the proof of Lemma 3.1.

We know that $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ is a subquotient of $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$, and that this irreducible representation satisfies two conditions from (iii) (see Theorems 3.2 and 3.3). This, together with (i) and (ii), implies (iii). $\square$

4.2. Theorem. *Let* $n, m \in \mathbb{Z}$, $m \geq n \geq 0$.
(i) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ *contains exactly two irreducible subquotients* π *which satisfy* $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{((n+m+2)p)}(\pi)$. *One of these subquotients is* $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$. *The other one we denote by*

$$\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma).$$

Then $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma) \not\cong \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$.
(ii) The multiplicity of $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ *in*

$$\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \nu^{-n+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma$$

is one.
(iii) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ *can be characterized as a unique irreducible subquotient* π *of* $\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \nu^{-n+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma$ *which satisfies conditions*

$$\delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{((n+m+2)p)}(\pi)$$
$$\delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \not\leq s_{((n+m+2)p)}(\pi).$$

(iv)
$$s.s.\left(s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)\right)\right)$$
$$= \sum_{i=0}^{n} \delta([\nu^{-i-1/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{i+3/2}\rho, \nu^{n+1/2}\rho]) \otimes \sigma.$$

(v) If $m > n$, then the representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ is square integrable.
(vi) $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)\tilde{} \cong \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \tilde{\sigma})$.

We define $\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ to be $\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)$. This convention is useful bellow in the proofs by induction.

Proof. From the previous lemma, one directly gets (i) and (ii).

Recall that the multiplicity of $\delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ in

$$s_{((n+m+2)p)}(\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \nu^{-n+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma)$$

and $s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma\right)$ is 2 in both cases, while multiplicity in

$$s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)\right)$$

is 1.

We now prove (iii) and (iv) by induction on $n+m$. For $n = m$ we know that both claims hold (Theorem 3.2). Therefore, it is enough to consider the case $n < m$. We assume this, and we assume that the claims (iii) and (iv) hold for m', n' such that $m' + n' < m + n$.

From Theorem 3.3 and the previous lemma we see that there exists a unique subquotient π of $\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma$ such that

$$\delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{((n+m+2)p)}(\pi),$$
$$\delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \not\leq s_{((n+m+2)p)}(\pi).$$

The previous lemma implies that π is a subquotient of $\delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$. Otherwise, $\delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ would have multiplicity at least 3 in $s_{((n+m+2)p)}(\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma)$, what can not be by the previous lemma.

If $n > 0$, then (1-4) and Theorems 3.2 and 3.3 imply

$$\begin{aligned} \text{(4-1)} \quad & \delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \\ & \qquad \leq s_{((n+m+2)p)}\left(\nu^{n+1/2}\rho \rtimes \delta([\nu^{-n+1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)\right), \end{aligned}$$

$$\begin{aligned} \text{(4-2)} \quad & \delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \\ & \qquad \not\leq s_{((n+m+2)p)}\left(\nu^{n+1/2}\rho \rtimes \delta([\nu^{-n+1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)\right); \end{aligned}$$

and

$$\begin{aligned} \text{(4-3)} \quad & \delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \\ & \qquad \leq s_{((n+m+2)p)}\left(\nu^{m+1/2}\rho \rtimes \delta([\nu^{-n-1/2}\rho, \nu^{m-1/2}\rho]_-, \sigma)\right), \end{aligned}$$

$$\text{(4-4)}\quad \delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma \not\leq s_{((n+m+2)p)}\left(\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m-1/2}\rho]_-,\sigma)\right).$$

We shall now consider the case $n=0$. Observe that (iii) is obvious for $n=0$. We shall now prove (iv) by induction with respect to m. For $n=0$, the formulas (4-3) and (4-4) hold (and also (4-1) holds, but (4-2) does not hold). This implies that $\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$ is a subquotient of $\nu^{m+1/2}\rho\rtimes\delta([\nu^{-1/2}\rho,\nu^{m-1/2}\rho]_-,\sigma)$. Now the inductive assumption and (1-4) imply

$$\text{(4-5)}\quad s_{((m+2)p)}\left(\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)\right) \leq (\nu^{m+1/2}\rho+\nu^{-m-1/2}\rho)\times\delta([\nu^{-1/2}\rho,\nu^{m-1/2}\rho])\otimes\sigma.$$

Note that

$$\text{(4-6)}\quad \text{s.s.}\left(s_{((m+2)p)}\left(\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\rtimes L(\nu^{1/2}\rho,\sigma)\right)\right) = \left[\sum_{i=-1}^{m}\delta([\nu^{-i-1/2}\rho,\nu^{-1/2}\rho])\times\delta([\nu^{i+3/2}\rho,\nu^{m+1/2}\rho])\right]\times\nu^{-1/2}\rho\otimes\sigma.$$

The above formula and Lemma 4.1 imply that $\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$ is a subquotient of $\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\rtimes L(\nu^{1/2}\rho,\sigma)$. This implies

$$s_{((m+2)p)}\left(\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-\sigma)\right)\leq s_{((m+2)p)}\left(\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\rtimes L(\nu^{1/2}\rho,\sigma)\right)$$

From this and formulas (4-5) and (4-6), now one can easily get the following estimate

$$s_{((m+2)p)}\left(\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)\right)\leq\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma.$$

Obviously, in the above relation the equality must hold. This finishes the proof for $n=0$.

Suppose now $n>0$. Relations (4-1), (4-2), (4-3) and (4-4) imply that π is a subquotient of $\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$ and $\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m-1/2}\rho]_-,\sigma)$. Now in the same way as in the proof of Theorem 3.3, one gets

$$\text{(4-7)}\quad s_{((n+m+2)p)}(\pi)\leq\sum_{i=0}^{n}\delta([\nu^{-i-1/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{i+3/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma.$$

One checks directly that $\delta([\nu^{3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ has multiplicity ≥ 1 in $s_{((2m+2)p)}\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\pi\right)$. Since

$$\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])^2\otimes\sigma\not\leq s_{((2m+2)p)}\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\pi\right)$$

(we can see it from (4-7)), we conclude that

$$\delta([\nu^{-m-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)\leq\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\pi.$$

From (4-7) and (1-4) follow easily that $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\leq s_{((n+m+2)p)}(\pi)$. Now in the same way as in the end of proof of Theorem 3.3 (see the last section of that proof), one gets

$$s_{((n+m+2)p)}(\pi)\geq\sum_{i=0}^{n}\delta([\nu^{-i-1/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{i+3/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma$$

The above two inequalities for $s_{((n+m+2)p)}(\pi)$ imply that in (4-7) we have an equality. This implies that $\pi=\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$, what is the claim of (iii). Now (iv) is obvious. Further, (iv) implies (v).

One can get (vi) considering $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ in the Jacquet module of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$, using (i) of Lemma 4.1, Theorems 3.2, 3.3, and Corollary 4.2.5 of [C]. We could also get (vi) using Proposition 3.6 of [J]. This finishes the proof of the theorem. $\square$

The following theorem gives a simple characterization of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ and $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$.

4.3. Theorem. *Let $n,m\in\mathbb{Z}$, $m>n\geq 0$. Then*
(i) $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ and $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$ are (isomorphic to) irreducible subrepresentations of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma$. Further, $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma$ does not contain any other irreducible subrepresentation.
(ii) The representation $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ (resp. $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$) is a unique irreducible subrepresentation of $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)$ (resp. $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho]_-,\sigma)$).

Proof. Denote $\pi=\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ (resp. $\pi_-=\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$). Now (i) of Theorem 3.3 (resp. (iv) of Theorem 4.2) and Theorem 7.3.2 of [C] imply that $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ is a direct summand in $s_{GL}(\pi)$ (resp. $s_{GL}(\pi_-)$). Frobenius reciprocity implies that there exists an embedding $\phi:\pi\hookrightarrow\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma$ (resp. $\phi_-:\pi_-\hookrightarrow\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma$). Suppose that π' is an irreducible subrepresentation of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma$, such that $\mathrm{Im}(\phi)\cap\pi'=\{0\}$ and $\mathrm{Im}(\phi_-)\cap\pi'=\{0\}$. Frobenius reciprocity implies that $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ is a quotient of $s_{GL}(\pi')$. Therefore, multiplicity of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma\right)$ is at least 3 (we use also here the last claim of (i) in Theorem 4.2). This multiplicity is 2 by (i) of Lemma 4.1. This contradiction completes the proof of (i).

In the same way as before, one checks that multiplicity of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{n+1/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\right)$ is 2, and

$$\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\leq s_{GL}\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)\right),$$

$\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma \le s_{GL}\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho]_-,\sigma)\right).$

Therefore, multiplicity of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ in the right hand sides of the above two inequalities is 1. Further, $\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ is a subquotient of $s_{GL}\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)\right)$

Since $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\hookrightarrow\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])$ ([Z1]), we have $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma\hookrightarrow\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$. The last representation is isomorphic to $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)\oplus\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho]_-,\sigma)$. Now we can conclude that π embeds into $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)$ and π_- into $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho]_-,\sigma)$. It remains to see the uniqueness of the irreducible subrepresentations in (ii). Frobenius reciprocity implies that it is enough to show that multiplicity of $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\otimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)$, and also of $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\otimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho]_-,\sigma)$ in $\mu^*\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\right)$, is 1. For this, one needs only to prove that the multiplicity is ≤ 1 (Frobenius reciprocity implies that the converse inequalities hold). In the continuation of this paper we shall prove a much more general fact about uniqueness of irreducible subrepresentation (Proposition 9.2, (ii)), which implies the second claim in (ii). Therefore, we shall only sketch here the proof that the multiplicity is ≤ 1. Write

(4-8)

$$M^*\left(\delta([\nu^{3/2+n}\rho,\nu^{1/2+m}\rho])\right)=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\circ m^*\left(\delta([\nu^{3/2+n}\rho,\nu^{1/2+m}\rho])\right)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\left(\sum_{a=n}^{m}\delta([\nu^{a+3/2}\rho,\nu^{1/2+m}\rho])\otimes\delta([\nu^{3/2+n}\rho,\nu^{1/2+a}\rho])\right)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\left(\sum_{a=n}^{m}\delta([\nu^{3/2+n}\rho,\nu^{1/2+a}\rho])\otimes\delta([\nu^{a+3/2}\rho,\nu^{1/2+m}\rho])\right)$$

$$=\sum_{a=n}^{m}\sum_{b=a}^{m}\delta([\nu^{-1/2-a}\rho,\nu^{-3/2-n}\rho])\times\delta([\nu^{b+3/2}\rho,\nu^{1/2+m}\rho])\otimes\delta([\nu^{3/2+a}\rho,\nu^{1/2+b}\rho]).$$

Compute now $\mu^*\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\right)$ using (1-4). To obtain $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\otimes\tau$ in $\mu^*\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\right)$ when we compute it using (4-1), we must take from (4-8) the term corresponding to $a=n$. From

$$\mu^*\left(\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\right)\le\mu^*\left(\left(\prod_{i=-n-1/2}^{n+1/2}\nu^i\rho\right)\rtimes\sigma\right)$$

$$=\left(\prod_{i=-n-1/2}^{n+1/2}(1\otimes\nu^i\rho+\nu^i\rho\otimes 1+\nu^{-i}\rho\otimes 1)\right)\rtimes(1\otimes\sigma)$$

(the above product runs over $i\in(1/2)+\mathbb{Z}$, $-n-1/2\le i\le n+1/2$), we get directly that b must be n. Thus, $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\otimes\tau$ can appear as a subquotient only from the term $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\otimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$ (which corresponds to $a=b=n$). Now (iii) of Theorem 3.2 implies our claim about multiplicities. This finishes the proof of the theorem. $\square$

4.4. Proposition. *Let $n\in\mathbb{Z}$, $n\geq 0$ and $\alpha\in\mathbb{R}$.*
(i) Assume that (ρ,σ) satisfies (C1/2). Suppose that $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ contains an irreducible square integrable subquotient, say π. Then π is equivalent either to a representation listed in Theorem 2.1, or Theorem 3.3, or Theorem 4.2.
(ii) If $\rho\not\cong\tilde{\rho}$, then $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ can not contain a square integrable subquotient.

Proof. Suppose that $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ contains a square integrable subquotient.

If $\nu^\alpha\delta([\rho,\nu^n\rho])$ is unitarizable, obviously we can not get a square integrable subquotient (this follows directly from the Frobenius reciprocity). Therefore, we can assume that $\nu^\alpha\delta([\rho,\nu^n\rho])$ is not unitarizable.

If $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ is irreducible, then it is not square integrable (the Langlands quotient coming from a proper parabolic subgroup, is never square integrable). Therefore, we can assume that $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ reduces. Theorem 9.1 of [T7] implies $\rho\cong\tilde{\rho}$ and

$$\nu^\alpha\delta([\rho,\nu^n\rho])\in\Big\{\delta([\nu^{-n-1/2}\rho,\nu^{-1/2}\rho]),\delta([\nu^{-n+1/2}\rho,\nu^{1/2}\rho]),\delta([\nu^{-n+3/2}\rho,\nu^{3/2}\rho]),\\ \dots,\delta([\nu^{-1/2}\rho,\nu^{n-1/2}\rho]),\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\Big\}.$$

Suppose that this is the case (and $\nu^\alpha\delta([\rho,\nu^n\rho])$ is not unitarizable, as we already have assumed). Note that at each reducibility point the Langlands quotient is not square integrable. Recall that $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ and $\nu^{-\alpha}\delta([\nu^{-n}\rho,\rho])\rtimes\sigma$ have the same Jordan-Hölder series (see (1-3)). Further, note that by Proposition 3.6 of [J], applying the involution constructed in [A2] (one can apply also [ScSt]), these representations have multiplicity one, and they have length 3, except if

$$\nu^\alpha\delta([\rho,\nu^n\rho])\in\Big\{\delta([\nu^{-n-1/2}\rho,\nu^{-1/2}\rho]),\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\Big\},$$

when the length is two. This implies the proposition. $\square$

5. Reducibility at 0

In this section we fix an irreducible unitarizable cuspidal representation ρ of $GL(p,F)$ and an irreducible cuspidal representation σ of S_q. We shall assume that $\rho\rtimes\sigma$ reduces (then $\rho\cong\tilde{\rho}$) and that $\nu^\alpha\rho\rtimes\sigma$ does not reduce for $\alpha\in\mathbb{R}^\times$ (in other words, we assume that (ρ,σ) satisfies (C0)).

From the Jacquet module $s_{(p)}(\rho\rtimes\sigma)$ one gets that $\rho\rtimes\sigma$ is a sum of two irreducible representations. Further, the Frobenius reciprocity implies that $\rho\rtimes\sigma$ is a multiplicity one representation. Write $\rho\rtimes\sigma=\tau_1\oplus\tau_2$ where τ_1 and τ_2 are irreducible ($\tau_1\not\cong\tau_2$).

First we shall recall Lemma 5.1 from [T10] (which is proved there).

5.1. Lemma. *The representation $\nu\rho \rtimes \tau_i$ contains a unique irreducible subrepresentation, which we denote by $\delta([\rho,\nu\rho]_{\tau_i},\sigma)$. This subrepresentation is square integrable and it is the only square integrable subquotient of $\nu\rho \rtimes \tau_i$. We have*

$$\mu^*\left(\delta([\rho,\nu\rho]_{\tau_i},\sigma)\right) = 1\otimes\delta([\rho,\nu\rho]_{\tau_i},\sigma) + \nu\rho\otimes\tau_i + \delta([\rho,\nu\rho])\otimes\sigma,$$
$$\delta([\rho,\nu\rho]_{\tau_i},\sigma)\tilde{} \cong \delta([\rho,\nu\rho]_{\tilde\tau_i},\tilde\sigma), \qquad \delta([\rho,\nu\rho]_{\tau_1},\sigma) \not\cong \delta([\rho,\nu\rho]_{\tau_2},\sigma). \quad \square$$

It will be convenient to us to use the following notation in further:

$$\delta([\rho,\rho]_{\tau_i},\sigma) = \tau_i, \qquad \delta(\emptyset_{\tau_i},\sigma) = \sigma.$$

5.2. Theorem. *Suppose that (ρ,σ) satisfies (C0). Write $\rho\rtimes\sigma = \tau_1\oplus\tau_2$ where τ_1 and τ_2 are irreducible. For $m\ge 1$ the representation $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu\rho\times\tau_i$ contains a unique irreducible subrepresentation, which we denote $\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$. Then:*
(i) $\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$ is square integrable.
(ii) $\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)\tilde{} \cong \delta([\rho,\nu^m\rho]_{\tilde\tau_i},\tilde\sigma)$.

$$(iii) \qquad \mu^*\left(\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)\right) = \sum_{k=0}^{n+1} \delta([\nu^k\rho,\nu^m\rho])\otimes\delta([\rho,\nu^{k-1}\rho]_{\tau_i},\sigma).$$

(iv) We may characterize $\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$ as a unique irreducible subquotient π of $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu\rho\times\tau_i$ for which $\delta([\rho,\nu^m\rho])\otimes\sigma$ is a subquotient of $s_{(p(m+1))}(\pi)$.
(v) $\delta([\rho,\nu^m\rho]_{\tau_1},\sigma)\not\cong\delta([\rho,\nu^m\rho]_{\tau_2},\sigma)$.

Proof. This is essentially Proposition 5.2 of [T10]. Since the claims of that proposition and the above theorem are not completely the same, we shall roughly recall of the proof. From $\mu^*(\tau_i) = 1\otimes\tau_i + \rho\otimes\sigma$, follows inductively

$$s_{((m+1)p)}(\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^2\rho\times\nu\rho\rtimes\tau_i) = \sum_{(\varepsilon_i)\in\{\pm1\}^m} \nu^{\varepsilon_m m}\rho\times\cdots\times\nu^{\varepsilon_2 2}\rho\times\nu^{\varepsilon_1}\rho\times\rho\otimes\sigma,$$

which implies that $s_{(p)^{m+1}}(\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^2\rho\times\nu\rho\rtimes\tau_i)$ is a multiplicity one representation. Therefore $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^2\rho\times\nu\rho\rtimes\tau_i$ has a unique irreducible subrepresentation. We shall prove the theorem by induction. The theorem holds for $m=1$ by Lemma 5.1 (for (iv) see (5-3) in [T10]). Let us suppose that the theorem holds up to $m\ge 1$. Now for the representation $\nu^{m+1}\rho\rtimes\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$ the inductive assumption gives

$$\text{s.s.}\left(s_{((m+2)p)}\left(\nu^{m+1}\rho\rtimes\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)\right)\right)$$
$$= \nu^{m+1}\rho\times\delta([\rho,\nu^m\rho])\otimes\sigma + \nu^{-(m+1)}\rho\times\delta([\rho,\nu^m\rho])\otimes\sigma,$$

and

$$\text{s.s.}\left(s_{((m+2)p)}\left(\delta([\nu^m\rho,\nu^{m+1}\rho])\rtimes\delta([\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)\right)\right)$$
$$= \delta([\nu^{-(m+1)}\rho,\nu^{-m}\rho])\times\delta([\rho,\nu^{m-1}\rho])\otimes\sigma$$
$$+\nu^{-m}\rho\times\nu^{m+1}\rho\times\delta([\rho,\nu^{m-1}\rho])\otimes\sigma + \delta([\nu^m\rho,\nu^{m+1}\rho])\times\delta([\rho,\nu^{m-1}\rho])\otimes\sigma.$$

This implies that the two considered representations have exactly one irreducible subquotient in common. It has in the Jacquet module $\delta([\rho,\nu^{m+1}\rho])\otimes\sigma$. One gets easily that this irreducible subquotient is $\delta([\rho,\nu^{m+1}\rho]_{\tau_i},\sigma)$. This also implies (i). The characterization of $\delta([\rho,\nu^{m+1}\rho]_{\tau_i},\sigma)$ as a unique irreducible subquotient of $\nu^{m+1}\rho\rtimes\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$ and $\delta([\nu^m\rho,\nu^{m+1}\rho])\rtimes\delta([\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)$ implies (ii). Claim (iii) follows in a standard way using the inductive assumption and characterization of essentially square integrable representations of general linear groups by Jacquet modules. Since the multiplicity of $\delta([\rho,\nu^m\rho])\otimes\sigma$ in the corresponding Jacquet module of $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu\rho\times\tau_i$ is one, we have (iv). One gets (v) from (iii). □

We continue with assumptions from the beginning of this section.

5.3. Lemma. *Let $n,m\in\mathbb{Z}, m\geq n\geq 0$. The representation*

$$(\nu\rho\times\nu^2\rho\times\cdots\times\nu^n\rho)\times(\nu\rho\times\nu^2\rho\times\cdots\times\nu^m\rho)\rtimes\tau_i \tag{5-1}$$

contains a unique irreducible subquotient π such that $s_{((n+m+1)p)}(\pi)$ contains

$$\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^m\rho])\otimes\sigma \tag{5-2}$$

as a subquotient. We denote π by $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$. The multiplicity of π in (5-1) is one.

Proof. We have

$$\text{s.s.}\left(s_{((n+m+1)p)}\left((\nu\rho\times\nu^2\rho\times\cdots\times\nu^n\rho)\times(\nu\rho\times\cdots\times\nu^m\rho)\rtimes\tau_i\right)\right)= \tag{5-3}$$
$$\sum_{\substack{(\varepsilon_j)\in\{\pm1\}^n\\(\mu_j)\in\{\pm1\}^m}}(\nu^{\varepsilon_1}\rho\rtimes\nu^{2\varepsilon_2}\rho\times\nu^{3\varepsilon_3}\rho\times\cdots\times\nu^{n\varepsilon_n}\rho)\times(\nu^{\mu_1}\rho\times\nu^{2\mu_2}\rho\times\cdots\times\nu^{m\mu_m}\rho)\times\rho\otimes\tau.$$

If some $\varepsilon_j\neq1$ or $\mu_j\neq1$, then the corresponding member in the sum have different GL-support from (5-2). If all ε_j are one, then the multiplicity of (5-2) in (5-3) is one ([Z1]). This proves the lemma. □

The representation $\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i$ contains a unique irreducible subrepresentation $\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)$ which we have already studied.

In the following theorem we continue with the previous notation. The theorem considers non square integrable tempered representations which are useful in the construction of square integrable representations.

5.4. Theorem. *(i) The representations $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ and $\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)$ have exactly one irreducible subquotient in common. This factor is $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$.*
(ii) $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma=\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_1},\sigma)\oplus\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_2},\sigma)$ and the representations on the right hand side are nonequivalent.
(iii) s.s. $\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\right)\right)=\sum_{k=0}^n\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma$.
(iv)
$$\delta([\nu^{-1}\rho,\nu^n\rho]_{\tau_i},\sigma)\tilde{}\cong\delta([\nu^{-1}\rho,\nu^n\rho]_{\tilde\tau_i},\tilde\sigma).$$

(v) One can characterize $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ also as a unique common irreducible subquotient of $\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i$ and $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$.

Proof. We consider the representation

$$\delta([\nu^{-n}\rho,\nu^{-1}\rho])\times\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma. \tag{5-4}$$

Obviously, in the Grothendieck group we have

$$\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\leq\delta([\nu^{-n}\rho,\nu^{-1}\rho])\times\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma, \tag{5-5}$$

$$\delta([\nu\rho\times\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\leq\delta([\nu^{-n}\rho,\nu^{-1}\rho])\times\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma. \tag{5-6}$$

Compute

$$\begin{aligned}\text{s.s.}\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{-1}\rho])\times\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma\right)\right)&\\ =2\rho\times\left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^{-1}\rho])\times\delta([\nu^{k+1}\rho,\nu^n\rho])\right]^2\otimes\sigma,&\end{aligned} \tag{5-7}$$

(5-8)

$$\text{s.s.}\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)\right)=2\left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\right]\otimes\sigma,$$

$$\begin{aligned}&\text{s.s.}\left(s_{((2n+1)p)}\left(\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\right)\right)\\ &=\left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^{-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\right]\times\delta([\rho,\nu^n\rho])\otimes\sigma.\end{aligned} \tag{5-9}$$

We shall now obtain same consequences from the above formulas. The multiplicity of $\delta([\nu^{-n}\rho,\nu^n\rho])\otimes\sigma$ in (5-8) is two (look at the support of GL-part of the representation). The Frobenius reciprocity now implies that the dimension of the intertwining algebra of the (unitarizable) representation $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ is at most two. Therefore, $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ is a multiplicity one representation of length ≤ 2. Also, if π is an irreducible subrepresentation of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$, then $\delta([\nu^{-n}\rho,\nu^n\rho])\otimes\sigma$ is a subquotient of $s_{((2n+1)p)}(\pi)$.

Considering $2\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ and taking into account supports, one gets that in the Grothendieck group

$$s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)\not\leq s_{((2n+1)p)}\left(\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\right).$$

Thus

$$\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\not\leq\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma). \tag{5-10}$$

In a similar way considering $\delta([\nu^{-n}\rho,\nu^{-1}\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma$ one gets

$$\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\not\leq\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma. \tag{5-11}$$

Note that the multiplicity of $\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^n\rho])$ in $\rho\times\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])$ is one (both of these representations are non-degenerate, and the highest derivatives are the same). We can now conclude that the multiplicity of $\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma$ in (5-7) is 2, in (5-8) is 2 and in (5-9) is 1. From the last multiplicities we can conclude that $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ and $\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)$ have non-disjoint Jordan-Hölder series. Further, from the above multiplicities follows that some common subquotient must have $\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma$ for a subquotient of corresponding Jacquet modules (the multiplicity must be one). Furthermore, (5-10) implies that $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ is reducible. Since $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ is a multiplicity one representation of length two, (5-10) and (5-11) imply that $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ and $\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)$ have exactly one irreducible subquotient in common. All this implies that the common irreducible subquotient must be $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$. Therefore, (i) holds.

Next we shall see that $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_1},\sigma)\not\cong\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_2},\sigma)$. Suppose that we have an isomorphism. Write $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma=\pi_1\oplus\pi_2$ where π_1 and π_2 are irreducible. We know that $\pi_1\not\cong\pi_2$. It is easy to conclude from (5-8) that $\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma\le s_{((2n+1)p)}(\pi_i)$ for some i. Lemma 5.3 and its proof imply that the multiplicity is one, so the inequality holds for $i=1$ and 2. Now $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_1},\sigma)\cong\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_2},\sigma)$ implies that there exists $i\in\{1,2\}$ such that $2\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_1},\sigma)+\pi_i\le(\rho\times\nu\rho\times\nu^2\rho\times\cdots\times\nu^m\rho)\times(\nu\rho\times\nu^2\rho\times\cdots\times\nu^n\rho)\rtimes\sigma$. Lemma 5.3 implies that this can not happen (look at the Jacquet modules corresponding to s_{GL}). This finishes the proof of (ii).

From the Jacquet modules of $\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)$ we know $\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\hookrightarrow\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i$. Thus $\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\hookrightarrow\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i$. Note that

$$\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\times\rho\rtimes\sigma\cong\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes(\tau_1\oplus\tau_2). \tag{5-12}$$

One gets directly that $s_{((2n+1)p)}\left(\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i\right)$ is just a half of the right hand side of (5-7). Looking at (5-8) we can now conclude that

$$\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\not\le\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i \tag{5-13}$$

Now it is clear that $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ may be characterized as a unique common irreducible subquotient of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ and $\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i$. This and(1-3) imply directly the formula for contragredients. Thus (iv) and (v) hold.

From (i), (5-8) and (5-9) we obtain easily that

$$\text{s.s.}\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\right)\right)\le\sum_{k=0}^{k=n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$$

Since the sum of s.s. $\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\right)\right)$ for $i=1,2$, equals to (5-8) by (ii), in the above inequality we must have the equality. This proves (iii). □

5.5. Theorem. *Suppose that (ρ,σ) satisfies (C0). Let $n,m\in\mathbb{Z}, 0<n<m$. Then:*
(i) There exists a unique common irreducible subquotient of $\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)$ and $\nu^n\rho\rtimes\delta([\nu^{-(n-1)}\rho,\nu^m\rho]_{\tau_i},\sigma)$. That subquotient is $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$.

(ii) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$ *is square integrable.*
(iii) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)\tilde{} \cong \delta([\nu^{-n}\rho,\nu^m\rho]_{\tilde{\tau}_i},\tilde{\sigma})$
(iv) *s.s.* $\left(s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)\right)\right)=\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$
(v) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_1},\sigma)\not\cong\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_2},\sigma)$

Proof. We consider the lexicographic ordering on pairs $\{(n,m)\in\mathbb{Z}\times\mathbb{Z},\quad 0<n<m\}$. We shall prove the theorem by induction with respect to this ordering. Write first

(5-14)
$$\begin{aligned}&\text{s.s.}\left(s_{((n+m+1)p)}\left(\nu^n\rho\times\nu^m\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)\right)\right)\\&=(\nu^n\rho\times\nu^m\rho+\nu^{-n}\rho\times\nu^m\rho+\nu^n\rho\times\nu^{-m}\rho+\nu^{-n}\rho\times\nu^{-m}\rho)\\&\times\left[\sum_{k=0}^{n-1}\delta([\nu^{-k}\rho,\nu^{m-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^{n-1}\rho])\right]\otimes\sigma,\end{aligned}$$

(5-15)
$$\begin{aligned}&\text{s.s.}\left(s_{((n+m+1)p)}\left(\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho]_{\tau_i},\sigma)\right)\right)\\&=(\nu^n\rho+\nu^{-n}\rho)\times\left[\sum_{k=0}^{n-1}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^{n-1}\rho])\right]\otimes\sigma,\end{aligned}$$

(5-16)
$$\begin{aligned}&\text{s.s.}\left(s_{((n+m+1)p)}\left(\nu^m\rho\times\delta([\nu^n\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)\right)\right)\\&=(\nu^m\rho\times\nu^{-m}\rho)\times\left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^{m-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\right]\otimes\sigma.\end{aligned}$$

We shall first find all common irreducible subquotients of (5-15) and (5-16). Since $\nu^{-m}\rho$ does not appear in GL-support of any irreducible representation in (5-15), this term after multiplication in (5-16) will not give anything in common. From the other side, if we fix a member of the sum in (5-16), and consider all $\alpha\in\mathbb{Z}$, such that $\nu^\alpha\rho$ is in the GL-support of that member, then they form a $\mathbb{Z}-$segment. Using this observation we can see that factor $\nu^{-n}\rho$ can give after multiplication in (5-15) something in common with (5-16) only when it is multiplied with $\delta([\nu^{-n+1}\rho,\nu^m\rho])$.

Comparing GL-supports, we see that the following pairs can have something in common:

$$\nu^{-n}\rho\times\delta([\nu^{-n+1}\rho,\nu^m\rho])\otimes\sigma\quad\text{and}\quad\nu^m\rho\times\delta([\nu^{-n}\rho,\nu^{m-1}\rho])\otimes\sigma;$$

$$\begin{aligned}&\nu^n\rho\times\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^{n-1}\rho])\otimes\sigma\quad\text{and}\\&\qquad\nu^m\rho\times\delta([\nu^{-k}\rho,\nu^{m-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma,\ \text{for } k=0,1,\dots,n-1.\end{aligned}$$

From the description of subquotients of generalized principal series representations ([Z1], see also [T1]), we get that irreducible subquotients which are in common are

(5-17)
$$\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma,\ \text{when } k=0,1,\dots,n.$$

Multiplicities with which these representations appear in (5-15) and (5-16) are one.

We shall now see the multiplicities of the above representations in (5-14). Considering supports, by a similar analysis as above, we can easily get that they can appear only in the following terms

$$\nu^{-n}\rho \times \nu^m\rho \times \delta([\nu^{-n+1}\rho, \nu^{m-1}\rho]) \otimes \sigma, \quad \text{and}$$
$$\nu^n\rho \times \nu^m\rho \times \delta([\nu^{-k}\rho, \nu^{m-1}\rho]) \times \delta([\nu^{1+k}\rho, \nu^{n-1}\rho]) \otimes \sigma, \text{ when } k = 0, 1, \dots, n-1.$$

This implies that the multiplicities of representations of (5-17) in (5-14) are one.

We now claim

$$\text{(5-18)}\quad \nu^n\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^m\rho]_{\tau_i}, \sigma) \leq \nu^n\rho \times \nu^m\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^{m-1}\rho]_{\tau_i}, \sigma),$$
$$\text{(5-19)}\quad \nu^m\rho \rtimes \delta([\nu^{-n}\rho, \nu^{m-1}\rho]_{\tau_i}, \sigma) \leq \nu^n\rho \times \nu^m\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^{m-1}\rho]_{\tau_i}, \sigma).$$

If $m > n+1$, then both relations follow from the inductive assumptions. Suppose that $m = n+1$. Then the first relation is again a consequence of the inductive assumption. For (5-19) it is enough to prove that $\delta([\nu^{-n}\rho, \nu^n\rho]_{\tau_i}, \sigma) \leq \nu^n\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^n\rho]_{\tau_i}, \sigma)$. Note that the right hand side of the inequality is $\leq \nu^n\rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^{n-1}\rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^n\rho \rtimes \tau_i$. Further, using the inductive assumption we see that $s_{((2n+1)p)}\left(\nu^n\rho \rtimes \delta([\nu^{n-1}\rho, \nu^n\rho]_{\tau_i}, \sigma)\right)$ contains $\delta([\rho, \nu^n\rho] \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma$ as a subquotient. This proves the second inequality in the case $m = n+1$.

At this point we can draw same conclusions. Denote $\pi_1 = \nu^n\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^m\rho]_{\tau_i}, \sigma)$, $\pi_2 = \nu^m\rho \rtimes \delta([\nu^{-n}\rho, \nu^{m-1}\rho]_{\tau_i}, \sigma)$, and $\pi_3 = \nu^n\rho \times \nu^m\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^{m-1}\rho]_{\tau_i}, \sigma)$. If π is an irreducible subquotient of π_1 and π_2, then $s_{((2n+m+1)p)}(\pi)$ has for a subquotient at least one representation from (5-17). Conversely, if π is a subquotient of π_3 which has at least one representation from (5-17) as a subquotient of $s_{((n+m+1)p)}(\pi)$, then π has multiplicity one in π_3, and it is a subquotient of both π_1 and π_2. We used that $\pi_1 \leq \pi_3$ (what is just inequality (5-18)), $\pi_2 \leq \pi_3$ ((5-19)), and that all multiplicities of representation from (5-17) in $s_{(2n+m+1)p)}(\pi_i)$ are one. Denote all common irreducible subquotients of π_1 and π_2 by $\vartheta_1, \dots, \vartheta_\ell$, where $\vartheta_i \not\cong \vartheta_j$ for $i \neq j$. We now know that s.s.$(s_{((n+m+1)p)}(\vartheta_1 + \cdots + \vartheta_\ell)) = \sum_{k=0}^{n} \delta([\nu^{-k}\rho, \nu^m\rho]) \times \delta([\nu^{1+k}\rho, \nu^n\rho]) \otimes \sigma$. From this we see easily that all $\vartheta_1, \dots, \vartheta_\ell$ are square integrable using the square integrability criterion.

It remains to prove $\ell = 1$. This would prove (i) and (iii). Then the formula for the contragredient follows directly from the inductive assumption, Theorems 5.2, 5.4, and the characterization of $\delta([\nu^{-n}\rho, \nu^m\rho]_{\tau_i}, \sigma)$ in (i).

Take $\vartheta \in \{\vartheta_1, \dots, \vartheta_\ell\}$ which has $\delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$ as a subquotient of $s_{((n+m+1)p)}(\vartheta)$. Then $\delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$ it is actually a direct summand (see the central character). Therefore, $\vartheta \hookrightarrow \delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$. This implies $\vartheta \hookrightarrow \nu^m\rho \times \nu^{m-1}\rho \times \cdots \times \nu^{-n}\rho \rtimes \sigma$. Take $0 \leq k < n$. Then

$$\nu^m\rho \times \nu^{m-1}\rho \times \cdots \times \nu^{-n+1}\rho \times \nu^{-n}\rho \rtimes \sigma \cong \nu^m\rho \times \nu^{m-1}\rho \times \cdots \times \nu^{-n+1}\rho \times \nu^n\rho \rtimes \sigma$$
$$\cong \nu^m\rho \times \nu^{m-1}\rho \times \cdots \times \nu^{-k}\rho \times \nu^n\rho \times \nu^{-k-1}\rho \times \nu^{-k-2}\rho \times \cdots \times \nu^{-n+1}\rho \rtimes \sigma \cong \dots$$
$$\cong \nu^m\rho \times \nu^{m-1}\rho \times \cdots \times \nu\rho \times \rho \times \nu^{-1}\rho \times \cdots \times \nu^{-k}\rho \times \nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu^{k+1}\rho \rtimes \sigma.$$

Thus $\nu^m\rho\otimes\cdots\otimes\nu\rho\otimes\rho\otimes\nu^{-1}\rho\otimes\cdots\otimes\nu^{-k}\rho\otimes\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu^{k+1}\rho\otimes\sigma$ is a subquotient of $s_{(p)^{n+m+1}}(\vartheta)$. Therefore, $s_{((n+m+1)p)}(\vartheta)$ has an irreducible subquotient which has GL-support $(\nu^m\rho,\cdots,\nu\rho,\rho,\nu^{-1}\rho,\cdots,\nu^{-k}\rho,\nu^n\rho,\nu^{n-1}\rho,\cdots,\nu^{k+1}\rho)$. The only representation in (5-17) with such GL-support, is $\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{k+1}\rho,\nu^n\rho])\otimes\sigma$. Thus, the above representation must be subquotient of $s_{((n+m+1)p)}(\vartheta)$. Since $0\le k<n$ was arbitrary, we get that $\ell=1$ (the proof of ℓ =1 we could start from any ϑ_i, any irreducible quotient of $s_{((n+m+1)p)}(\vartheta_i)$; in a similarly way as above we would get that all representations in (5-17) are subquotients of $s_{((n+m+1)p)}(\vartheta_i)$; this would again imply $\ell=1$).

The claim (v) follows from the following lemma in a similar way as (iv) of Theorem 5.4 followed from the fact that $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ is a multiplicity one representation. $\square$

5.6. Lemma. *If $0\le n\le m$, then $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ is a multiplicities one representation.*

Proof. For $n=m$ we know that the lemma holds (Theorem 5.4). It is enough to consider the case $n<m$. We shall prove the lemma by induction on $n+m$. For $n=0$ and $m=1$ the lemma follows from the formula for $\mu^*\left(\delta([\rho,\nu\rho])\rtimes\sigma\right)$ in the proof of Lemma 5.1. Fix $n+m>1$ and suppose that the lemma holds for $n'+m'<n+m$. Observe that $M^*(\delta([\nu^{-n}\rho,\nu^m\rho]))$ can be written as

$$\left[1\otimes\delta([\nu^{-n}\rho,\nu^m\rho])\right]+\left[\nu^m\rho\otimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho])+\nu^n\rho\otimes\delta([\nu^{-n+1}\rho,\nu^m\rho])\right]+X,$$

where X is a sum of members of the form $x_i\otimes y_i$ such that x_i is a representation of some $GL(p\,k,F)$ with $k\ge 2$. This implies

$$\text{s.s.}\left(s_{(p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma\right)\right)=\nu^m\rho\otimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho])\rtimes\sigma+\nu^n\rho\otimes\delta([\nu^{-n+1}\rho,\nu^m\rho])\rtimes\sigma.$$

The inductive assumption and $n\ne m$, imply that the above representation is a multiplicity one representation (observe that $\delta([\nu\rho,\nu^m\rho])\rtimes\sigma$ is irreducible by Theorem 9.1 of [T7]). Now the lemma follows directly since each irreducible subquotient π of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ must have $s_{(p)}(\pi)\ne 0$. $\square$

The above lemma follows also from Proposition 3.10 of [J], using [A2] or [ScSt].

5.7. Remark. One can easily see that $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)\cong\delta([\nu^{-n'}\rho',\nu^{m'}\rho']_{\tau'_{i'}},\sigma')$ implies $\rho\cong\rho'$, $n=n'$, $m=m'$ and $\sigma\cong\sigma'$ (then we have shown that also $\tau_i\cong\tau'_{i'}$).

5.8. Theorem. *Let $n,m\in\mathbb{Z}$, $m>n\ge 0$.. Write $\rho\rtimes\sigma=\tau_1\oplus\tau_2$, with τ_1 and τ_2 irreducible. Then*
(i) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$, $(i=1,2)$, is a subrepresentation of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$. There are no other irreducible subrepresentations of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$.
(ii) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$ is a subrepresentation of $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$, and there is no other irreducible subrepresentation in $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$.

Proof. The proof is a variation of the proof of Theorem 4.3. We shall give only the main points of the proof. Set $\pi_i=\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$. Theorems 5.2 ((iii)), 5.5

((iv)) and [C] (Theorem 7.3.2 imply that $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ is a direct summand in $s_{GL}(\pi)$ and $s_{GL}(\pi_-)$. Therefore, we have embeddings $\pi_i\hookrightarrow\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$, $i=1,2$. Assume that there is an irreducible subrepresentation π' of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ different from (images of) π_i, $i=1,2$. Then $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ is a quotient of $s_{GL}(\pi')$, which implies (using also (v) of Theorem 5.5) that multiplicity of $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma\right)$ is at least 3. One checks directly that this multiplicity is 2. This completes the proof of (i).

Multiplicity of $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)$ is 2, and in $s_{GL}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\right)$ is at least 1 ($i=1,2$). Thus, these two multiplicities are both 1. The fact $\delta([\nu^{-n}\rho,\nu^m\rho])\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho])$ and the above discussion, imply that either $\pi_i\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ for $i=1,2$, or $\pi_i\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_{3-i}},\sigma)$ for $i=1,2$. We shall see that the last possibility can not occur. Note that $\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\le\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\le\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\times\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu\rho\times\tau_i$ by (i) of Theorem 5.4 and Theorem 5.2. Now Lemma 5.3 implies that π_i is a subquotient of $\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ for $i=1,2$. The above discussion about multiplicities implies now $\pi_i\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ for $i=1,2$.

The uniqueness in (ii) one gets in the same way as in Theorem 4.3 from

$$
\begin{aligned}
\text{(5-20)}\qquad M^*\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\right) &= (m\otimes 1)\circ(\sim\otimes m^*)\circ s\circ m^*\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\right)\\
&= (m\otimes 1)\circ(\sim\otimes m^*)\circ s\left(\sum_{a=n}^{m}\delta([\nu^{a+1}\rho,\nu^m\rho])\otimes\delta([\nu^{n+1}\rho,\nu^a\rho])\right)\\
&= (m\otimes 1)\circ(\sim\otimes m^*)\left(\sum_{a=n}^{m}\delta([\nu^{n+1}\rho,\nu^a\rho])\otimes\delta([\nu^{a+1}\rho,\nu^m\rho])\right)\\
&= \sum_{a=n}^{m}\sum_{b=a}^{m}\delta([\nu^{-a}\rho,\nu^{-n-1}\rho])\times\delta([\nu^{b+1}\rho,\nu^m\rho])\otimes\delta([\nu^{a+1}\rho,\nu^b\rho]),
\end{aligned}
$$

and

$$
\text{(5-21)}\quad \mu^*\left(\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)\le\left(\prod_{i=-n}^{n}(1\otimes\nu^i\rho+\nu^i\rho\otimes 1+\nu^{-i}\rho\otimes 1)\right)\rtimes(1\otimes\sigma).\qquad\square
$$

5.9. Proposition. *Let $n\in\mathbb{Z}$, $n\ge 0$ and $\alpha\in\mathbb{R}$.*
(i) Assume that (ρ,σ) satisfies (C0). Suppose that $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ contains an irreducible square integrable subquotient, say π. Then π is equivalent either to a representation listed in Theorem 5.2 or Theorem 5.5.
(ii) If $\rho\not\cong\tilde\rho$, then $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ can not contain a square integrable subquotient.

Proof. One proves the above proposition in a similar way as Proposition 4.4. One needs only to use Proposition 3.11 of [J] instead of Proposition 3.6 from the same paper, which was used in the proof of Proposition 4.4. $\square$

6. Reducibility at 1, I

In this section ρ will be an irreducible unitarizable cuspidal representation of $GL(p,F)$ and σ an irreducible cuspidal representation of S_q such that $\nu\rho\rtimes\sigma$ reduces and $\nu^\alpha\rho \rtimes \sigma$ is irreducible for $\alpha \in \mathbb{R}\backslash\{\pm 1\}$. In other words, we assume that (ρ,σ) satisfies (C1).

6.1. Theorem. *For a positive integer n the representation $\rho\times\delta([\nu\rho,\nu^n\rho],\sigma)$ splits into a sum of two non-equivalent irreducible tempered representations. They are not square integrable. Denote them by π_1 and π_2. Then $\delta([\rho,\nu^n\rho])\otimes\sigma$ is a subquotient either of $s_{((n+1)p)}(\pi_1)$ or of $s_{((n+1)p)}(\pi_2)$. Denote the irreducible tempered representation which has $\delta([\rho,\nu^n\rho])\otimes\sigma$ for a subquotient of the Jacquet module by $\delta([\rho,\nu^n\rho],\sigma)$. The other irreducible tempered representation will be denoted by $\delta([\rho,\nu^n\rho]_-,\sigma)$. Then:*
(i) s.s. $\left(s_{((n+1)p)}\left(\delta([\rho,\nu^n\rho],\sigma)\right)\right)=\delta([\rho,\nu^n\rho])\otimes\sigma+\delta([\nu\rho,\nu^n\rho])\times\rho\otimes\sigma.$
(ii) s.s. $\left(s_{((n+1)p)}\left(\delta([\rho,\nu^n\rho]_-,\sigma)\right)\right)=L\left(\rho,\delta([\nu\rho,\nu^n\rho])\right)\otimes\sigma.$
(iii) $\delta([\rho,\nu^n\rho],\sigma)\tilde{}\cong\delta([\rho,\nu^n\rho],\tilde\sigma)$, $\delta([\rho,\nu^n\rho]_-,\sigma)\tilde{}\cong\delta([\rho,\nu^n\rho]_-,\tilde\sigma)$.
(iv) The representation $\delta([\rho,\nu^n\rho],\sigma)$ can be characterized as a unique common irreducible subquotient of $\delta([\rho,\nu^n\rho])\rtimes\sigma$ and $\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$.

Proof. Write

$$\mu^*\left(\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)\right)=1\otimes\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$$
$$+\left[2\rho\otimes\delta([\nu\rho,\nu^n\rho],\sigma)+\nu^n\rho\otimes\rho\rtimes\delta([\nu\rho,\nu^{n-1}\rho],\sigma)\right]+\cdots+\left[2\rho\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma\right],$$
$$\text{s.s.}\left(s_{((n+1)p)}\left(\delta([\rho,\nu^n\rho])\rtimes\sigma\right)\right)=\left[\sum_{k=0}^{n+1}\delta([\nu^{-k+1}\rho,\rho])\times\delta([\nu^k\rho,\nu^n\rho])\right]\otimes\sigma.$$

From the Frobenius reciprocity we can conclude that $\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$ is a multiplicity one representation of length ≤ 2. Then, the common irreducible factors in the Jacquet modules $s_{((n+1)p)}\left(\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)\right)$ and $s_{((n+1)p)}\left(\delta([\rho,\nu^n\rho])\rtimes\sigma\right)$ are $\delta([\rho,\nu^n\rho])\otimes\sigma$ and $L\left(\rho,\delta([\nu\rho,\nu^n\rho])\right)\otimes\sigma$. The multiplicity of $\delta([\rho,\nu^n\rho])\otimes\sigma$ in both Jacquet modules is 2. The multiplicity of $L\left((\rho,[\nu\rho,\nu^n\rho])\right)\otimes\sigma$ in the first Jacquet module is two, while in the second one is one. Note that

$$\delta([\rho,\nu^n\rho])\rtimes\sigma\not\leq\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma),\quad \rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)\not\leq\delta([\rho,\nu^n\rho])\rtimes\sigma,$$
$$\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)\leq\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma,\quad \delta([\rho,\nu^n\rho])\rtimes\sigma\leq\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma,$$
$$\text{s.s.}\left(s_{((n+1)p)}\left(\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma\right)\right)=2\rho\times\left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^{-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\right]\otimes\sigma.$$

The multiplicities of $\delta([\rho,\nu^n\rho])$ and $L\left(\rho,\delta([\nu\rho,\nu^n\rho])\right)$ in the above Jacquet module are both equal to two. We can now conclude that $\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$ and $\delta([\rho,\nu^n\rho])\rtimes\sigma$ have exactly one irreducible subquotient in common, say π_1, and that s.s. $\left(s_{((n+1)p)}(\pi_1)\right)=\delta([\rho,\nu^n\rho])\otimes\sigma+\rho\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$. Denote the other summand of $\rho\times\delta([\nu\rho,\nu^n\rho],\sigma)$ by π_2. Then $s_{((n+1)p)}(\pi)=L\left(\rho,\delta([\nu\rho,\nu^n\rho])\right)$. All the remaining claims of the theorem now follow automatically. $\square$

We need the following lemma for a lower estimate of a Jacquet module in the following theorem.

6.2. Lemma. *Suppose that (ρ,σ) satisfies (C1). Let π be an irreducible subquotient of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$. Then*

$$s.s.\left(s_{((2n+1)p)}(\pi)\right)\geq\sum_{k=1}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{k+1}\rho,\nu^n\rho])\otimes\sigma.$$

Proof. Note that each term of the sum on the right hand side of the above inequality is irreducible.

Recall that π must be a subrepresentation of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$. Frobenius reciprocity implies $s_{((2n+1)p)}(\pi)\geq\delta([\nu^{-n}\rho,\nu^n\rho])\otimes\sigma$. If $n=1$, then the lemma is proved. Thus suppose that $n>1$. Now

$$\begin{aligned}\pi\hookrightarrow\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma&\hookrightarrow\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^{-n}\rho\rtimes\sigma\\&\cong\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{-n+2}\rho\times\nu^{-n+1}\rho\times\nu^n\rho\rtimes\sigma.\end{aligned}$$

Thus $s_{(p,p,\dots,p)}(\pi)$ has $\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu^{-n+2}\rho\otimes\nu^{-n+1}\rho\otimes\nu^n\rho\otimes\sigma$ as subquotient. Since this subquotient of the Jacquet module must come as a subquotient of the Jacquet module of an irreducible subquotient of $s_{((2n+1)p)}(\pi)$ (because of the transitivity of process of taking Jacquet modules), and $\pi\leq\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$, we see that $\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu^{-n+2}\rho\otimes\nu^{-n+1}\rho\otimes\nu^n\rho\otimes\sigma$ must came from Jacquet module of $s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)$. Recall that (5-8) gives a formula for semi simplification of the last representation. Considering the right hand side of (5-8), looking at the GL-supports, we see that $\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu^{-n+2}\rho\otimes\nu^{-n+1}\rho\otimes\nu^n\rho\otimes\sigma$ can come only from $\delta([\nu^{-n+1}\rho,\nu^n\rho])\times\nu^n\rho\otimes\sigma$. This implies $s_{((2n+1)p)}(\pi)\geq\delta([\nu^{-n+1}\rho,\nu^n\rho])\times\nu^n\rho\otimes\sigma$. In the some way, one proves for the other terms the inequality. This completes the proof of the lemma. $\square$

6.3. Theorem. *Suppose that (ρ,σ) satisfies (C1). Let n be a positive integer. Then representation $\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^{-n}\rho\rtimes\sigma$ contains a unique irreducible subquotient $\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)$ which has $\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ for a subquotient of $s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\right)$. Further:*

(i) The multiplicity of $\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ in $s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\right)$ is two.

(ii) The multiplicity of $\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)$ in $\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^{-n}\rho\rtimes\sigma$ is one.

(iii) The representation $\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)$ may be characterized as a unique common irreducible subquotient of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ and $\delta([\rho,\nu^n\rho])\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$.

(iv) $\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\tilde{}\cong\delta([\nu^{-n}\rho,\nu^n\rho],\tilde{\sigma})$.

(v)

$$\begin{aligned}s.s.\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\right)\right)&=\sum_{k=-1}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma\\&=2\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma+\sum_{k=1}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.\end{aligned}$$

(vi) The representation $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ is a multiplicity one representation of length two. Denote the other irreducible subquotient by $\delta([\nu^{-n}\rho,\nu^n\rho]_-,\sigma)$ (see (iii)). Then

$$s.s.\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho]_-,\sigma)\right)\right)=\sum_{k=1}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$$

Proof. The proof of the theorem is similar to the proof of Theorem 3.2 (and Theorem 5.4). Therefore, we shall only sketch the proof (the complete proof can be found in [T6]). Denote $\pi_1=\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma$, $\pi_2=\delta([\rho,\nu^n\rho])\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$, $\pi_3=\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$. Then $\pi_2,\pi_3\le\pi_1$. From the formula for s.s. $\left(s_{((2n+1)p)}(\pi_3)\right)$, we see that π_3 is a multiplicity one representation of length ≤ 2. Further, s.s. $\left(s_{((2n+1)p)}(\pi_3)\right)\not\le$ s.s. $\left(s_{((2n+1)p)}(\pi_2)\right)$ implies $\pi_3\not\le\pi_2$. The multiplicity of $\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma$ in $s_{((2n+1)p)}(\pi_i)$, $i=1,2,3$, is 2. From this we conclude that π_3 reduces., and that there exists a common irreducible subquotient π of π_2 and π_3 which has $\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma$ in the Jacquet module. We can also conclude that the multiplicity in the Jacquet module is 2. Now the last lemma and the formula for s.s. $\left(s_{((2n+1)p)}(\pi_3)\right)$ (see (5-8)) imply (v). All other claims follow now easily. □

6.4. Proposition. *Let $n,m\in\mathbb{Z}$, $m\ge n\ge 0$. Then*
(i) $s_{((n+m+1)p)}(\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma)$ contains $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient. The multiplicity is two.
(ii) If π is a subquotient of $\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$ such that $s_{((n+m+1)p)}(\pi)$ contains $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient, then $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ has in $s_{((n+m+1)p)}(\pi)$ multiplicity two. The multiplicity of π in $\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$ is one. We denote π by $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)$ (note that the above definition in the cases of $n=m$ or $n=0$ agrees with our old definitions in that cases).

Proof. We have proved (i) already. For (ii), it is enough to see that if π is a subquotient of $\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$ such that $s_{((n+m+1)p)}(\pi)$ contains $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient, then the multiplicity of $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ in $s_{((n+m+1)p)}(\pi)$ is two. Theorems 6.1 and 6.3 imply that it is enough to consider only the case $0<n<m$. Suppose that there exists a subquotient π_1 of $\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$ such that $s_{((n+m+1)p)}(\pi_1)$ contains $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient with multiplicity one. Set $\vartheta=\nu^{-n}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$. Then there exists a subrepresentation $\vartheta_1\subseteq\vartheta_2\subseteq\vartheta$ such that $\vartheta_2/\vartheta_1\cong\pi_1$. Because of (i), there exists a subquotient π_2 of ϑ_1 or ϑ/ϑ_2 which has $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ for a subquotient of $s_{((n+m+1)p)}(\pi_2)$ (note that we do not claim that $\pi_1\not\cong\pi_2$). We now know $\pi_1+\pi_2\le\vartheta$ and $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma\le s_{((n+m+1)p)}(\pi_i)$ for $i=1,2$.

Consider now $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\vartheta$. Set $\varphi=\sum_{k=n}^{m}\delta([\nu^{-k}\rho,\nu^{-(n+1)}\rho])\times\delta([\nu^{k+1}\rho,\nu^m\rho])$. Now $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes(\pi_1+\pi_2)\le\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^{-m}\rho\rtimes\sigma$. This implies

$$\text{(6-1)}\quad s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes(\pi_1+\pi_2)\right)\le s_{((2m+1)p)}(\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^{-m}\rho\rtimes\sigma).$$

From the other side (using (1-4)), we have

$$\begin{aligned} s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\pi_i\right) &\geq \varphi\times\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma \\ &\geq \delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^m\rho])\otimes\sigma. \end{aligned}$$

This fact, (i) and (6-1) imply that $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^m\rho])\otimes\sigma$ has multiplicity one in $s_{((2n+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\pi_i\right)$. This contradicts to (ii) in the case $m=n$ which we know that holds (Theorem 6.4). $\square$

6.5. Lemma. *For $0\leq n\leq m$ we have*

$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\leq\sum_{k=-1}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho]).$$

6.6. Remark. Note that in the above sum only the first term (corresponding to $k=-1$) is not always irreducible. It is reducible when $n<m$. In that case, that term is a multiplicity one representation of length two. In the Grothendieck group we have $\delta([\nu\rho,\nu^m\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma=\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])+L\left(\delta([\nu\rho,\nu^m\rho]),\delta([\rho,\nu^n\rho])\right)$.

Proof. For $n=m$ or $n=0$ we know that the lemma holds (Theorems 6.3 and 6.4). Therefore, it is enough to consider the case of $m>n>0$. We shall prove this case by induction (the lexicographic ordering is considered on pairs (n,m)). First we can conclude from Jacquet modules that $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\leq\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho],\sigma)$ and $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\leq\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho],\sigma)$. Now we can write a natural upper bound for $s_{((n+m+1)p)}\left(\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho],\sigma)\right)$ using the inductive assumption, and also a natural upper bound for $s_{((n+m+1)p)}\left(\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho],\sigma)\right)$ using the inductive assumption. Then we find all common irreducible subquotients of that upper bounds, and also the multiplicities of that common irreducible subquotients. As a consequence, we get the estimate of the lemma. Since we have already done estimates of this type in the proofs of Theorems 3.3 and 5.5, we omit here details (all details can be found in the preprint [T6]). $\square$

6.7. Lemma. *For $0<n<m$ we have*

$$\begin{aligned} &s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right) \\ &\quad\geq 2\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma+\sum_{k=1}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma. \end{aligned}$$

Proof. We know $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\geq2\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ from Proposition 6.4. Suppose that we know $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\geq\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$. We shall first show that this implies the lemma. One needs to consider only the case of $n>1$. Since $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ has different central character from the other irreducible subquotients in the Jacquet module $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)$, we conclude that it is a direct summand in the Jacquet module (use Lemma 6.5). Thus $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\hookrightarrow\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$. Now we shall use an argument similar to the one that we have already used in the proof of Lemma 6.2. We have

$$\begin{aligned}\delta([\nu^{-n}\rho,\nu^m\rho],\sigma) &\hookrightarrow \delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma\\ &\hookrightarrow \nu^m\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^{-n}\rho\rtimes\sigma\cong\nu^m\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^n\rho\rtimes\sigma.\end{aligned}$$

This implies that $\nu^m\rho\otimes\cdots\otimes\nu^{-n+1}\rho\otimes\nu^n\rho\otimes\sigma$ is in the Jacquet module. Further, the second term $\delta([\nu^{-n+1}\rho,\nu^m\rho])\times\nu^n\rho\otimes\sigma$ in the sum must be in the Jacquet module (this is the only possible term by Lemma 6.5 which is in the Jacquet module and which has $\nu^m\rho\otimes\cdots\otimes\nu^{-n+1}\rho\otimes\nu^n\rho\otimes\sigma$ in suitable Jacquet module). One gets further terms in a similar fashion.

Note that $\delta([\nu^{-m}\rho,\nu^m\rho])\otimes\sigma\le s_{((2m+1)p)}\left(\delta([\nu^{-m}\rho,\nu^m\rho],\sigma)\right)$, and $\delta([\nu^{-m}\rho,\nu^m\rho],\sigma)\le\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)$. This implies

$$\delta([\nu^{-m}\rho,\nu^m\rho])\otimes\sigma\le s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right). \tag{6-2}$$

Write $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)=\vartheta\otimes\sigma$. Then

$$\begin{aligned}&\text{s.s.}\left(s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\right)\\ &\qquad=\left[\sum_{k=n}^{m}\delta([\nu^{-k}\rho,\nu^{-n-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^m\rho])\right]\times\vartheta\otimes\sigma.\end{aligned}$$

Lemma 6.5 and (6-2) imply that $\delta([\nu^{-n}\rho,\nu^m\rho])\le\vartheta$ (consider the term $\delta([\nu^{-m}\rho,\nu^m\rho])\otimes\sigma$). This ends the proof of the lemma. □

6.8. Theorem. *Suppose that (ρ,σ) satisfies (C1) (then $\rho\cong\tilde\rho$). For $n,m\in\mathbb{Z}$, $0<n<m$, the representation $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)$ is square integrable. Further, $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}\cong\delta([\nu^{-n}\rho,\nu^m\rho],\tilde\sigma)$ and*

$$\begin{aligned}&\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])+\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma\\ &\le s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\le\sum_{k=-1}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.\end{aligned}$$

Proof. It remains only to prove the formula for the contragredient. We proceed by induction. Suppose that $m>n>0$ and that the theorem holds for $m'+n'<m+n$. Note that $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}$ is a common irreducible subquotient of $\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho],\tilde\sigma)$ and $\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho],\tilde\sigma)$. The last two representations can have at most two common irreducible subquotients (this follows from the proof of Lemma 6.5). One is $\delta([\nu^{-n}\rho,\nu^m\rho],\tilde\sigma)$. If there is only one irreducible subquotient in common, then the proof is complete. If there are two, denote the second one by

π. Then $s_{((n+m+1)p)}(\pi)$ is irreducible. Therefore, for the proof in this case, it is enough to show that $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}\right)$ is not irreducible. Recall that

$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}\right)\cong\left[\underline{s}_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\right]\tilde{}$$

where $\underline{s}$ denotes the Jacquet module with respect to the choice of lower triangular matrices for standard minimal parabolic subgroup ([C], Corollary 4.2.5). But the lower parabolic subgroup ${}^tP_{((n+m+1)p)}$ is conjugated to $P_{((n+m+1)p)}$. Therefore, the length is the same. This implies that $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}\right)$ is reducible. Now the proof is complete. $\square$

6.9. Remark. Proposition 3.10 of [J], together with [A2] or [ScSt], imply

$$\text{s.s.}\left(s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\right)=\sum_{k=-1}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$$

7. Reducibility at 1, II

We continue in this section to denote by ρ an irreducible unitarizable cuspidal representation of $GL(p,F)$ and by σ an irreducible cuspidal representation of S_q, such that (ρ,σ) satisfies (C1).

7.1. Proposition. *There exists a unique irreducible subquotient $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$ of $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ which satisfies conditions*

$$\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma\leq s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right), \tag{7-1}$$

$$\nu\rho\times\delta([\rho,\nu^2\rho])\otimes\sigma\not\leq s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right). \tag{7-2}$$

Multiplicity of $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$ in $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ is 1. Further $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\not\cong\delta([\nu^{-1}\rho,\nu^2\rho],\sigma)$, $s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)=\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma$ and $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$ is square integrable.

Proof. Write

$$\begin{aligned}\text{s.s.}\left(s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma\right)\right)=&\left[\delta([\nu^{-2}\rho,\nu\rho])+\delta([\nu^{-1}\rho,\nu\rho])\times\nu^2\rho\right.\\&\left.+\delta([\rho,\nu\rho])\times\delta([\nu\rho,\nu^2\rho])+\nu\rho\times\delta([\rho,\nu^2\rho])+\delta([\nu^{-1}\rho,\nu^2\rho])\right]\otimes\sigma.\end{aligned} \tag{7-3}$$

We see from this formula that the multiplicity of $\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma$ in (7-3) is 2. Now we can conclude from Theorem 6.8 that there exists a unique irreducible subquotient $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$ of $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ satisfying (7-1) and (7-2).

Since

$$\text{s.s.}\left(s_{(4p)}\left(\nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma\right)\right)=(\nu^{-2}\rho+\nu^2\rho)\times2\left(\delta([\nu^{-1}\rho,\nu\rho])+\nu\rho\times\delta([\rho,\nu\rho])\right)\otimes\sigma,$$

multiplicity of $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ in the above representation is 2. Since

$$\begin{aligned}\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma &\le \nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma,\\ \nu^2\rho\rtimes\delta([\nu^{-1}\rho,\nu\rho]_-,\sigma) &\le \nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma,\\ \delta([\nu^{-1}\rho,\nu^2\rho],\sigma) &\le \nu^2\rho\rtimes\delta([\nu^{-1}\rho,\nu\rho],\sigma),\\ \delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma &\le s_{(4p)}\big(\delta([\nu^{-1}\rho,\nu^2\rho],\sigma)\big),\\ \delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma &\le s_{(4p)}\big(\nu^2\rho\rtimes\delta([\nu^{-1}\rho,\nu\rho]_-,\sigma)\big),\\ \delta([\nu^{-1}\rho,\nu\rho],\sigma) &\not\cong \delta([\nu^{-1}\rho,\nu\rho]_-,\sigma),\end{aligned}$$

we see that it must be $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\le\nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho]_-,\sigma)$. This implies

$$s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)\le(\nu^{-2}\rho+\nu^2\rho)\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma.$$

Since $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\le\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$, we can conclude further

$$s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)\le\delta([\nu^{-2}\rho,\nu\rho])\otimes\sigma+\nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma. \tag{7-4}$$

Consider now the unique irreducible quotient $L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)$ of $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ (such irreducible quotient is unique by the properties of the Langlands classification). Clearly, $L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)\not\cong\delta([\nu^{-1}\rho,\nu^2\rho],\sigma)$, since the later representation is square integrable. From Frobenius reciprocity we can conclude that $\delta([\nu^{-2}\rho,\nu\rho])\otimes\sigma$ is a subquotient of $s_{(4p)}\left(L\left(\delta([\nu^{-1}\rho,\nu^2\rho],\sigma\right)\right)$. Multiplicity of $\delta([\nu^{-2}\rho,\nu\rho])\otimes\sigma$ in (7-3) is one. Therefore, one can characterize $L\left(\delta([\nu^{-1}\rho,\nu^2\rho],\sigma\right)$ using this subquotient of the Jacquet module.

Consider $\delta([\rho,\nu^2\rho])\times\nu\rho\rtimes\sigma$. Clearly $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma\le\delta([\rho,\nu^2\rho])\times\nu\rho\rtimes\sigma$. Further

$$\begin{aligned}&\text{s.s.}(s_{(4p)}(\delta([\rho,\nu^2\rho])\rtimes\delta(\nu\rho,\sigma)))=\\ &\quad\big(\delta([\nu^{-2}\rho,\rho])+\delta([\nu^{-1}\rho,\rho])\times\nu^2\rho+\rho\times\delta([\nu\rho,\nu^2\rho])+\delta([\rho,\nu^2\rho])\big)\times\nu\rho\otimes\sigma.\end{aligned}$$

Note that the multiplicities of $\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma$ and $\nu\rho\times\delta([\rho,\nu^2\rho])\otimes\sigma$ in the above representation are 1 and 2 respectively. This implies $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\le\delta([\rho,\nu^2\rho])\rtimes L(\nu\rho,\sigma)$ (use Theorem 6.8). Applying Jacquet functor to this inequality, we get

$$\begin{aligned}&s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)\\ &\le\big(\delta([\nu^{-2}\rho,\rho])+\delta([\nu^{-1}\rho,\rho])\times\nu^2\rho+\rho\times\delta([\nu^1\rho,\nu^2\rho])+\delta([\rho,\nu^2\rho])\big)\times\nu^{-1}\rho\otimes\sigma.\end{aligned}$$

Multiplicity of $\delta([\nu^{-2}\rho,\nu\rho])\otimes\sigma$ in the above representation is 0. Therefore, we can conclude $L\left(\delta([\nu^{-1}\rho,\nu^2\rho],\sigma\right)\not\cong\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$.

One has the following embeddings

$$L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)\hookrightarrow\delta([\nu^{-2}\rho,\nu\rho])\rtimes\sigma\hookrightarrow\nu\rho\times\rho\times\nu^{-1}\rho\times\nu^{-2}\rho\rtimes\sigma\cong\nu\rho\times\rho\times\nu^{-1}\rho\times\nu^2\rho\rtimes\sigma$$

(passing to contragredients one gets the first embedding). Therefore, $\nu\rho\otimes\rho\otimes\nu^{-1}\rho\otimes\nu^2\rho\otimes\sigma$ is a subquotient of the Jacquet module of $L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)$. From (7-3) we can now conclude that $L\left(\delta([\nu^{-1}\rho,\nu\rho]),\nu^2\rho\right)\otimes\sigma$ is a subquotient of $s_{(4p)}\left(L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)\right)$. This, (7-4) and (7-3) imply $s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)=\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma$. This implies square integrability. $\square$

7.2. Proposition. *Suppose $m \geq 3$. Then there exists a unique irreducible subquotient $\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)$ of $\delta([\nu^{-1}\rho, \nu^m\rho]) \rtimes \sigma$ which satisfies following conditions*

$$\delta([\nu^{-1}\rho, \nu^m\rho]) \otimes \sigma \leq s_{((m+2)p)} \left(\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)\right) \tag{7-5}$$

$$\nu\rho \times \delta([\rho, \nu^m\rho]) \otimes \sigma \not\leq s_{((m+2)p)} \left(\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)\right). \tag{7-6}$$

Multiplicity of $\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)$ in $\delta([\nu^{-1}\rho, \nu^m\rho]) \rtimes \sigma$ is 1. Also $\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma) \not\cong \delta([\nu^{-1}\rho, \nu^m\rho], \sigma)$ and $s_{((m+2)p)} \left(\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)\right) = \delta([\nu^{-1}\rho, \nu^m\rho]) \otimes \sigma$. The representation $\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)$ is square integrable.

Proof. We prove the lemma by induction. Write

$$\text{s.s.} \left(s_{((m+2)p)}(\delta([\nu^{-1}\rho, \nu^m\rho]) \rtimes \sigma)\right) = \sum_{i=-2}^{m} \delta([\nu^{-i}\rho, \nu\rho]) \times \delta([\nu^{i+1}\rho, \nu^m\rho]) \otimes \sigma. \tag{7-7}$$

Multiplicity of $\delta([\nu^{-1}\rho, \nu^m\rho]) \otimes \sigma$ in (7-7) is 2. Now Theorem 6.8 implies that there exists a unique irreducible subquotient $\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)$ of $\delta([\nu^{-1}\rho, \nu^m\rho]) \rtimes \sigma$ which satisfies (7-5) and (7-6).

Further, $\delta([\nu^{-1}\rho, \nu^m\rho]) \rtimes \sigma \leq \nu^m\rho \times \delta([\nu^{-1}\rho, \nu^{m-1}\rho]) \rtimes \sigma$. We see again easily that the multiplicity of $\delta([\nu^{-1}\rho, \nu^m\rho]) \otimes \sigma$ in $s_{((m+2)p)} \left(\nu^m\rho \times \delta([\nu^{-1}\rho, \nu^{m-1}\rho]) \rtimes \sigma\right)$ is 2 since

$$\begin{aligned} &\text{s.s.} \left(s_{((m+2)p)} \left(\nu^m\rho \times \delta([\nu^{-1}\rho, \nu^{m-1}\rho]) \rtimes \sigma\right)\right) \\ &\qquad = (\nu^{-m}\rho + \nu^m\rho) \times \left[\sum_{i=-2}^{m-1} \delta([\nu^{-i}\rho, \nu\rho]) \times \delta([\nu^{i+1}\rho, \nu^{m-1}\rho])\right] \otimes \sigma. \end{aligned}$$

Now

$$\begin{aligned} \delta([\nu^{-1}\rho, \nu^m\rho], \sigma) &\leq \nu^m \rtimes \delta([\nu^{-1}\rho, \nu^{m-1}\rho], \sigma), \\ \delta([\nu^{-1}\rho, \nu^m\rho]) \otimes \sigma &\leq s_{((m+2)p)} \left(\delta([\nu^{-1}\rho, \nu^m\rho], \sigma)\right), \\ \delta([\nu^{-1}\rho, \nu^m\rho]) \otimes \sigma &\leq s_{((m+2)p)} \left(\nu^m\rho \rtimes \delta([\nu^{-1}\rho, \nu^{m-1}\rho]_-, \sigma)\right), \\ \delta([\nu^{-1}\rho, \nu^{m-1}\rho]_-, \sigma) &\not\cong \delta([\nu^{-1}\rho, \nu^{m-1}\rho], \sigma) \end{aligned}$$

imply

$$\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma) \leq \nu^m\rho \times \delta([\nu^{-1}\rho, \nu^{m-1}\rho]_-, \sigma).$$

This, together with the inductive assumption, or the preceding proposition if $m = 3$, implies

$$s_{((m+2)p)} \left(\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)\right) \leq (\nu^{-m}\rho + \nu^m\rho) \times \delta([\nu^{-1}\rho, \nu^{m-1}\rho]) \otimes \sigma.$$

Since $\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma) \leq \delta([\nu^{-1}\rho, \nu^m\rho]) \rtimes \sigma$, we get easily using (7-7)

$$s_{((m+2)p)} \left(\delta([\nu^{-1}\rho, \nu^m\rho]_-, \sigma)\right) \leq \delta([\nu^{-1}\rho, \nu^m\rho]) \otimes \sigma$$

(here we needed the assumption $m \geq 3$). This implies square integrability. Also, obviously the equality must hold in the above relation. This finishes the proof. □

7.3. Theorem. *Let $n, m \in \mathbb{Z}, 1 < n < m$. Then there exists a unique irreducible subquotient $\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)$ of $\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$ which satisfies conditions*

(7-8)
$$\delta([\nu^{-1}\rho, \nu^m\rho]) \times \delta([\nu^2\rho, \nu^n\rho]) \otimes \sigma \leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)\right),$$

(7-9)
$$\delta([\rho, \nu^m\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma \not\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)\right).$$

Multiplicity of $\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)$ in $\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$ is 1 and $\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma) \not\cong \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)$. Further

(7-10)
$$s.s.\left(s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)\right)\right) = \sum_{i=1}^{n} \delta([\nu^{-i}\rho, \nu^m\rho]) \times \delta([\nu^{i+1}\rho, \nu^n\rho]) \otimes \sigma.$$

The representation $\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)$ is square integrable.

Proof. We shall prove the lemma by induction on n and m. Note that the claim of the theorem holds if $n = 1$ or $n = m$, except that in the later case the representation $\delta([\nu^{-n}\rho, \nu^n\rho]_-, \sigma)$ is not square integrable. First we shall prove the following inequality

(7-11)
$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)\right) \leq \sum_{i=-1}^{n} \delta([\nu^{-i}\rho, \nu^m\rho]) \times \delta([\nu^{i+1}\rho, \nu^n\rho]) \otimes \sigma.$$

Write

(7-12)
$$\text{s.s.}\left(s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma\right)\right) = \sum_{i=-m}^{n+1} \delta([\nu^{i}\rho, \nu^n\rho]) \times \delta([\nu^{-i+1}\rho, \nu^m\rho]) \otimes \sigma.$$

Observe that the multiplicity of $\delta([\nu^{-1}\rho, \nu^m\rho]) \times \delta([\nu^2\rho, \nu^n\rho]) \otimes \sigma$ in (7-12) is 2. Note that the multiplicity is 2 also when $n = m$ (see the formula (6-3) also). Now Theorem 6.8 implies that there exists a unique irreducible subquotient $\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)$ of $\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$ satisfying (7-8) and (7-9).

Obviously

(7-13)
$$\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma \leq \nu^m\rho \times \nu^n\rho \times \delta([\nu^{-n+1}\rho, \nu^{m-1}\rho]) \rtimes \sigma.$$

Since $n > m \geq 2$, it is easy to see that the multiplicity of $\delta([\nu^{-1}\rho, \nu^m\rho]) \times \delta([\nu^2\rho, \nu^n\rho]) \otimes \sigma$ in $s_{((n+m+1)p)}\left(\nu^m\rho \times \nu^n\rho \times \delta([\nu^{-n+1}\rho, \nu^{m-1}\rho]) \rtimes \sigma\right)$ is 2. Now

$$\delta([\nu^{-1}\rho, \nu^m\rho]) \times \delta([\nu^2\rho, \nu^n\rho]) \otimes \sigma \leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)\right),$$
$$\delta([\nu^{-1}\rho, \nu^m\rho]) \times \delta([\nu^2\rho, \nu^n\rho]) \otimes \sigma \leq s_{((n+m+1)p)}\left(\nu^n\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^m\rho]_-, \sigma)\right),$$
$$\delta([\nu^{-1}\rho, \nu^m\rho]) \times \delta([\nu^2\rho, \nu^n\rho]) \otimes \sigma \leq s_{((n+m+1)p)}\left(\nu^m\rho \rtimes \delta([\nu^{-n}\rho, \nu^{m-1}\rho]_-, \sigma)\right),$$
$$\nu^n\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^m\rho]_-, \sigma) \leq \nu^m\rho \times \nu^n\rho \times \delta([\nu^{-n+1}\rho, \nu^{m-1}\rho]) \rtimes \sigma,$$
$$\nu^m\rho \rtimes \delta([\nu^{-n}\rho, \nu^{m-1}\rho]_-, \sigma) \leq \nu^m\rho \times \nu^n\rho \times \delta([\nu^{-n+1}\rho, \nu^{m-1}\rho]) \rtimes \sigma,$$
$$\delta([\rho, \nu^m\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma \not\leq s_{((n+m+1)p)}\left(\nu^m\rho \rtimes \delta([\nu^{-n}\rho, \nu^{m-1}\rho]_-, \sigma)\right),$$
$$\delta([\rho, \nu^m\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma \not\leq s_{((n+m+1)p)}\left(\nu^n\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^m\rho]_-, \sigma)\right),$$

imply that it must be

$$\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\leq \nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho]_-,\sigma), \tag{7-14}$$

$$\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\leq \nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^{m}\rho]_-,\sigma). \tag{7-15}$$

Now in the same way as in the proof of Lemma 6.5, using induction follows the inequality (7-11). Since on the right hand sides of (7-14) and (7-15) there are no representations which in the support have only representations of type $\nu^\alpha\rho$ with $\alpha\geq 0$, we get the stronger inequality

$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right)\leq\sum_{i=1}^{n}\delta([\nu^{-i}\rho,\nu^m\rho])\times\delta([\nu^{i+1}\rho,\nu^n\rho])\otimes\sigma. \tag{7-16}$$

Consider $\delta([\nu^{-n}\rho,\nu^m\rho])\times\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\sigma$. One gets easily that the multiplicity of $\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^{2}\rho,\nu^m\rho])\otimes\sigma$ in $s_{((2m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho])\times\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\sigma\right)$ is 2, since the last representation is equal to

$$\left[\sum_{i=n}^{m}\delta([\nu^{-i}\rho,\nu^{-n-1}\rho])\times\delta([\nu^{i+1}\rho,\nu^m\rho])\right]\times\left[\sum_{i=-m}^{n+1}\delta([\nu^{i}\rho,\nu^{n}\rho])\times\delta([\nu^{-i+1}\rho,\nu^m\rho])\right]\otimes\sigma.$$

Further

$$\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^{2}\rho,\nu^m\rho])\otimes\sigma\leq s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right),$$
$$\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^{2}\rho,\nu^m\rho])\otimes\sigma\leq s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right),$$
$$\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma\not\leq s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right).$$

One can easily conclude

$$\delta([\nu^{-m}\rho,\nu^m\rho]_-,\sigma)\leq\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma).$$

Using the last relation and $\delta([\nu^{-m}\rho,\nu^m\rho])\otimes\sigma\leq s_{((2m+1)p)}\left(\delta([\nu^{-m}\rho,\nu^m\rho]_-,\sigma)\right)$, one can in the same way as in the second half of the proof of Lemma 6.7 prove that

$$\delta([\nu^{-n}\rho,\nu^m\rho]\otimes\sigma\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right).$$

Using the inequality (7-16), we see that $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ must be a direct summand in $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right)$. Now in the same way as in the first half of the proof of Lemma 6.7 it follows that the formula in the lemma for $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right)$ holds. This finishes the proof.

7.4. Corollary. *The representation $\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)$ can be characterized as a unique irreducible subquotient of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ which satisfies conditions*

$$\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right),$$
$$\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^m\rho])\otimes\sigma\not\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right). \quad\square$$

7.5. Theorem. *Let $n, m \in \mathbb{Z}$, $m > n > 0$. Then*
(i) $\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)$ and $\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)$ are subrepresentations of $\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$, and $\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$ does not contain any other irreducible subrepresentation.
(ii) $\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)$ (resp. $\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)$) is a unique irreducible subrepresentation of $\delta([\nu^{n+1}\rho, \nu^m\rho]) \rtimes \delta([\nu^{-n}\rho, \nu^n\rho], \sigma)$ (resp. $\delta([\nu^{n+1}\rho, \nu^m\rho]) \rtimes \delta([\nu^{-n}\rho, \nu^n\rho]_-, \sigma)$).

Proof. The proof is a simple modification of the proof of Theorem 4.3. We shall only outline the proof. Put $\pi = \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)$ (resp. $\pi_- = \delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)$). Again we conclude from Theorems 6.8 and 7.3, using Theorem 7.3.2 of [C], that $\delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$ is a direct summand in $s_{GL}(\pi)$ and $s_{GL}(\pi_-)$. Therefore, there exist embeddings $\pi \hookrightarrow \delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$ and $\pi_- \hookrightarrow \delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$. If there is an irreducible subrepresentation π' of $\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$ different from (images of) π and π_-, then $\delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$ would be a quotient of $s_{GL}(\pi')$. Therefore, the multiplicity of $\delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$ in $s_{GL}(\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma)$ would be at least 3. This contradicts to the fact that multiplicity of $\delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$ in $s_{GL}(\delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma)$ is 2, what one easily prove using (1-4).

Multiplicity of $\delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$ in $s_{GL}\left(\delta([\nu^{n+1}\rho, \nu^m\rho]) \times \delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma\right)$ is 2, in $s_{GL}\left(\delta([\nu^{n+1}\rho, \nu^m\rho]) \times \delta([\nu^{-n}\rho, \nu^n\rho], \sigma)\right)$ and $s_{GL}\left(\delta([\nu^{n+1}\rho, \nu^m\rho]) \times \delta([\nu^{-n}\rho, \nu^n\rho]_-, \sigma)\right)$ is at least 1. Because of this, the last two multiplicities are both 1. Note that $\delta([\rho, \nu^m\rho]) \times \delta([\nu\rho, \nu^m\rho]) \otimes \sigma$ is a subquotient of $s_{GL}\left(\delta([\nu^{n+1}\rho, \nu^m\rho]) \times \delta([\nu^{-n}\rho, \nu^n\rho]\sigma)\right)$

Now from $\delta([\nu^{-n}\rho, \nu^m\rho]) \hookrightarrow \delta([\nu^{n+1}\rho, \nu^m\rho]) \times \delta([\nu^{-n}\rho, \nu^n\rho])$ and the above discussion we can conclude that π embeds in $\delta([\nu^{n+1}\rho, \nu^m\rho]) \times \delta([\nu^{-n}\rho, \nu^n\rho], \sigma)$, and π_- in $\delta([\nu^{n+1}\rho, \nu^m\rho]) \times \delta([\nu^{-n}\rho, \nu^n\rho]_-, \sigma)$. At the end, one gets uniqueness in (ii) in the same way as in Theorem 4.3, using formulas (5-20) and (5-21). $\square$

7.6. Proposition. *Let $n \in \mathbb{Z}$, $n \geq 0$ and $\alpha \in \mathbb{R}$.*
(i) Assume that (ρ, σ) satisfies (C1). Suppose that $\nu^\alpha\delta([\rho, \nu^n\rho]) \rtimes \sigma$ contains an irreducible square integrable subquotient, say π. Then π is equivalent either to a representation listed in Theorem 2.1, or Theorem 6.8 or Theorem 7.3.
(ii) If $\rho \not\cong \tilde{\rho}$, then $\nu^\alpha\delta([\rho, \nu^n\rho]) \rtimes \sigma$ can not contain a square integrable subquotient.

Proof. Now one uses Proposition 3.10 of [J]. $\square$

References

[A] Aubert, A.-M., *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p-adique*, Trans. Amer. Math. Soc. **347** (1995), 2179-2189 (and *Erratum*, Trans. Amer. Math. Soc **348** (1996), 4687-4690).

[BeZ] Bernstein, J. and Zelevinsky, A.V., *Induced representations of reductive p-adic groups I*, Ann. Sci. École Norm Sup. **10** (1977), 441-472.

[Bo] Borel, A., *Automorphic L-functions*, Symp. Pure Math. 33, part 2, Amer. Math. Soc., Providence, Rhone Island, 1979, pp. 27-61.

[C] Casselman, W., *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint.

[G] Goldberg, D., *Reducibility of induced representations for $Sp(2n)$ and $SO(n)$*, Amer. J. Math. **116** (1994), no. 5, 1101-1151.

[G] Herb, R.A., *Elliptic representations for $Sp(2n)$ and $SO(n)$*, Pacific J. Math. **161** (1993), 347-358.

[J] Jantzen, C., *Degenerate principal series for symplectic and odd-orthogonal groups*, Mem. Amer. Math. Society **590** (1996), 1-100.

[LMuT] Lapid, E., Muić G. and Tadić, M., *On the generic unitary dual of quasisplit classical groups*, Int. Math. Res. Not. **26** (2004), 1335-1354.

[Mœ] Mœglin, C., *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité.*, J. Eur. Math. Soc. **4** (2002), no. 2, 143–200.

[MœT] Mœglin, C. and Tadić, M., *Construction of discrete series for classical p-adic groups*, J. Amer. Math. Soc. **15** (2002), 715-786.

[Mu1] Muić, G., *On generic irreducible representations of $Sp(n, F)$ and $SO(2n+1, F)$*, Glasnik Mat. **33(53)** (1998), 19-31.

[Mu2] ______, *Composition Series of Generalized Principal Series; The Case of Strongly Positive Discrete Series*, Israel J. Math. **140** (2004), 157–202.

[Mu3] ______, *Reducibilities of Generalized Principal Series*, Canad. J. Math. (to appear).

[R] Rodier, F., *Décomposition de la série principale des groupes réductifs p-adiques*, Non-Commutative Harmonic Analysis, Lecture Notes in Math. 880, Springer-Verlag, Berlin, 1981.

[SaT] Sally, P.J. and Tadić, M., *Induced representations and classifications for $GSp(2, F)$ and $Sp(2, F)$*, Mémoires Soc. Math. France **52** (1993).

[ScSt] Schneider, P. and Stuhler, U., *Representation theory and sheaves on the Bruhat-Tits building*, Publ. Math. IHES (1997), 97-191.

[Sh1] Shahidi, F., *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. **132** (1990), 273-330.

[Sh2] ______, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J. **66** (1992), 1-41.

[T1] Tadić, M., *Induced representations of $GL(n, A)$ for p-adic division algebras A*, J. reine angew. Math. **405** (1990), 48-77.

[T2] ______, *Representations of p-adic symplectic groups*, Compositio Math. **90** (1994), 123-181.

[T3] ______, *Representations of classical p-adic groups*, Representations of Lie groups and quantum groups (V. Baldoni and M. Picardello, eds.), Pitman Research Notes in Mathematics Series 311, Longman, Essex, 1994, pp. 129-204.

[T4] ______, *Structure arising from induction and Jacquet modules of representations of classical p-adic groups*, Journal of Algebra **177** (1995), no. 1, 1-33.

[T5] ______, *On regular square integrable representations of p-adic groups*, Amer. J. Math. **120** (1998), no. 1, 159-210.

[T6] ______, *Construction of square-integrable representations of classical p-adic groups*, preprint in Mathematica Goettingensis (1993), no. 11.

[T7] ______, *On reducibility of parabolic induction*, Israel J. Math. **107** (1998), 29-91.

[T8] ______, *Square integrable representations of classical p-adic groups of segment type*, preprint in Mathematica Goettingensis (1994), no. 15.

[T9] ______, *On square integrable representations of classical p-adic groups*, preprint (1996).

[T10] ______, *Square integrable representations of classical p-adic groups corresponding to segments*, Representation Theory **3** (1999), 58-89.

[T11] ______, *A family of square integrable representations of classical p-adic groups*, Glasnik Mat. **37(57)** (2002), 21-57.

[V1] Vogan, D.A., *The local Langlands conjecture*, Contemp. Math. **145** (1993), 305-379.

[V2] ______, *Personal comunication.*

[W] Waldspurger, J.-L., *Un exercice sur $GSp(4, F)$ et les représentations de Weil*, Bull. Soc. Math. France **115** (1987), 35-69.

[Z1] Zelevinsky, A.V., *Induced representations of reductive p-adic groups II, On irreducible representations of $GL(n)$*, Ann. Sci École Norm Sup. **13** (1980), 165-210.

[Z2] ______, *p-adic analogue of the Kazhdan-Lusztig conjecture*, Funct. Anal. Appl. **15** (1981), 82-93.

Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia