GENERALIZED SPHERICAL FUNCTIONS ON REDUCTIVE p-ADIC GROUPS

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Abstract. Generalized spherical functions on a reductive p-adic group G are eigenfunctions for the action of the Bernstein center, which satisfy a transformation property for the action of a maximal parabolic subgroup of G. In this paper we show that spaces of generalized spherical functions are finite dimensional. We compute dimensions of spaces of generalized spherical functions on a Zariski open dense set of infinitesimal characters. As a consequence, we get that on that Zariski open dense set of infinitesimal characters, the dimension of a space of generalized spherical functions is constant on each connected component of infinitesimal characters.

1. Introduction

In the case of real reductive group G, zonal spherical functions can be introduced as eigenfunctions for the action of the center $\mathfrak{Z}(G)$ of the enveloping algebra of G which are constant on double K-classes, where K denotes a maximal compact subgroup of G. In this way, one deals with differential equations and eigenfunctions of differential operators. Generalized spherical functions on real reductive groups are natural generalization of zonal spherical functions (see [HOW]).

Bernstein center ([BD]) plays in the representation theory of reductive p-adic groups the role that has the center of the enveloping algebra of a Lie algebra in the representation theory of real reductive groups. Bernstein center shows up in a number of important problems, but its role in the representation theory is far from being well understood.

The construction of Bernstein center of a reductive p-adic group ([BD]) opened possibility to study generalized spherical functions also in p-adic case. In this paper we define and study basic properties of generalized spherical functions in the case of a reductive p-adic group (they are defined as eigenfunctions of the Bernstein center, which satisfy a transformation property for the action of the maximal compact subgroup). This paper may be viewed as a contribution to the further understanding of the Bernstein center.

Let G be the group of rational points of a connected reductive group defined over a p-adic field F. Fix a minimal parabolic subgroup $P_\emptyset$ defined over F and let K be a maximal

1991 *Mathematics Subject Classification.* 22E50, 22E35.

Key words and phrases. reductive p-adic group, generalized spherical function, Bernstein center, infinitesimal character.

The first author is partially supported by RGC Competitive Earmarked Research Grant HKUST 713/96P. The second author is partly supported by Croatian Ministry of Science and Technology grant # 37108.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

compact subgroup of G satisfying $G = P_\emptyset K$. Let (τ_1, V_{τ_1}), (τ_2, V_{τ_2}) be finite dimensional representations of K. Denote

$$V = \operatorname{Hom}_{\mathbb{C}}(V_{\tau_2}, V_{\tau_1})$$

and

$$\tau = (\tau_1, \tau_2).$$

Then generalized spherical functions of type τ (or τ-spherical functions) are mappings $f : G \to V$ which satisfy $f(k_1 x k_2) = \tau_1(k_1) f(x) \tau_2(k_2)$ for all $k_1, k_2 \in K$, $g \in G$, and which are eigenfunctions for the action of the Bernstein center. If ω is corresponding infinitesimal character of the Bernstein center of G, then the space of all generalized spherical functions of type τ corresponding to the infinitesimal character ω will be denoted by

$$\mathcal{E}_\omega(G, \tau).$$

In this paper we first show that these spaces are finite dimensional (Corollary 5.5). We show how one can get all the generalized spherical functions from a single admissible representation with contragredient infinitesimal character $\tilde{\omega}$ (Propositions 5.4 and 6.1). After this, we show that on a Zariski open dense set of infinitesimal characters, one can choose the above representation to be irreducible (in this case there exists exactly one such representation and it has a simple description). We also compute dimensions of spaces of generalized spherical functions for infinitesimal characters in this Zariski open dense set (Theorem 6.4). If ω is an infinitesimal character of parabolically induced representation $\operatorname{Ind}_P^G(\rho)$, where ρ is irreducible cuspidal representation of Levi factor M of P, then the dimension of generalized spherical functions for ω in Zariski open dense set of infinitesimal characters is given by

$$\dim_{\mathbb{C}}(\mathcal{E}_\omega(G, \tau)) = \dim_{\mathbb{C}}(\operatorname{Hom}_{M \cap K}(\tau_1, \tilde{\rho})) \dim_{\mathbb{C}}(\operatorname{Hom}_{M \cap K}(\tau_2, \tilde{\rho})),$$

where $\tilde{\rho}$ denotes the contragredient representation of ρ. On this Zariski open dense set of infinitesimal characters, the dimension of a space of generalized functions is constant on each connected component of infinitesimal characters.

The description of generalized spherical functions that we have obtained can be used to describe them by integrals similar to Eisenstein integrals. On the Zariski open dense set of infinitesimal characters that we have mentioned above, these integrals then give the formula for all the generalized spherical functions.

This paper is the p-adic analogue of a part of the basic results for the real reductive groups obtained in [HOW]. It is chime in with what Harish-Chandra called the Lefschetz principal, which says that whatever is true for real reductive groups is also true for p-adic groups. There are many not so surprising similarities between our proofs for the p-adic case and that for the real case in [HOW] (but there is a number of differences). The Hecke algebra $\mathcal{H}(G)$ of compactly supported smooth functions on p-adic group often plays a similar role as the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the real group. The Bernstein center $\mathfrak{Z}(G) \cong \operatorname{End}_{G \times G}(\mathcal{H}(G))$ plays the same role as the center $Z(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ in the case of real groups. The main tool that we use here are the techniques of Bernstein center ([BD] and [BDK]).

At the end of this introduction we shall give a brief account of the paper. In the second section we introduce notation regarding actions of the group G on spaces of functions on G. The third section recalls the definition of the Bernstein center. We prove here also some simple facts about actions of Bernstein center needed in the sequel. Generalized spherical functions are defined in the fourth section. The fifth section starts with the study of spherical functions and their dimensions. In this section we prove that spaces of generalized spherical functions are finite dimensional. In the last section we prove the formula for dimension of spaces of generalized spherical functions for infinitesimal characters in the Zariski open dense set of infinitesimal characters

2. Actions on functions

In this section we shall denote by G a locally compact totally disconnected group. Further, V will denote a finite dimensional complex vector space. The dimension will be denoted by n. The dual space of V will be denoted by V'. We shall fix a basis $e_1, e_2, \dots, e_n$ of V. By $e_1', e_2', \dots, e_n'$ we shall denote the basis of V' biorthogonal to the basis $e_1, e_2, \dots, e_n$.

The space of all continuous (resp. locally constant) functions from G into V will be denoted by $C(G,V)$ (resp $C^\infty(G,V)$). The space $C(G,\mathbb{C})$ will be simply denoted by $C(G)$. We shall consider left and right actions L and R of G on $C(G,V)$ (here $(L_x\varphi)(g) = \varphi(x^{-1}g)$ and $(R_x\varphi)(g) = \varphi(gx)$. The smooth part of this $G \times G$-representation will be denoted by $C(G,V)^{(s,s)}$. This space consists of all the functions $\varphi : G \to V$ for which there exists an open compact subgroup H of G such that φ is constant on each double H-class.

For a linear operator $A : W_1 \to W_2$ we shall denote by $\Lambda_A : C(G,W_1) \to C(G,W_2)$ the linear operator defined by $(\Lambda_A(\varphi))(g) = A(\varphi(g))$ (i.e. $\Lambda_A(\varphi) = A \circ \varphi$). Then Λ_A intertwines the left actions L on $C(G,W_1)$ and $C(G,W_2)$, as well as it intertwines the right actions R (for example, $(L_x(\Lambda_A(\varphi)))(g) = (\Lambda_A(\varphi))(x^{-1}g) = A(\varphi(x^{-1}g)) = A((L_x\varphi)(g)) = (\Lambda_A(L_x\varphi))(g))$.

The following two simple particular cases of above linear operators will be important for us. Let $v' \in V'$. Then

$$\Lambda_{v'} : C(G,V) \to C(G), \quad \varphi \ \mapsto \ v' \circ \varphi. \tag{2-1}$$

Further, for $v \in V$ consider the mapping $v^\# : \mathbb{C} \to V,\ c \mapsto cv$. Then

$$\Lambda_{v^\#} : C(G) \to C(G,V), \quad f \ \mapsto \ (g \mapsto f(g)v). \tag{2-2}$$

The bilinear mapping $(f,v) \ \mapsto \ (g \mapsto f(g)v), \ \ C(G)\times V \to C(G,V)$ induces a natural isomorphism

$$I_\otimes : C(G) \otimes V \to C(G,V), \ \ I_\otimes(\sum f_i \otimes v_i)(g) = \sum f_i(g)v_i,$$

i.e.

$$I_\otimes(\sum f_i \otimes v_i) = \sum \Lambda_{v_i^\#}(f_i).$$

Obviously,

$$\varphi = I_{\otimes}(\sum_{i=1}^{n} e_i' \circ \varphi \otimes e_i) = I_{\otimes}(\sum_{i=1}^{n} \Lambda_{e_i'}(\varphi) \otimes e_i) = \sum_{i=1}^{n} \Lambda_{e_i^{\#}}(\Lambda_{e_i'}(\varphi)). \tag{2-3}$$

We shall consider the trivial action of G on V. Then G acts on $C(G) \otimes V$ in two natural ways, one coming from the left and the other from the right action on $C(G)$. Then $I_{\otimes}$ is an intertwining for both actions. Using $I_{\otimes}$, we shall identify $C(G) \otimes V$ with $C(G, V)$

For a smooth representation (π, X) of G the contragredient representation is denoted by $(\tilde{\pi}, \tilde{X})$. Recall of a simple lemma a proof of which can be found at several places.

2.1. Lemma. *For $f \in C(G)^{(s,s)}$ the following conditions are equivalent:*

(1) *The subrepresentation generated by f with respect to the right action of G is admissible.*
(2) *The subrepresentation generated by f with respect to the left action of G is admissible.*

This is equivalent to the fact that there exists an admissible representation (π, X) of G, $x_i \in X$ and $\tilde{x}_i \in \tilde{X}$, $i = 1, \ldots, k$ such that $f(g) = \sum_{i=1}^{k} \tilde{x}_i(\pi(g)x_i)$ for $g \in G$ (i.e. f is a matrix coefficient of an admissible representation).

The subspace generated by all the matrix coefficients

$$g \mapsto < \pi(g^{-1})v, \tilde{v} >$$

of all admissible representations (π, X) of G is denoted by $\mathcal{A}(G)$. Note that (1) and (2) characterize $\mathcal{A}(G)$ in a different ways.

Denote by $\mathcal{A}(G, V)$ the space of all $\varphi \in C(G, V)^{(s,s)}$ such that φ generates admissible representation for the left action. Obviously, by Lemma 2.1 $\mathcal{A}(G) \otimes V \subseteq \mathcal{A}(G, V)$ since $\Lambda_{v^{\#}}$ is an intertwining. From the other side, by Lemma 2.1 we know that $\Lambda_{v'}(\varphi) \in \mathcal{A}(G)$ if $\varphi \in \mathcal{A}(G, V)$, since $\Lambda_{v'}$ intertwines (left) representations. Thus

$$\mathcal{A}(G) \otimes V = \mathcal{A}(G, V).$$

In the same way we could get that $\mathcal{A}(G, V)$ is the set of all $\varphi \in C(G, V)^{(s,s)}$ such that φ generates admissible representation for the right action. Obviously, $\mathcal{A}(G) \otimes V$ and $\mathcal{A}(G, V)$ are invariant for both actions of G (and isomorphic as $G \times G$-representations).

For an admissible representation (π, X) of G we shall denote by

$$\mathcal{A}(\pi)$$

the vector subspace spanned by all matrix coefficients of π. Then $\mathcal{A}(\pi)$ is invariant for the left and the right action of G.

Let (π, X) be a smooth representation of G and let Y be a complex vector space. Then we consider an action L on $\mathrm{Hom}_{\mathbb{C}}(X, Y)$ defined by

$$(L(g)A)(x) = A(\pi(g^{-1})x), \quad g \in G, A \in \mathrm{Hom}_{\mathbb{C}}(X, Y), x \in X.$$

The smooth part of this representation will be denoted by $\mathrm{Hom}_{\mathbb{C}}(X,Y)^{(s)}$. Note that $(L, \mathrm{Hom}_{\mathbb{C}}(X,\mathbb{C})^{(s)}) = (\tilde{\pi}, \tilde{X})$.

Sometimes we shall consider the action $\mathcal{L}$ of G on $\mathrm{Hom}_{\mathbb{C}}(Y,X)$ given by

$$(\mathcal{L}_g A)(y) = \pi(g)A(y), \quad g \in G, A \in \mathrm{Hom}_{\mathbb{C}}(Y,X), y \in Y.$$

Note that $\mathrm{Hom}_{\mathbb{C}}(Y,X)$ is a smooth representation of G if Y is finite dimensional.

3. Bernstein center $\mathfrak{Z}(G)$ and infinitesimal characters

In the rest of the paper we shall denote by G the group of rational points of a connected reductive group over a local field F. The modulus character of F will be denoted by $|\ |_F$. Let G^0 be the group of all $g \in G$ such that $|\chi(g)|_F = 1$ for all rational characters χ of G. Then

$$G/G^0$$

is a free $\mathbb{Z}$-module of finite rank (the rank is equal to the dimension of a maximal split torus in the center of G).

A character $\chi : G \longrightarrow \mathbb{C}^\times$ is called unramified if it is trivial on G^0. Obviously, one can identify the group of all unramified characters of G with the group

$$\Psi(G) = \mathrm{Hom}_{\mathbb{Z}}(G/G^0, \mathbb{C}^\times).$$

This group has a natural structure of a commutative complex algebraic group (it is isomorphic to $(\mathbb{C}^\times)^{\mathrm{rank}\ (G/G^0)}$.

Let $\tilde{G}$ be the set of equivalence classes of all the irreducible smooth representations of G. Fix $\pi \in \tilde{G}$. Consider

$$\chi \mapsto \chi\pi, \quad \Psi(G) \longrightarrow \tilde{G}.$$

Denote

$$\Psi(G)_\pi = \{\chi \in \Psi(G); \chi\pi \cong \pi\}.$$

Then $\Psi(G)_\pi$ is a finite group and $\chi\Psi(G)_\pi \mapsto \chi\pi$ is a one-to-one map from $\Psi(G)/\Psi(G)_\pi$ onto $\Psi(G)\pi \subseteq \tilde{G}$.

Let $\Omega(G)$ be the set of all conjugacy classes of the pairs (M,ρ) where M is a Levi subgroup of G and ρ is an irreducible cuspidal representation of M. The sets

$$\Omega = \{(M, \chi\rho) : \chi \in \Psi(M)\} \subseteq \Omega(G)$$

are called connected components of $\Omega(G)$. The mapping

$$\chi \mapsto (M, \chi\rho), \quad \Psi(M) \longrightarrow \Omega$$

has finite fibers (usually we shall not make distinction between a pair (M,ρ) and its conjugacy class in $\Omega(G)$). One defines the structure of a complex variety on Ω in a natural way (a function $f : \Omega(G) \to \mathbb{C}$ is called regular, if the restriction of f to any connected component of $\Omega(G)$ is regular). The algebra of all regular functions on $\Omega(G)$ will be denoted by $\mathfrak{Z}(G)$. For a subset $X \subseteq \Omega(G)$ we shall say that it is Zariski open if $X \cap \Omega$ is a Zariski open set in Ω for any connected component Ω of $\Omega(G)$. Zariski open sets in $\Omega(G)$ define a topology on $\Omega(G)$, which will be called Zariski topology on $\Omega(G)$. Since connected components of $\Omega(G)$ are irreducible varieties, an open subset $O \subseteq \Omega(G)$ is dense (with respect to the Zariski topology) if and only if $O \cap \Omega \neq \emptyset$ for any connected component Ω of $\Omega(G)$. Further, the intersection of a finite number of open dense subsets in $\Omega(G)$ is an open dense subset of $\Omega(G)$.

Let $\pi \in \tilde{G}$. Then there is a parabolic subgroup $P = MN$ of G and an irreducible cuspidal representation ρ of M such that π is a subquotient of $\mathrm{Ind}_P^G(\rho)$ ($\mathrm{Ind}_P^G(\rho)$ denotes the representation of G parabolically induced by ρ from P; the induction that we consider is normalized). The class of (ρ, M) in $\Omega(G)$ is uniquely determined by π. In this way we get a canonical projection

$$\Pi : \pi \mapsto (M, \rho), \quad \tilde{G} \to \Omega(G). \tag{3-1}$$

The canonical projection has finite fibers.

We fix a Haar measure dg on G. The convolution algebra of all compactly supported locally constant functions on G will be denoted by $\mathcal{H}(G)$.

We shall recall of the Bernstein center in the rest of this section (see [BD] for more details).

Denote by I the set of idempotents in $\mathcal{H}(G)$. Consider the order $\ll$ on I: $e \ll f$ if $e \in f * \mathcal{H}(G) * f$. For $e \in I$, denote by $Z(e * \mathcal{H}(G) * e)$ the center of the subalgebra $e * \mathcal{H}(G) * e$. One considers the family $Z(e * \mathcal{H}(G) * e), e \in I$, as a projective system, where the transition maps are given in the following way: if $e \ll f$, then the transition map $Z(f * \mathcal{H}(G) * f) \to Z(e * \mathcal{H}(G) * e)$ sends $z \mapsto e * z$. Denote the projective limit by $\mathcal{Z}(G)$. It consists of all systems of elements $h = (h(e))_{e \in I}$, $h(e) \in Z(e * \mathcal{H}(G) * e)$, which satisfy $h(e) = h(f) * e$ if $e \ll f$.

Let (π, X) be a smooth representation of G, and let $h \in \mathcal{Z}(G)$, $x \in X$. Take $e \in I$ such that $\pi(e)x = x$. Define $\pi(h)x$ to be $\pi(h(e))x$. Then $\pi(h)x$ does not depend on e as above. In this way we get a representation of the algebra $\mathcal{Z}(G)$ on X. The actions of G and $\mathcal{Z}(G)$ commutes, i.e. for each $h \in \mathcal{Z}(G)$ the mapping $\pi(h) : X \to X$ is G-intertwining. Further, if $A : X_1 \to X_2$ is an intertwining of two smooth representations of G, then it is also an intertwining of the corresponding representations of $\mathcal{Z}(G)$.

If π is irreducible, then $\mathcal{Z}(G)$ acts by scalars. The character of $\mathcal{Z}(G)$ that we obtain in this way is denoted by

$$\chi_\pi.$$

We shall call χ_π the infinitesimal character of π. We denote by $\mathfrak{Z}^0(G)$ the ideal of $\mathfrak{Z}(G)$, consisting of all functions supported on a finite number of components. An infinitesimal character ω can be interpreted as an algebra homomorphism

$$\omega : \mathfrak{Z}(G) \to \mathbb{C},$$

which is non-trivial on $\mathfrak{Z}^0(G)$.

Note that for each $h \in \mathcal{Z}(G)$ we obtain a function $h' : \tilde{G} \longrightarrow \mathbb{C}$, $\pi \mapsto \chi_\pi(h)$. This function factors through the canonical projection (3-1) $\Pi : \tilde{G} \longrightarrow \Omega(G)$ by a function $\hat{h} : \Omega(G) \longrightarrow \mathbb{C}$. Then $\hat{h} \in \mathfrak{Z}(G)$ and the mapping $h \mapsto \hat{h}$ is an isomorphism of $\mathcal{Z}(G)$ onto $\mathfrak{Z}(G)$. Let (π, X) be a smooth representation of G. We define a representation π of $\mathfrak{Z}(G)$ on X by $\pi(\hat{h})x = \pi(h)x, h \in \mathcal{Z}(G), x \in X$. Clearly, $\mathfrak{Z}(G)$ acts by scalars in an irreducible representation π. The corresponding character of $\mathfrak{Z}(G)$ will be again denoted by χ_π. Obviously, $\pi(\mathfrak{z})x = \chi_\pi(\mathfrak{z})x = \mathfrak{z}(\Pi(\pi))x$ for $\mathfrak{z} \in \mathfrak{Z}(G)$, $x \in X$ (thus the character χ_π is given by evaluation of functions of $\mathfrak{Z}(G)$ in $\Pi(\pi)$).

We shall say that a smooth representation (π, X) has an infinitesimal character if there exists (M, ρ) such that $\pi(\mathfrak{z})x = \mathfrak{z}((M, \rho))x$ for $\mathfrak{z} \in \mathfrak{Z}(G)$, $x \in X$.

For a function $\varphi \in \mathcal{H}(G)$, we denote by $\check{\varphi}$ a function defined by $\check{\varphi}(g) = \varphi(g^{-1})$. Recall that $(\varphi_1 * \varphi_2)\check{} = \check{\varphi}_2 * \check{\varphi}_1$.

Let $h = (h(e))_{e \in I} \in \mathcal{Z}(G)$. Then $h(e) \in Z(e * \mathcal{H}(G) * e)$ and $h(e) = h(f) * e$ if $e \ll f$. Obviously $h(\check{e})\check{} \in Z(e * \mathcal{H}(G) * e)$. Note that $e \ll f$ if and only if $\check{e} \ll \check{f}$. Further, for $e \ll f$ we have $h(\check{e})\check{} = e * h(\check{f})\check{} = h(\check{f})\check{} * e$. Therefore, $(h(\check{e})\check{})_{e \in I} \in \mathcal{Z}(G)$. This element will be denoted by $\check{h}$. Note that by definition

$$\check{h}(e) = h(\check{e})\check{}, \quad h \in \mathcal{Z}(G).$$

From this follows $(\check{h})\check{} = h$ for $h \in \mathcal{Z}(G)$.

Let (π, X) be a smooth representation of G and let Y be a finite dimensional complex vector space. Take $A \in \mathrm{Hom}_{\mathbb{C}}(X, Y)^{(s)}$, $x \in X$ and $\varphi \in \mathcal{H}(G)$. Then

$$(L(\varphi)A)(x) = \left(\int_G \varphi(g)L(g)A\, dg\right)(x) = \int_G \varphi(g)(L(g)A)(x)dg = \int_G \varphi(g)A(\pi(g^{-1})x)dg$$

$$= \int_G A(\varphi(g)\pi(g^{-1})x)dg = A\left(\int_G \varphi(g)\pi(g^{-1})xdg\right) = A\left(\int_G \varphi(g^{-1})\pi(g)xdg) = A(\pi(\check{\varphi})x\right).$$

From this follows

3.1. Lemma. *For a smooth representation (π, X) of G, a finite dimensional complex vector space Y, $A \in Hom_{\mathbb{C}}(X, Y)^{(s)}$, $x \in X$ and $h \in \mathcal{Z}(G)$ we have*

$$(L(h)A)(x) = A(\pi(\check{h})x).$$

In particular, $(\tilde{\pi}(h)\tilde{x})(x) = \tilde{x}(\pi(\check{h})x)$ for $x \in X$ and $\tilde{x} \in \tilde{X}$.

Proof. Chose an open compact subgroup H of K such that x and A are invariant for the action of H. Denote by e_H the characteristic function of H divided by the Haar measure of H. Then $e_H \in I$, $\check{e}_H = e_H$, $L(e_H)A = A$ and $\pi(e_H)x = x$. Now

$$\begin{aligned}(L(h)A)(x) &= (L(h(e_H))A)(x) = A(\pi(h(e_H)\check{}))(x) \\ &= A(\pi(h(\check{e}_H)\check{}))(x) = A(\pi(\check{h}(e_H))(x) = A(\pi(\check{h})(x).\end{aligned}$$

This completes the proof. □

The above lemma implies that for an irreducible representation π we have

$$\chi_\pi(h) = \chi_{\tilde{\pi}}(\check{h}), \quad h \in \mathcal{Z}(G).$$

For $\mathfrak{z} \in \mathfrak{Z}(G)$ we shall denote by $\tilde{\mathfrak{z}} \in \mathfrak{Z}(G)$ the function defined by

$$\tilde{\mathfrak{z}}((M, \rho)) = \mathfrak{z}((M, \tilde{\rho})).$$

3.2. Lemma. *(i) For $h \in \mathcal{Z}(G)$ we have*

$$(\check{h})\hat{} = (\hat{h})\tilde{}.$$

(ii) Let (π, X) be a smooth representation of G and let Y be a finite dimensional complex vector space. Then for $A \in Hom_{\mathbb{C}}(X, Y)^{(s)}$, $x \in X$ and $\mathfrak{z} \in \mathfrak{Z}(G)$ we have

$$(L(\mathfrak{z})A)(x) = A(\pi(\tilde{\mathfrak{z}})x).$$

Proof. (i) Let $\pi \in \tilde{G}$. Suppose that the image of π under the canonical projection $\Pi : G \to \Omega(G)$ is (M, ρ). Then $\tilde{\pi}$ goes to $(M, \tilde{\rho})$ under this projection. Using Lemma 3.1 we get

$$(\check{h})\hat{}((M, \rho)) = \chi_\pi(\check{h}) = \chi_{\tilde{\pi}}(h) = \hat{h}((M, \tilde{\rho})) = (\hat{h})\tilde{}(M, \rho)).$$

This ends the proof of (i).
(ii) Let $\mathfrak{z} = \hat{h}$. Now from (i) we get

$$(L(\mathfrak{z})A)(x) = (L(h)A)(x) = A(\pi(\check{h})x) = A(\pi((\check{h})\hat{})x) = A(\pi((\hat{h})\tilde{})x) = A(\pi(\tilde{\mathfrak{z}})x).$$

Thus (ii) also holds. $\square$

3.3. Lemma. *Let $z \in \mathfrak{Z}(G)$ and $f \in C(G)^{(s,s)}$. Then*

$$R_{\mathfrak{z}} f = L_{\tilde{\mathfrak{z}}} f.$$

Proof. Note that for $\psi \in \mathcal{H}(G)$ and $f \in C(G)^{(s,s)}$ we have

$$L_\psi f = \psi * f \quad \text{and} \quad R_\psi f = f * \check{\psi}.$$

Let H be an open compact subgroup of G such that f is constant on double H-classes. For $\varphi \in Z(e_H * \mathcal{H}(G) * e_H)$ we shall show that $\varphi * f = f * \varphi$.

Fix $x \in G$. Introduce the function f_x on G:

$$f_x(y) = \begin{cases} f(y) \text{ for } y \in \operatorname{supp}(\varphi)^{-1} x\, H \cup Hx \operatorname{supp}(\varphi)^{-1}, \\ 0 \quad \text{ for } y \notin \operatorname{supp}(\varphi)^{-1} x\, H \cup Hx \operatorname{supp}(\varphi)^{-1}. \end{cases}$$

Obviously, $f_x \in e_H * \mathcal{H}(G) * e_H$. Now

$$(\varphi * f)(x) = \int_G \varphi(g) f(g^{-1}x) dg = \int_G \varphi(g) f_x(g^{-1}x) dg = (\varphi * f_x)(x)$$
$$(f_x * \varphi)(x) = \int_G f_x(xg^{-1}) \varphi(g) dg = \int_G f(xg^{-1}) \varphi(g) dg = (f * \varphi)(x).$$

Thus, $\varphi * f = f * \varphi$.

Take $h \in \mathcal{Z}(G)$ such that $\mathfrak{z} = \hat{h}$. Now the above calculation implies

$$L_{\mathfrak{z}} f = L_{\hat{h}} f = L_h f = L_{h(e_H)} f = h(e_H) * f = f * h(e_H)$$
$$= f * (h(e_H)\check{})\check{} = R_{h(e_H)\check{}} f = R_{h(\check{e}_H)\check{}} f = R_{\check{h}(e_H)} f = R_{\check{h}} f = R_{(\check{h})\hat{}} f = R_{(\hat{h})\tilde{}} f = R_{\tilde{\mathfrak{z}}} f. \quad \square$$

3.4. Corollary. *Let V be a finite dimensional complex vector space. Then for $z \in \mathfrak{Z}(G)$ and $\varphi \in C(G,V)^{(s,s)}$ we have $R_{\mathfrak{z}}\varphi = L_{\tilde{\mathfrak{z}}}\varphi$.*

Proof. This follows from the fact that $C(G,V)^{(s,s)}$ and $C(G)^{(s,s)} \otimes V$ are isomorphic as $G \times G$-representations. $\square$

3.5. Remark. *The above corollary implies that a subspace of $C(G,V)^{(s,s)}$ is invariant for the action of $\mathfrak{Z}(G)$ induced by the right translations if and only if it is invariant for the action of $\mathfrak{Z}(G)$ induced by the left translations.*

Let us recall of the following result from [BD]:

3.6. Theorem. *Suppose that (π, X) is a finitely generated smooth representation of G and H is an open compact subgroup of G. Then the space X^H of all vectors in X fixed by elements of H is invariant for the action of $\mathfrak{Z}(G)$ and it is a finitely generated as $\mathfrak{Z}^0(G)$-module.*

The following immediate consequence of the above result will be useful for us.

3.7. Corollary. *Each finitely generated smooth representation of G which has an infinitesimal character, is admissible. Moreover, it has finite length.*

Let $M < G$ be a standard Levi subgroup. Define the morphism $i_{GM} : \Omega(M) \to \Omega(G)$ by $(L,\rho) \to (L,\rho)$. This is a finite morphism of algebraic varieties. The morphism i_{GM} is not in general an inclusion, since cuspidal pairs conjugate under G may be non-conjugate under M. We follow [BDK] to call the corresponding map $i^*_{GM} : \mathfrak{Z}(G) \to \mathfrak{Z}(M)$ the Harish-Chandra homomorphism. As for the real groups, $\mathfrak{Z}(M)$ is a finitely generated $\mathfrak{Z}(G)$-module.

Let $P = MN$ be a parabolic subgroup of G. The functor of normalized parabolic induction was denoted by Ind_P^G. The normalized Jacquet functor going in the opposite direction will be denoted by r_{MG}.

3.8. Proposition ([BD]). *Suppose that π and σ are smooth representations of G and M respectively. Let $\mathfrak{z} \in \mathfrak{Z}(G)$. Then*

(1)
$$Ind_P^G(\sigma(i^*_{GM}(\mathfrak{z}))) = (Ind_P^G(\sigma))(\mathfrak{z}),$$

(2)
$$r_{GM}(\pi(\mathfrak{z})) = (r_{GM}(\pi))(i^*_{GM}(\mathfrak{z})).$$

We shall say that $(M,\rho) \in \Omega(G)$ is regular in G if there does not exist $w \in G\backslash M$ which normalizes M such that $\rho \cong w.\rho$ where $w.\rho$ denotes a representation $m \mapsto \rho(w^{-1}mw)$ of M. We shall say that $(M,\rho) \in \Omega(G)$ is irreducible in G if $\mathrm{Ind}_P^G(\rho)$ is irreducible. Further, we shall say that $(M,\rho) \in \Omega(G)$ is semi simple in G, if each admissible representation (π, X) of G which has an infinitesimal character equal to $\mathfrak{z} \mapsto \mathfrak{z}((M,\rho))$, is semi simple.

3.9. Proposition. *There exists an open dense subset $\Omega(G)'$ of $\Omega(G)$ (with respect to the Zariski topology) such that each $(M,\rho) \in \Omega(G)'$ is irreducible, regular and semi simple.*

In the first version of this paper we gave a direct proof of above proposition (based on Proposition 3.8). Later, P. Schneider has informed us that the claim of the proposition follows from Proposition 3.14 of [BD]. Because of this, we omit the proof here (although this proof can be of independent interest; it may be also used for determining explicitly the set whose existence is claimed in the above proposition).

3.10. Remark. *It is easy to determine a set $\Omega(G)'$ of the above proposition quite explicitly (from our proof of above proposition), if we know when the representations parabolically induced from cuspidal representations reduce. For example, one can take*

$$\Omega(SL(2,F))' = \Omega(SL(2,F)) \setminus \left(\{(M_\emptyset,\chi);\chi^2 = 1_{F^\times}\} \bigcup \{(M_\emptyset, |\ \ |_F)\}\right).$$

4. Definition of generalized spherical functions

In the rest of this paper we shall fix a maximal compact subgroups K of G (which is the group of rational points of a connected reductive group over a local non-archimedean field F) which satisfies

$$G = KP_\emptyset,$$

where $P_\emptyset$ is a minimal parabolic subgroup of G. Recall that K is an open subgroup of G. In the sequel, we shall assume that the Haar measure of G is normalized on K.

Let τ_1 and τ_2 be continuous finite dimensional representations of K on V_{τ_1} and V_{τ_2} respectively. Denote $\tau = (\tau_1, \tau_2)$ and let

$$V = \mathrm{Hom}_{\mathbb{C}}(V_{\tau_2}, V_{\tau_1}).$$

We shall consider (τ, V) as a double representation of K:

$$\tau(k_1,k_2)\varphi = \tau_1(k_1)\,\varphi\,\tau_2(k_2)$$

for $k_1, k_2 \in K$ (it means that $(k_1,k_2) \mapsto \tau(k_1,k_2^{-1})$ is a representation of $K \times K$).

Denote by $C(G,\tau)$ the space of all functions $\varphi : G \longrightarrow V$ satisfying

$$\varphi(k_1 g k_2) = \tau_1(k_1)\varphi(g)\tau_2(k_2)$$

for all $k_1, k_2 \in K, g \in G$, i.e.

$$L_{k_1^{-1}} R_{k_2}\varphi = \Lambda_{\tau_1(k_1)\circ}\Lambda_{\circ\tau_2(k_2)}\varphi$$

where $\tau_1(k_1)\circ,\ \circ\tau_2(k_2) : V \longrightarrow V$ are the mappings $A \mapsto \tau_1(k_1)\circ A$ and $A \circ \tau_2(k_2)$ respectively. Clearly,

$$C(G,\tau) \subseteq C(G,V)^{(s,s)}.$$

Set

$$\mathcal{A}(G,\tau) = C(G,\tau) \cap \mathcal{A}(G,V).$$

4.1. Lemma. *The space $C(G,\tau)$ is invariant for the action of $\mathfrak{Z}(G)$ with respect to the left and right action.*

Proof. To see this, take $\varphi \in C(G,V)$, $\mathfrak{z} \in \mathfrak{Z}(G)$ and $k_1, k_2 \in K$. Then

$$L_{k_1^{-1}} R_{k_2}(L_{\mathfrak{z}}\varphi) \;=\; L_{\mathfrak{z}}(L_{k_1^{-1}} R_{k_2}(\varphi)) \;=\; L_{\mathfrak{z}}(\Lambda_{\tau_1(k_1)\circ}\Lambda_{\circ\tau_2(k_2)}\varphi) \;=\; \Lambda_{\tau_1(k_1)\circ}\Lambda_{\circ\tau_2(k_2)}(L_{\mathfrak{z}}\varphi).$$

Thus, $L_{\mathfrak{z}}\varphi \in C(G,\tau)$. In a similar way one shows that $R_{\mathfrak{z}}\varphi \in C(G,\tau)$. □

Let ω be an infinitesimal character of G, i.e. a character of $\mathfrak{Z}(G)$ which is non-trivial on $\mathfrak{Z}^0(G)$. Denote

$$\mathcal{E}_\omega(G,V) = \mathcal{E}_\omega^L(G,V) = \{\varphi \in C(G,V)^{(s,s)}; L_{\mathfrak{z}}\varphi = \omega(\mathfrak{z})\varphi \text{ for } \mathfrak{z} \in \mathfrak{Z}(G)\},$$
$$\mathcal{E}_\omega^R(G,V) = \{\varphi \in C(G,V)^{(s,s)}; R_{\mathfrak{z}}\varphi = \omega(\mathfrak{z})\varphi \text{ for } \mathfrak{z} \in \mathfrak{Z}(G)\}.$$

4.2. Lemma. *Each of the spaces $\mathcal{E}_\omega(G,V)$ and $\mathcal{E}_\omega^R(G,V)$ is invariant for the left and the right action of G.*

Proof. Let $\varphi \in \mathcal{E}_\omega(G,V)$, $g \in G$ and $\mathfrak{z} \in \mathfrak{Z}(G)$. Then $L_{\mathfrak{z}}L_g\varphi = L_gL_{\mathfrak{z}}\varphi = \omega(\mathfrak{z})L_g\varphi$ and $L_{\mathfrak{z}}R_g\varphi = R_gL_{\mathfrak{z}}\varphi = \omega(\mathfrak{z})R_g\varphi$. Thus, $L_g\varphi, R_g\varphi \in \mathcal{E}_\omega(G,V)$. In the same way one shows the invariance of $\mathcal{E}_\omega^R(G,V)$. □

4.3. Corollary. *The spaces $\mathcal{E}_\omega(G,V)$ and $\mathcal{E}_\omega^R(G,V)$ are contained in $\mathcal{A}(G,V)$.*

Proof. By the above lemma, each φ in $\mathcal{E}_\omega(G,V)$ (and in $\mathcal{E}_\omega^R(G,V)$) generates a (finitely generated) representation with an infinitesimal character. By Corollary 3.7, it is admissible. Thus, $\varphi \in \mathcal{A}(G,V)$. □

Denote

$$\mathcal{E}_\omega(G,\tau) = \mathcal{E}_\omega(G,V) \cap C(G,\tau),$$
$$\mathcal{E}_\omega^R(G,\tau) = \mathcal{E}_\omega^R(G,V) \cap C(G,\tau).$$

These spaces are contained in $\mathcal{A}(G,V)$ and $C(G,\tau)$. We shall denote

$$\mathcal{A}(G,\tau) = \mathcal{A}(G,V) \cap C(G,\tau).$$

Then $\mathcal{E}_\omega(G,\tau)$ and $\mathcal{E}_\omega^R(G,\tau)$ are contained in $\mathcal{A}(G,\tau)$.

5. The mapping $\varphi \mapsto T_\varphi$

Denote

$$C(G,V)_{\tau_2}^{(s)} = \{\varphi \in C(G,V)^{(s,s)}; \varphi(gk) = \varphi(g)\tau_2(k) \text{ for all } g \in G, k \in K\}.$$

The above condition can be expressed as

$$R_k\varphi = \Lambda_{\circ\tau_2(k)}\varphi,$$

for all $k \in K$. Clearly, $C(G,V)_{\tau_2}^{(s)}$ is invariant for the left action of the group and the representation defined by this action is smooth. From this follows that $C(G,V)_{\tau_2}^{(s)}$ is invariant for the left action of $\mathfrak{Z}(G)$). Therefore, it is also invariant for the right action of $\mathfrak{Z}(G)$ by Remark 3.5 (one can also see this directly from the fact that $R_kR_{\mathfrak{z}}\varphi = R_{\mathfrak{z}}R_k\varphi = R_{\mathfrak{z}}\Lambda_{\circ\tau_2(k)}\varphi = \Lambda_{\circ\tau_2(k)}R_{\mathfrak{z}}\varphi$ for $\mathfrak{z} \in \mathfrak{Z}(G),\ g \in G,\ k \in K$ and $\varphi \in C(G,V)_{\tau_2}^{(s)}$).

We shall look at the Hecke algebra of all locally constant compactly supported functions $\mathcal{H}(G)$ as a left $\mathcal{H}(G)$-module in an obvious way. This is the same action that we get if we look at the representation L of G on $\mathcal{H}(G)$ by left translations, and integrate it to the representation of the Hecke algebra $\mathcal{H}(G)$.

We consider $\mathcal{H}(K) \subseteq \mathcal{H}(G)$ as a subalgebra. We shall consider $\mathcal{H}(G)$ as a right $\mathcal{H}(K)$-module in a natural way (the action of $\mathcal{H}(K)$ is given by the multiplication from the right-hand side).

We shall consider the action of G on $\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_1}$ which comes from the action of G on the first factor.

5.1. Lemma. *The representation of G on $\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_1}$ is finitely generated.*

Proof. Let H be the kernel of τ_2. Denote by e_H the characteristic function of H, divided by the measure of H. It is enough to show that $e_H \otimes V_{\tau_1}$ generates $\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_1}$ as an $\mathcal{H}(G)$-module. Note that

$$\begin{aligned}\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_1} &= \{h \otimes v; h \in \mathcal{H}(G), v \in V_{\tau_1}\}\\ &= \{h \otimes e_H v; h \in \mathcal{H}(G), v \in V_{\tau_1}\} = \{h * e_H \otimes v; h \in \mathcal{H}(G), v \in V_{\tau_1}\}.\end{aligned}$$

This proves the lemma. □

5.2. Lemma. *Let (π, X) be a smooth representation of G and let ω be an infinitesimal character.*

(i) The subspace $X_{[\omega]}$ spanned by

$$\pi(\mathfrak{z})x - \omega(\mathfrak{z})x,\ \mathfrak{z} \in \mathfrak{Z}(G), x \in X,$$

is G-invariant. The subrepresentation on this space will be denoted by

$$(\pi_{[\omega]}, X_{[\omega]}).$$

(ii) The space $X/X_{[\omega]}$ will be denoted by $X^{<\omega>}$. The quotient representation on this space will be denoted by

$$(\pi^{<\omega>}, X^{<\omega>}).$$

Then the representation $(\pi^{<\omega>}, X^{<\omega>})$ has an infinitesimal character, and the infinitesimal character is ω (i.e. $(\pi^{<\omega>}, X^{<\omega>})$ is an ω-representation). If π is finitely generated, then $(\pi^{<\omega>}, X^{<\omega>})$ is admissible (ω-representation).

(iii) If $\mathfrak{Z}(G)$ acts on a complex vector space W, we shall denote

$$W_\omega = \{w \in W; \mathfrak{z}.w = \omega(\mathfrak{z})w \text{ for all } \mathfrak{z} \in \mathfrak{Z}(G)\}.$$

The space X_ω is G-invariant and the subrepresentation on this subspace will be denoted by

$$(\pi_\omega, X_\omega).$$

(iv) Let U be a finite dimensional complex vector space. There is a natural representation L of G on $Hom_{\mathbb{C}}(X,U)$ (see section 2.). The smooth part of this representation will be denoted by $Hom_{\mathbb{C}}(X,U)^{(s)}$.

If $T : X \longrightarrow U$ is a linear operator which vanishes on $X_{[\tilde{\omega}]}$, then T factors through $X^{<\tilde{\omega}>} = X/X_{[\tilde{\omega}]}$ by the operator which will be denoted by $T^\flat$. Then the mapping $T \mapsto T^\flat$ defines an isomorphism of G-representations

$$Hom_{\mathbb{C}}(X,U)^{(s)}_\omega \longrightarrow Hom_{\mathbb{C}}(X^{<\tilde{\omega}>},U)^{(s)}.$$

Proof. The invariance in (i) and (iii) is clear.

To prove (ii), denote the quotient map with Φ and the quotient representation with π'. Then

$$\pi^{<\omega>}(\mathfrak{z})\Phi(x) - \omega(\mathfrak{z})\Phi(x) = \Phi(\pi(\mathfrak{z})x) - \omega(\mathfrak{z})\Phi(x) = \Phi(\pi(\mathfrak{z})x - \omega(\mathfrak{z})x) = 0.$$

The last claim of (ii) follows from Corollary 3.7.

Now we shall prove (iv). Let $T \in \mathrm{Hom}_{\mathbb{C}}(X,U)^{<\omega>}_\omega$. Then $(L(\mathfrak{z})T)(x) = \omega(\mathfrak{z})T(x)$, which implies $T(\pi(\check{\mathfrak{z}})x) = \omega(\mathfrak{z})T(x)(= \tilde{\omega}(\check{\mathfrak{z}})T(x))$ for all $\mathfrak{z} \in \mathfrak{Z}(G)$ and $x \in X$. Thus T vanishes on all $\pi(\mathfrak{z})x - \tilde{\omega}(\mathfrak{z})x$. Therefore, T induces an operator $T^\flat \in \mathrm{Hom}_{\mathbb{C}}(X/X_{[\omega]},U)$. The mapping $T \mapsto T^\flat$, from $\mathrm{Hom}_{\mathbb{C}}(X,U)_\omega$ into $\mathrm{Hom}_{\mathbb{C}}(X/X_{[\tilde{\omega}]},U)$ is obviously G-intertwining. Therefore, $T^\flat$ is in the smooth part $\mathrm{Hom}_{\mathbb{C}}(X/X_{[\tilde{\omega}]},U)^{(s)}$ of $\mathrm{Hom}_{\mathbb{C}}(X^{<\tilde{\omega}>},U) = \mathrm{Hom}_{\mathbb{C}}(X/X_{[\tilde{\omega}]},U)$.

From the other side, $\mathrm{Hom}_{\mathbb{C}}(X/X_{[\tilde{\omega}]},U)_*$ is a representation with an infinitesimal character ω (we see it in the same way as above). There is natural embedding $S \mapsto S^\sharp$ of the space $\mathrm{Hom}_{\mathbb{C}}(X/X_{[\tilde{\omega}]},U)^{(s)}$ into $\mathrm{Hom}_{\mathbb{C}}(X,U)^{(s)}_\omega$ (which is G-intertwining). Obviously, the mappings $T \mapsto T^\flat$ and $S \mapsto S^\#$ are inverses each other. Therefore, we can identify $\mathrm{Hom}_{\mathbb{C}}(X,U)^{(s)}_\omega$ with $\mathrm{Hom}_{\mathbb{C}}(X/X_{[\tilde{\omega}]},U)^{(s)}$ in this way. $\square$

We define an action L of G on $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}$ in a natural way:

$$(L(g)A)(h \otimes v) = A(L_{g^{-1}}h \otimes v)$$

(see section 2.). Then for $f \in \mathcal{H}(G)$

$$(L(f)A)(h \otimes v) = A(\check{f} * h \otimes v).$$

In this way we have also an action of $\mathfrak{Z}(G)$ on $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}$.

Proposition 5.3. *(i) Let $\varphi \in C(G,V)^{(s)}_{\tau_2}$. Then the formula*

$$\sum_i h_i \otimes v_i \mapsto \sum_i ((\check{h}_i * \varphi)(1))(v_i) = \sum_i ((L_{\check{h}_i}\varphi)(1))(v_i)$$

defines a linear mapping

$$T_\varphi : \mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2} \to V_{\tau_1}$$

(in particular, the above mapping is well-defined).

(ii) The mapping

$$T : C(G,V)^{(s)}_{\tau_2} \to Hom_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)},$$
$$\varphi \mapsto T_\varphi$$

is an injective G-intertwining.

(iii) We have

$$Hom_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1}) \subseteq Hom_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}.$$

Further, $Hom_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$ is invariant for the action of $\mathfrak{Z}(G)$.

(iv) If $\varphi \in C(G,\tau)$, then $T_\varphi \in Hom_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$.

Proof. (i) Let $\varphi \in C(G,V)^{(s)}_{\tau_2}$. Define

$$T'_\varphi : \mathcal{H}(G) \times V_{\tau_2} \to V_{\tau_1}, \quad (h,v) \mapsto ((\check{h} * \varphi)(1))(v) = ((L_{\check{h}}\varphi)(1))(v),$$

where $\check{h}(x) = h(x^{-1})$ (recall $(h_1 * h_2)\check{} = \check{h}_2 * \check{h}_1$). Now for $b \in \mathcal{H}(K)$

$$T'_\varphi(h * b, v) = ((L_{(h*b)\check{}}\varphi)(1))(v) = ((L_{\check{b}*\check{h}}\varphi)(1))(v) = ((L_{\check{b}}(L_{\check{h}}\varphi))(1))(v)$$
$$\int_K b(k) L_k(L_{\check{h}}\varphi)(1)v\ dk = \int_K b(k)(L_{\check{h}}\varphi)(k)v\ dk = \int_K b(k)(L_{\check{h}}\varphi)(1)\tau_2(k)v\ dk$$
$$= (L_{\check{h}}\varphi)(1)\int_K b(k)\tau_2(k)v\ dk = (L_{\check{h}}\varphi)(1)\tau_2(b)v = T'_\varphi(h, \tau_2(b)v).$$

Therefore, T'_φ can be factored through

$$T_\varphi : \mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2} \to V_{\tau_1}, \quad \sum_i h_i \otimes v_i \mapsto \sum_i ((\check{h}_i * \varphi)(1))(v_i) = \sum_i ((L_{\check{h}_i}\varphi)(1))(v_i).$$

In this way we get that the mapping

$$T : C(G,V)^{(s)}_{\tau_2} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1}), \quad \varphi \mapsto T_\varphi$$

is well-defined. This mapping is obviously linear.

(ii) First we shall show that this mapping is injective. Suppose $T_\varphi = 0$. Then

$$0 = T_\varphi(h \otimes v) = ((L_{\check{h}}\varphi)(1))(v),$$

for all $h \in \mathcal{H}(G)$ and $v \in V_{\tau_2}$. Thus

$$(L_h\varphi)(1) = 0 \text{ for all } h \in \mathcal{H}(G).$$

Take $h \in \mathcal{H}(G)$ such that $L_h\varphi = \varphi$. Then for $x \in G$

$$\varphi(x) = (L_h\varphi)(x) = \left(\int_G h(g)\,(L_g\varphi)\,dg\right)(x) = \int_G h(g)\varphi(g^{-1}x)dg$$
$$= \int_G h(xg)\varphi(g^{-1})dg = \int_G (L_{x^{-1}}h)(g)\varphi(g^{-1})dg = (L_{L_{x^{-1}}h}\varphi)(1).$$

This implies $\varphi = 0$. Thus, T is injective.

Now take $\varphi \in C(G,V)^{(s)}_{\tau_1}$. Then for $f \in \mathcal{H}(G)$

$$T_{L_f\varphi}(h \otimes v) = T_{f*\varphi}(h \otimes v) = (\check{h} * f * \varphi)(1)v$$
$$= T_\varphi(\check{f} * h \otimes v) = T_\varphi(L_{\check{f}}h \otimes v) = (L(f)T_\varphi)(h \otimes v).$$

This shows that $\varphi \mapsto T_\varphi$ is G-intertwining.

Since T is G-intertwining, the image of T is in the smooth part

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}$$

of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$.

(iii) The inclusion in (iii) follows directly from the definition of $\mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$.

Let B be a linear operator on V_{τ_1}. Then the operator

$$\Lambda_{B\circ} : \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1}), A \mapsto BA$$

is a G-intertwining since

$$(L(g)(\Lambda_{B\circ}A))(h \otimes v) = (\Lambda_{B\circ}A)(L_{g^{-1}}h \otimes v) = B(A(L_{g^{-1}}h \otimes v))$$
$$= B((L(g)A)(h \otimes v)) = (\Lambda_{B\circ}(L(g)A))(h \otimes v)).$$

We shall show now that $L(\mathfrak{z})A \in \mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$ if A is in the space $\mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$ and $\mathfrak{z} \in \mathfrak{Z}(G)$. To see this, take $k \in K$ and consider

$$(L(\mathfrak{z})A)(L_kh \otimes v) = (L(k^{-1})(L(\mathfrak{z})A))(h \otimes v) = (L(\mathfrak{z})L(k^{-1})A)(h \otimes v)$$
$$= L(\mathfrak{z})(L(k^{-1})A(h \otimes v)) = L(\mathfrak{z})(A(L_kh \otimes v)) = L(\mathfrak{z})(\tau_1(k)(A(h \otimes v)))$$
$$= L(\mathfrak{z})(\Lambda_{\tau_1(k)}A)(h \otimes v) = \Lambda_{\tau_1(k)}L(\mathfrak{z})A(h \otimes v) = \tau_1(k)((L(\mathfrak{z})A)(h \otimes v)).$$

This shows that $\mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$ is invariant for the action of $\mathfrak{Z}(G)$.

(iv) Let $\varphi \in C(G,\tau)$. For $h \in \mathcal{H}(G)$, $v \in V_{\tau_2}$ and $k \in K$ we have

$$T_\varphi(L_kh \otimes v) = (((L_kh)\check{} * \varphi)(1))v = \int_G L_kh(g)\varphi(g)vdg = \int_G h(k^{-1}g)\varphi(g)vdg$$
$$= \int_G h(g)\varphi(kg)vdg = \tau_1(k)\int_G h(g)\varphi(g)vdg = \tau_1(k)(\check{h} * \varphi)(1)v = \tau_1(k)T_\varphi(h \otimes v).$$

This shows $T_\varphi \in \mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$. $\square$

5.4. Proposition. *There exists a one-to-one linear mapping*

$$\mathcal{E}_\omega(G,\tau) \quad \rightarrow \quad Hom_{\mathcal{H}(K)}((\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1}).$$

Proof. Recall that for $\varphi \in C(G,\tau)$, $T_\varphi \in \mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$, and further for $\varphi \in \mathcal{E}_\omega(G,\tau)$, T_φ is from

$$\begin{aligned}(5\text{-}1)\quad \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}_\omega \cap \mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})&\\ = \mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})_\omega.&\end{aligned}$$

We know also that $\varphi \mapsto T_\varphi$ is a one-to-one mapping. Therefore, we have a one-to one mapping from $\mathcal{E}_\omega(G,\tau)$ into $\mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})_\omega$. This one-to-one mapping goes from $\mathcal{E}_\omega(G,\tau)$ into

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})_\omega\\ &= \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}_\omega \cap \mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})\\ &\cong \mathrm{Hom}_{\mathbb{C}}((\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1})^{(s)} \cap \mathrm{Hom}_{\mathcal{H}(K)}((\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1})\\ &= \mathrm{Hom}_{\mathcal{H}(K)}((\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1}).\end{aligned}$$

This proves the proposition. $\square$

5.5. Corollary. *The spaces of generalized spherical functions $\mathcal{E}_\omega(G,\tau)$ are finite dimensional.*

Proof. Note that $(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}$ is an admissible representation by Corollary 3.7, Lemma 5.1 and Proposition 5.2. This and the above proposition imply the corollary. $\square$

6. The mapping $T \mapsto \varphi_T$

For $T \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}$ we define $\varphi_T : G \rightarrow V$ by the formula:

$$\varphi_T(g)(v) = T(L_g(e_{K_0}\otimes v)), \quad v \in V_{\tau_2},$$

where L denotes the natural representation of G on $\mathcal{H}(G)\otimes V$ (the action is on the first factor), and

$$K_0 = \mathrm{Ker}(\tau_1)\cap \mathrm{Ker}(\tau_2).$$

6.1. Proposition. *(i) The mapping $T \mapsto \varphi_T$ is a one-to-one G-intertwining from*

$$Hom_{\mathbb{C}}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)} \textit{ into } C(G,V)^{(s)}_{\tau_2}.$$

(ii) There exists a one-to-one G-intertwining from

$$Hom_{\mathbb{C}}(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}_\omega \cong Hom_{\mathbb{C}}((\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1})^{(s)}$$

into

$$C(G,V)_{\tau_2}^{(s)} \cap \mathcal{E}_\omega(G,V).$$

(iii) If $T \in Hom_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$*, then* $\varphi_T \in C(G,\tau)$.

(iv) There exists a one-to-one linear mapping of

$$Hom_{\mathcal{H}(K)}((\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1})^{(s)} \text{ into } \mathcal{E}_\omega(G,\tau).$$

Proof. (i) For $k \in K$ and $\varphi \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}$ we have

$$\begin{aligned}\varphi_T(gk)(v) &= T(L_{gk}(e_{K_0} \otimes v)) = T(L_{gk}e_{K_0} \otimes v)) = T(L_g L_k e_{K_0} \otimes v) = T(L_g(L_k e_{K_0} \otimes v)) \\ &= T(L_g(\frac{1}{\mu(K_0)}\mathrm{ch}_{kK_0} \otimes v)) = T(L_g(e_{K_0} \otimes \tau_2(\frac{1}{\mu(K_0)}\mathrm{ch}_{kK_0})v)) = T(L_g(e_{K_0} \otimes \tau_2(k)v)) \\ &\qquad \varphi_T(g)(\tau_2(k)v)\end{aligned}$$

(we denoted in the above computation the characteristic function of a subset Y by ch_Y). This implies that $\varphi_T \in C(G,V)_{\tau_2}$.

We shall show now that $T \mapsto \varphi_T$ is a G-intertwining:

$$\begin{aligned}&(L_x\varphi_T)(g)v = \varphi_T(x^{-1}g)v = T(L_{x^{-1}g}(e_{K_0} \otimes v)) = T((L_{x^{-1}g}e_{K_0}) \otimes v) \\ &= T(L_{x^{-1}}(L_g e_{K_0}) \otimes v) = L(x)T(L_g(e_{K_0}) \otimes v)) = (L(x)T)(L_g(e_{K_0}) \otimes v)) = \varphi_{L(x)T}(g)v.\end{aligned}$$

Note that now we get also that $\varphi_T \in C(G,V)_{\tau_2}^{(s)}$.

Now we shall prove that $T \mapsto \varphi_T$ is injective. Suppose $T \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}$ and $\varphi_T(g)v = 0$ for all $g \in G$ and $v \in V_{\tau_2}$. Then $T(L_g(e_{K_0} \otimes v)) = T(L_g e_{K_0} \otimes v) = 0$. Since $L_g e_{K_0} \otimes v$ generate vector space $\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}$, we have $T = 0$.

(ii) From (i) follows directly (ii).

(iii) Suppose that $\varphi \in \mathrm{Hom}_{\mathcal{H}(K)}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})$. Then we have for $k \in K$:

$$\varphi_T(kg)(v) = T(L_{kg}(e_{K_0} \otimes v)) = \tau_1(k)T(L_g(e_{K_0} \otimes v)) = \tau_1(k)\varphi_T(g)v.$$

Thus, $\varphi_T \in C(G,\tau)$.

(iv) We get (iv) directly from (iii). □

6.2. Remarks. *(i) The study of generalized spherical functions in a natural way reduces to the case of* τ_1 *and* τ_2 *irreducible representations of* K*. For a smooth representation* (π, X) *of* G*, we shall denote by*

$$X[\tau_1]$$

the τ_1*-isotipic subspace of* X*.*

(ii) Note that

$$\begin{aligned} e_{K_0}\otimes \quad : \ & V_{\tau_2} \longrightarrow \mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, \\ & v \mapsto e_{K_0} \otimes v, \end{aligned}$$

which shows up in the definition of $\mapsto \varphi_T$ *on* $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}$*, is a* K*-intertwining. Therefore, if* τ_2 *is irreducible, then* $e_{K_0} \otimes v \in (\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})[\tau_2]$ *for* $v \in V_{\tau_2}$*).*

(iii) If we consider the mapping that $T \mapsto \varphi_T$ *defines on* $\mathrm{Hom}_{\mathbb{C}}((\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1})^{(s)}$ *(which is isomorphic to* $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2}, V_{\tau_1})^{(s)}_{\omega}$*), it is given by the formula*

$$g \mapsto \left(v \mapsto T(L_g(e_{K_0} \otimes v + (\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})_{[\tilde{\omega}]}\right).$$

Note that again

$$v \mapsto e_{K_0} \otimes v + (\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})_{[\tilde{\omega}]}$$

is a K*-intertwining of* V_{τ_2} *(now into* $(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}$*). This* K*-intertwining will be denoted by*

$$\bar{e}_{K_0} \otimes \quad .$$

(iv) Let the infinitesimal character ω *corresponds to* (M, ρ)*. Suppose that* τ_2 *is a* K*-type of* $\mathrm{Ind}_P^G(\tilde{\rho})$*. Then there exists a non-zero* K*-intertwining* $\mu : V_{\tau_2} \longrightarrow \mathrm{Ind}_P^G(\tilde{\rho})$*. Consider the mapping*

$$(f, v) \mapsto R_f(\mu(v)), \quad \mathcal{H}(G) \times V_{\tau_2} \longrightarrow \mathrm{Ind}_P^G(\tilde{\rho}).$$

This is obviously a non-trivial bilinear mapping. Consider now for $f \in \mathcal{H}(G), h \in \mathcal{H}(K)$ *and* $v \in V_{\tau_2}$

$$R_{f*h}(\mu(v)) = R_f R_h(\mu(v)) = R_f(\mu(\tau_2(h)v)).$$

This implies that $(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>} \neq \{0\}$*.*

Suppose now $(\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>} \neq \{0\}$*. Then there exists a non-trivial* G*-intertwining*

$$I : \mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2} \longrightarrow \mathrm{Ind}_P^G(\tilde{\rho}).$$

Since $I \neq 0$*, one sees directly that* $I(e_{K_0} \otimes v) \neq 0$ *for some* $v \in V_{\tau_2}$*. Now consider a non-trivial linear mapping*

$$I_{e_{K_0}} : v \mapsto I(e_{K_0} \otimes v), \quad V_{\tau_2} \longrightarrow \mathrm{Ind}_P^G(\tilde{\rho}).$$

Note that for $h \in \mathcal{H}(K)$ *we have*

$$\begin{aligned} I_{e_{K_0}}(\tau_2(h)v) = I(e_{K_0} \otimes \tau_2(h)v) &= I(e_{K_0} * h \otimes v) \\ &= I(h * e_{K_0} \otimes v) = R_h I(e_{K_0} \otimes v) = R_h I_{e_{K_0}}(v) \end{aligned}$$

(we have used above that $e_{K_0} * h = h * e_{K_0}$, which holds since K_0 is a normal subgroup of K). Thus, we have shown that τ_2 is a K-type of $\mathrm{Ind}_P^G(\tilde{\rho})$.

Now we shall consider a slightly modified setting. Let (π, X) be a smooth representation of G with infinitesimal character $\tilde{\omega}$. Consider a natural representation of G on $\mathrm{Hom}_{\mathbb{C}}(X, V_{\tau_1})$ (recall that $\mathrm{Hom}_{\mathbb{C}}(X, V_{\tau_1})^{(s)}$ denotes the smooth part of it). Fix $\alpha \in \mathrm{Hom}_K(V_{\tau_2}, X)$. For $T \in \mathrm{Hom}_{\mathbb{C}}(X, V_{\tau_1})^{(s)}$ define $\varphi_T^\alpha : G \to V$ by

$$\varphi_T^\alpha(g)v = T(\pi(g)(\alpha(v))), v \in V_{\tau_2}, g \in G.$$

Further, $T \mapsto \varphi_T^\alpha$ is a linear mapping from $\mathrm{Hom}_{\mathbb{C}}(X, V_{\tau_1})^{(s)}$ into $C(G, V)_{\tau_2}$. Since it is a G-intertwining, the image is in $C(G, V)_{\tau_2}^{(s)}$. Further, if $T \in \mathrm{Hom}_K(X, V_{\tau_1})^{(s)}$, then $\varphi_T^\alpha \in C(G, \tau)$. Thus, $\varphi_T^\alpha \in \mathcal{E}_\omega(G, V)$ (since π has infinitesimal character $\tilde{\omega}$). In this way we obtain a linear mapping

$$\varPhi_X : \mathrm{Hom}_K(V_{\tau_2}, X) \otimes \mathrm{Hom}_K(X, V_{\tau_1})^{(s)} \to \mathcal{E}_\omega(G, \tau), \quad \alpha \otimes T \mapsto \varphi_T^\alpha.$$

6.3. Lemma. *(i) If $X = X_1 \oplus X_2$ is a sum of G-subrepresentations, then $Im(\varPhi_{X_1}) + Im(\varPhi_{X_2}) = Im(\varPhi_X)$.*

(ii) If π is irreducible, then $\varPhi_X$ is injective.

Proof. (i) One gets $\mathrm{Im}(\varPhi_{X_i}) \subseteq \mathrm{Im}(\varPhi_X)$ directly (if $\varphi_T^\alpha \in \mathrm{Im}(\varPhi_{X_i})$, then extend the codomain of α and the domain of T in an obvious way).

Denote the action of G in X_i by π_i. Let $\varphi_T^\alpha \in \mathrm{Im}(\varPhi_X)$. Let α_i be the composition $V_{\tau_2} \xrightarrow{\alpha} X_1 \oplus X_2 \xrightarrow{\mathrm{pr}_i} X_i$ and T_i the composition $X_i \hookrightarrow X_1 \oplus X_2 \xrightarrow{T} X_1 \oplus X_2 \xrightarrow{\mathrm{pr}_i} X_i$. Now

$$\begin{aligned}\varphi_T^\alpha(g)v = T(\pi(g)(\alpha(v))) &= T(\pi(g)(\alpha_1(v) + \alpha_2(v))) = T(\pi_1(g)(\alpha_1(v)) + \pi_2(g)(\alpha_2(v))) \\ &= T_1(\pi_1(g)(\alpha_1(v))) + T_2(\pi_2(g)(\alpha_2(v))) = \varphi_{T_1}^{\alpha_1}(g)v + \varphi_{T_2}^{\alpha_2}(g)v.\end{aligned}$$

This completes the proof of (i).

(ii) Suppose that $\psi \in \mathrm{Hom}_K(V_{\tau_2}, X) \otimes \mathrm{Hom}_K(X, V_{\tau_1})^{(s)}$ is a non-zero element which is in the kernel of $\varPhi_X$. Write $\psi = \sum_{i=1}^n \alpha_i \oplus T_i$ in the shortest way. Then α_i are linearly independent, and also T_i are linearly independent. Now

$$\sum_{i=1}^n \varphi_{T_i}^{\alpha_i}(g)v = \sum_{i=1}^n T_i(\pi(g)(\alpha_i(v))) = 0, \forall g \in G, v \in T_{\tau_2}.$$

Writing in the above relation gg_1 instead of g, multiplying the relation by $f(g_1)$ and integrating the expression that we got in this way over G, we get

$$\sum_{i=1}^n T_i(\pi(g)\pi(f)(\alpha_i(v))) = 0, \quad \text{for all} \quad g \in G, f \in \mathcal{H}(G), v \in V_{\tau_2}.$$

First note that $\alpha_i(v)$ are linearly independent (observe that $\sum_{i=1}^n \mathrm{Im}(\alpha_i) = \oplus_{i=1}^n \mathrm{Im}(\alpha_i)$, since $n \leq \dim_{\mathbb{C}} \mathrm{Hom}(V_{\tau_2}, \oplus_{i=1}^n \mathrm{Im}(\alpha_i))$, and the last dimension is the length of $\oplus_{i=1}^n \mathrm{Im}(\alpha_i)$ by the Schur lemma). Fix $v \neq 0$. Chose f such that $\pi(f)(\alpha_1(v)) \neq 0$ and $\pi(f)(\alpha_i(v)) = 0$ for $i > 1$. Then $T_1(\pi(g)\pi(f)(\alpha_1(v))) = 0$ for all $g \in G$. The irreducibility of π implies $T_1 = 0$. This contradiction completes the proof of the lemma. □

In the rest of this section we shall assume that τ_1 and τ_2 are irreducible. As we already have mentioned, the case of generalized spherical functions corresponding to reducible τ_1 and (or) τ_2 reduces in a natural way to the case of irreducible τ_1 and τ_2.

Let ω be the infinitesimal character corresponding to $(M, \rho) \in \Omega(G)$, where M is a Levi subgroup of P. Then Frobenius reciprocity implies

$$\mathrm{Hom}_K(\tau_i, \mathrm{Ind}_P^G(\tilde{\rho})) \cong \mathrm{Hom}_{M\cap K}(\tau_i, \tilde{\rho}). \tag{6-1}$$

As a consequence of these considerations we get:

6.4. Theorem. *If $\omega = (M, \rho)$ is irreducible and semi simple infinitesimal character of G, then*

$$dim_{\mathbb{C}}(\mathcal{E}_\omega(G, \tau)) = dim_{\mathbb{C}}(Hom_{M\cap K}(\tau_1, \tilde{\rho}))\ dim_{\mathbb{C}}(Hom_{M\cap K}(\tau_2, \tilde{\rho})).$$

Proof. Observe first that Proposition 5.5 and (iv) of Proposition 6.1 imply that that there exists an isomorphism

$$\lambda : \mathrm{Hom}_{\mathcal{H}(K)}((\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1}) \to \mathcal{E}_\omega(G, \tau).$$

Using $(\bar{e}_{K_0} \otimes \ \ ) \otimes \ \ $, we shall identify

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{H}(K)}((\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1}) \\ &\quad \hookrightarrow \mathrm{Hom}_{\mathcal{H}(K)}(V_{\tau_2}, (\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}) \otimes \mathrm{Hom}_{\mathcal{H}(K)}((\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}, V_{\tau_1}). \end{aligned}$$

Now

$$\varPhi_{(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}}$$

extends λ (see (iii) of Remarks 6.2). Since λ is an isomorphism (in particular, an epimorphism), $\varPhi_{(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}}$ is an epimorphism.

Suppose now that τ_2 is no K-type of $\mathrm{Ind}_P^G(\tilde{\rho})$. Then the fact that λ is an isomorphism, (iv) of Remarks 6.2 and Frobenius reciprocity (6-1) imply the formula of the theorem.

Suppose now that τ_2 is a K-type of $\mathrm{Ind}_P^G(\tilde{\rho})$. Now (iv) of Remarks 6.2 imply that $\mathrm{Hom}_{\mathcal{H}(K)}(V_{\tau_2}, (\mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>})$ is a (finite) sum of representations isomorphic to $\mathrm{Ind}_P^G(\tilde{\rho})$. We shall fix one such irreducible subrepresentation, and identify it with $\mathrm{Ind}_P^G(\tilde{\rho})$. Having in mind this identification, (i) of Lemma 6.3 implies that the restriction of the mapping $\varPhi_{(\mathcal{H}(G)\otimes_{\mathcal{H}(K)} V_{\tau_2})^{<\tilde{\omega}>}}$ to $\mathrm{Hom}_{\mathcal{H}(K)}(\mathrm{Ind}_P^G(\tilde{\rho}), \mathrm{Ind}_P^G(\tilde{\rho}))$ is surjective. By (i) of Lemma 6.3, this restriction is injective. Thus, the restriction is an isomorphism. Now Frobenius reciprocity (6-1) implies the claim of the theorem in this case.

This completes the proof of the theorem. □

6.4. Remarks. *(i) Proposition 3.9 tells that the conditions required in the above theorem hold on a Zariski open dense subset of infinitesimal characters* $\Omega(G)$.

(ii) The formula in the theorem and Proposition 3.9 implies that the dimension of the space of generalized functions $\mathcal{E}_\omega(G,\tau)$ *is constant on a Zariski open dense subset of each connected component of* $\Omega(G)$.

(iii) Using isomorphism of this section one can easily write the formulas for generalized spherical functions. By these formulas generalized spherical functions are given by integrals (these are Eisenstein integrals). The integral in these formula for generalized spherical functions comes from the isomorphism $Ind_P^G(\rho)\tilde{} \cong Ind_P^G(\tilde{\rho})$*, which is given by integrating over* K.

Acknowledgments

The first author would like to thank Nolan Wallach and for stimulating discussions, and the second author would like to thank Dragan Miličić and Peter Schneider for useful observations related to the problems studied in this paper. Part of this work was done while the second author enjoying the hospitality of the Hong Kong University of Science and Technology.

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