ON THE REPRESENTATION THEORY OF $GL(n)$ OVER A p-ADIC DIVISION ALGEBRA AND UNITARITY IN THE JACQUET-LANGLANDS CORRESPONDENCE

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Abstract. Let F be a local non-archimedean field of characteristic 0, and let A be an F-central division algebra of dimension d_A over F. In this paper, we first develop some parts of the representation theory of $GL(m, A)$, assuming the conjecture that unitary parabolic induction is irreducible for $GL(m, A)$'s. Among others, we obtain the formula for characters of irreducible unitary representations of $GL(m, A)$ in terms of standard characters. Then we study the Jacquet-Langlands correspondence on the level of Grothendieck groups of $GL(pd_A, F)$ and $GL(p, A)$. Using the above character formula, we get explicit formulas for the Jacquet-Langlands correspondence of irreducible unitary representations of $GL(n, F)$ (assuming the conjecture to hold). As a consequence, we get that the Jacquet-Langlands correspondence sends irreducible unitary representations of $GL(n, F)$ either to zero or to irreducible unitary representations, up to a sign.

Introduction

A key aspect of the Langlands program is functoriality ([L]). One of the first examples of functoriality which were studied in the local case was the connection between representations of various inner forms of $GL(n)$ (see [KnRg]). The first example, studied already in [JL], was the connection between irreducible representations of $GL(2)$ over a local field F and irreducible representations of the multiplicative group of the quaternion algebra over F. (The L groups of these two groups are both $GL(2, \mathbb{C}) \times \mathrm{Gal}(\bar{F}/F)$, and the functoriality considered here corresponds to the identity mapping.)

Let F be a local non-archimedean field of characteristic 0 and let A be an F-central division algebra of rank d_A over F. For ecah positive integer p, P. Deligne, D. Kazhdan and M.-F. Vigneras established bijections

$$\mathrm{LJ}_{pd_A}$$

between irreducible essentially square-integrable representations of groups $GL(pd_A, F)$ and $GL(p, A)$ ([Rg] takes care of the case $p = 1$). The crucial requirement which holds for these

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bijections, and which characterizes them uniquely, is that characters Θ_δ and $\Theta_{\mathrm{LJ}_{pd_A}(\delta)}$ of the representations δ and $\mathrm{LJ}_{pd_A}(\delta)$ satisfy the following character indentity

$$(-1)^{pd_A}\Theta_\delta(g) = (-1)^p\Theta_{\mathrm{LJ}_{pd_A}(\delta)}(g')$$

whenever g and g' have the same characteristic polynomials, and when this polynomial is separable. These bijections are called Jacquet-Langlands correspondences.

A.I. Badulescu observed that Jacquet-Langlands correspondences extend in a very natural way to mappings between Grothendieck groups

$$\mathrm{LJ}_{pd_A} : \mathrm{Groth}(GL(pd_A, F)) \to \mathrm{Groth}(GL(p, A)),$$

such that the extensions are compatible with parabolic induction, i.e. that they commute with parabolic induction. (Essentially, such extensions are unique if we require that characters of $GL(n, F)$'s go to 0 if $d_A \not| n$.) Moreover, these extensions satisfy the above character identity on the level of formal characters (for a precise description of the extensions, see 6.1). We shall call these mappings Jacquet-Langlands correspondences on the level of Grothendieck groups.

For the group G of rational points of a reductive group defined over a local non-archimedean, we shall denote by $\tilde{G}$ the set of all the equvalence classes of irreducible smooth representations of G. The unitary dual $\hat{G}$ of G consists of all the unitarizable classes in $\tilde{G}$.

We consider $GL(n, F)\tilde{}$ as a subset of $\mathrm{Groth}(GL(n, F))$ in a natural way (it forms a $\mathbb{Z}$-basis). An interesting question is to understand what happens with irreducible representations under the Jacquet-Langlands correspondence on the level of Grothendieck groups, and in particular, what happens with irreducible unitary representations. Already, very simple examples will show that LJ_n will carry some irreducible representations to zero. Further, it is not hard to see that an irreducible (unitarizable) representation can go to the negative of an irreducible representation.

In this paper we study what happens with irreducible unitary representations under the Jacquet-Langlands correspondence, assuming the following conjecture for general linear groups over division algebras, introduced in [T5]:

(U0) unitary parabolic induction is irreducible for $GL(m, A)$'s

(i.e., if π_1 and π_2 are irreducible unitary representations of $GL(m_1, A)$ and $GL(m_2, A)$, then the parabolically induced representation $\mathrm{Ind}^{GL(m_1+m_2,A)}(\pi_1 \otimes \pi_2)$ is irreducible).

Note that D. Vogan's paper [Vo] implies that (U0) holds in the archimedean case. Also, J. Bernstein proved in [Be] that (U0) holds if $A = F$ (unfortunately the method of [Be] can not be extended to the division algebra case).

In this paper, we first develop some directions of the representation theory of $GL(n)$ over division algebras over a local non-archimedean field, to be able to obtain the formula for characters of irreducible unitary representations of $GL(m, A)$ in terms of standard characters. (Note that assuming (U0) to hold, [T5] and [BaRe] imply a classification of the unitary duals $GL(m, A)\hat{}$ of $GL(m, A)$ for all $m \geq 1$.) Using this character formula, we compute explicit formulas for $\mathrm{LJ}_n(\pi)$, $\pi \in GL(n, F)\hat{}$ (Propositions 7.3, 9.5 and section 11). As a consequence, we get the following:

Corollary. *Assume that (U0) holds. Then*

$$\mathrm{LJ}_{pd_A}(GL(pd_A,F)^\wedge) \subseteq \pm\, GL(p,A)^\wedge \cup \{0\},$$

i.e., Jacquet-Langlands correspodences send irreducible unitary representations of general linear groups over F either again to irreducible unitary representations of general linear groups over A, up to a sign, or to 0.

A direct consequence of the above corollary is the following observation, which may be of some interest. Let σ be an element of $GL(p,A)^\sim$ (resp. of $\mathrm{Groth}(GL(p,A))\backslash\{0\}$). Suppose that (U0) holds and suppose that there exists $\pi \in GL(pd_A,F)^\wedge$ such that the characters of σ and π are equal up to (the same) sign on elements with same characteristic polynomials. Then σ is unitarizable (resp. $\sigma \in \pm\, GL(p,A)^\wedge$).

Let us note that there are very strong formal similarities between the Jacquet-Langlands correspondences studied in this paper and the Kazhdan-Patterson lifting studied in [T8].

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We shall now give a description of the content of the paper according to sections. In the first section, the notation and basic results for general linear groups over a local non-archimedean field are introduced (which are needed in the sequel). The second section introduces notation and basic results for general linear groups over division algebras. In the third section, we show that the canonical involution on irreducible representations of $GL(m,A)$ (introduced by A.-M. Aubert, and by P. Schneider and U. Stuhler) preserves unitarity. Moreover, we obtain an explicit formula for the involution on irreducible unitary representations. (We assume (U0) to hold throughout.) In the fourth section we describe irreducible subquotients of ends of complementary series, obtaining in this way a character identity, which enable us to compute in the fifth section characters of irreducible unitary representations of $GL(m,A)$ in terms of standard characters. We recall the Jacquet-Langlands correspondence on the level of Grothendieck groups in the sixth section. In the seventh section we compute $\mathrm{LJ}(\pi)$ for one of four basic types of $\pi \in GL(n,F)^\wedge$, while the unitarity of $\mathrm{LJ}(\pi)$ is shown in the eighth section. Sections nine and ten study the same problem for the second basic type of π. In the last section, we compute the Jacquet-Langlands correspondence of the remaining two basic types of π, using canonical involutions.

1. Some facts from the representation theory of $GL(n,F)$

In this section, we shall introduce notation and basic results that we shall need for general linear groups over a local non-archimedean field.

1.1. We shall fix a local non-archimedean field F. The modulus character of F will be denoted by $|\ \ |_F$ (it satisfies $|x|_F \int_F f(xa)da = \int_F f(a)da$ for any continuous compactly supported complex-valued function f on F, where da denotes a Haar measure of the additive group $(F,+)$ of the field).

1.2. Let G be the group of rational points of a reductive group over F. The set of all equivalence classes of irreducible smooth representations of G will be denoted by

$$\tilde{G}.$$

The subset of unitarizable classes in $\tilde{G}$ will be denoted by

$$\hat{G}.$$

A representation $\pi \in \tilde{G}$ is called essentially square-integrable if there exists a character χ of G such that $\chi\pi$ is square-integrable representation modulo the center. All the essentially square-integrable classes in $\tilde{G}$ will be denoted by

$$\mathcal{D}(G).$$

The Grothendieck group of the category of all representations of G of finite length will be denoted by

$$\mathrm{Groth}(G).$$

1.3. Now we shall introduce the Bernstein and Zelevinsky notation for the general linear group $GL(n,F)$ (for more explanation regarding notation see [Z] and [T2]).

For two smooth representation π_1 and π_2 of $GL(n_1,F)$ and $GL(n_2,F)$, we shall consider $\pi_1 \otimes \pi_2$ as a representation of $GL(n_1,F) \times GL(n_2,F)$. Identifying in a natural way $GL(n_1,F) \times GL(n_2,F)$ with the Levi factor of the parabolic subgroup

$$\left\{ \begin{bmatrix} g_1 & * \\ 0 & g_2 \end{bmatrix} ; g_i \in GL(n_i,F), i = 1,2 \right\},$$

we shall denote by

$$\pi_1 \times \pi_2$$

the smooth representation of $GL(n_1+n_2,F)$ parabolically induced by $\pi_1 \otimes \pi_2$. Then

$$(\pi_1 \times \pi_2) \times \pi_3 \cong \pi_1 \times (\pi_2 \times \pi_3).$$

1.4. The characters of $F^\times$ will be identified with characters of $GL(n,F)$ using the determinant homomorphism. The character of $GL(n,F)$ corresponding to $|\ \ |_F$ will be denoted by

$$\nu.$$

For any character χ of $F^\times$,

$$\chi(\pi_1 \times \pi_2) \cong (\chi\pi_1) \times (\chi\pi_2).$$

1.5. Let

$$R_{n,F} = \text{Groth}(GL(n,F)).$$

Then $GL(n,F)\tilde{}$ is a $\mathbb{Z}$-basis of $R_{n,F}$. If π is an admissible smooth representation of $GL(n,F)$ of finite length, the semisimplification of π will be denoted by s.s.(π) (which is in $R_{n,F}$).

We can lift $\times$ to a $\mathbb{Z}$-bilinear mapping $\times : R_{n_1,F} \times R_{n_2,F} \to R_{n_1+n_2,F}$ since the semisimplification of $\pi_1 \times \pi_2$ depends only on semisimplifications of π_1 and π_2. Set

$$R_F = \oplus_{n\geq 0} R_{n,F}.$$

One extends $\times$ to an operation $\times : R_F \times R_F \to R_F$ in an obvious way, and R_F becomes an associative, commutative graded ring.

Fix a character χ of $F^\times$. Lift the mappings $\pi \mapsto \chi\pi : R_{n,F} \to R_{n,F}$ to a $\mathbb{Z}$-linear map $\chi : R_F \to R_F$. In this way we get an endomorphism of the graded ring R_F .

We have a natural partial ordering on $R_{n,F}$. ($GL(n,F)\tilde{}$ generates the cone of positive elements in $R_{n,F}$.) Then orderings on the $R_{n,F}$'s determine an ordering $\leq$ on R_F in a natural way. An additive homomorphism $\phi : R_F \to R_F$ will be called called positive if $x \in R_F$, $x \geq 0$ implies $\phi(x) \geq 0$.

1.6. Denote

$$\mathcal{D}_F = \underset{n\geq 1}{\cup} \mathcal{D}(GL(n,F)).$$

For $\delta \in \mathcal{D}_F$ there exists a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. The representation $\nu^{-e(\delta)}\delta$ will be denoted by δ^u. In this way,

$$\delta = \nu^{e(\delta)}\delta^u,$$

where $e(\delta) \in \mathbb{R}$ and δ^u is unitarizable.

1.7. Now we shall describe the Langlands classification for general linear groups. Let $M(\mathcal{D}_F)$ be the set of all finite multisets in $\mathcal{D}_F$ and $d = (\delta_1, \delta_2, \dots, \delta_k) \in M(\mathcal{D}_F)$. Let γ be a permutation of $\{1, 2, \dots, k\}$ such that $e(\delta_{\gamma(1)}) \geq e(\delta_{\gamma(2)}) \geq \cdots \geq e(\delta_{\gamma(k)})$. The representation

$$\lambda(d) = \delta_{\gamma(1)} \times \delta_{\gamma(2)} \times \cdots \times \delta_{\gamma(k)}$$

has a unique irreducible quotient. Its class depends only on d (not on γ as above). This unique irreducible quotient will be denoted by $L(d)$ or $L(\delta_1, \delta_2, \dots, \delta_k)$. From 1.4 it follows that for a character χ of $F^\times$,

$$\chi L(\delta_1, \delta_2, \dots, \delta_k) \cong L(\chi\delta_1, \chi\delta_2, \dots, \chi\delta_k).$$

To have shorter notation, we shall often denote s.s.$(\lambda(d)) \in R_F$ simply by $\lambda(d) \in R_F$. This will produce no confusion. From the properties of the Langlands classification it is well-known that the $\lambda(d) \in R_F$, $d \in M(\mathcal{D}_F)$, form a basis of R_F.

Proposition ([Z]). *The ring R_F is a polynomial ring over $\mathcal{D}_F$.* $\square$

1.8. One defines addition of elements of $M(\mathcal{D}_F)$ by

$$(\delta_1, \delta_2, \dots, \delta_k) + (\delta'_1, \delta'_2, \dots, \delta'_{k'}) = (\delta_1, \delta_2, \dots, \delta_k, \delta'_1, \delta'_2, \dots, \delta'_{k'}).$$

Then

Proposition ([Rd]). *For $d_1, d_2 \in M(\mathcal{D}_F)$, $L(d_1 + d_2)$ is a subquotient of $L(d_1) \times L(d_2)$. The multiplicity is one.* $\square$

1.9. Let $\mathcal{C}_F$ be the set of all the equivalence classes of irreducible cuspidal representations of all general linear groups $GL(n, F)$, $n \geq 1$. For $\rho \in \mathcal{C}_F$ and $k \in \mathbb{Z}, k \geq 0$, the set

$$[\rho, \nu^k \rho] = \{\rho, \nu\rho, \nu^2\rho, \dots, \nu^k\rho\}$$

is called a segment of irreducible cuspidal representations. We shall also write the segment

$$[\nu^{k_1}\rho, \nu^{k_2}\rho] = [k_1, k_2]^{(\rho)}$$

(where $k_1, k_2 \in \mathbb{R}$ are such that $k_2 - k_1 \in \mathbb{Z}$ and $k_2 - k_1 \geq 0$). The set of all such segments will be denoted by $\mathcal{S}_F$. The set of all finite multisets in $\mathcal{S}_F$ will be denoted by $M(\mathcal{S}_F)$. We consider the partial ordering $\leq$ on $M(\mathcal{S}_F)$ introduced in 7.1 of [Z], defined by linking segments.

1.10. Let $\Delta = [\rho, \nu^k\rho] = \{\rho, \nu\rho, \nu^2\rho, \dots, \nu^k\rho\} \in \mathcal{S}_F$. The representation

$$\rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^k\rho$$

has a unique irreducible quotient, denoted by

$$\delta([\rho, \nu^k\rho]),$$

and a unique irreducible subrepresentation, denoted by

$$\delta([\rho, \nu^k\rho])^t.$$

The representations $\delta([\rho, \nu^k\rho])$ are essentially square-integrable. Further, the representations $\delta([\rho, \nu^k\rho])^t$ are called Zelevinsky's segment representations.

By Bernstein-Zelevinsky theory, the mapping

$$\delta : \mathcal{S}_F \to \mathcal{D}_F, \quad \Delta \mapsto \delta(\Delta),$$

is a bijection.

We can state this also in the following way. For $n \in \mathbb{N}$ and $\rho \in \mathcal{C}_F$ denote

$$\delta(\rho, n) = \delta([-(n-1)/2, (n-1)/2]^{(\rho)}).$$

Then $(\rho, n) \mapsto \delta(\rho, n)$ is a bijection of $\mathcal{C}_F \times \mathbb{N}$ onto $\mathcal{D}_F$.

We lift $\Delta \mapsto \delta(\Delta)$ naturally to a bijection

$$M(\delta) : M(\mathcal{S}_F) \longrightarrow M(\mathcal{D}_F), \quad (\Delta_1, \dots, \Delta_k) \mapsto (\delta(\Delta_1), \dots, \delta(\Delta_k)).$$

Using the above bijection we get the Langlands classification in terms of $M(\mathcal{S}_F)$.

For $a \in M(\mathcal{S}_F)$ we denote

$$L(a) = L\big(M(\delta)(a)\big).$$

We also denote

$$\lambda(a) = \lambda\big(M(\delta)(a)\big).$$

1.11. Note that 1.10 and Proposition 1.7 imply that R_F is a polynomial algebra over $\delta(\Delta), \Delta \in \mathcal{S}_F$. Therefore the mapping

$$\delta(\Delta) \mapsto \delta(\Delta)^t, \quad \Delta \in \mathcal{S}_F,$$

extends uniquely to a ring morphism ${}^t : R_F \longrightarrow R_F$. This ring morphism is involutive. A fundamental fact is that it carries irreducible representations into irreducible ones ([A], [ScSt]).

Obviously, for a character χ of $F^\times$, $\big(\chi\delta(\Delta)\big)^t \cong \chi\big(\delta(\Delta)^t\big)$ for $\Delta \in \mathcal{S}_F$. Therefore, ${}^t : R_F \longrightarrow R_F$ and $\chi : R_F \longrightarrow R_F$ commute, since they commute on generators.

1.12. For an irreducible representation π of a general linear group, there exists a unique $(\rho_1, \dots, \rho_k) \in M(\mathcal{C}_F)$ such that

$$\pi \hookrightarrow \rho_1 \times \dots \times \rho_k.$$

The multiset $(\rho_1, \dots, \rho_k)$ is called the cuspidal support of π and it is denoted by

$$\mathrm{supp}(\pi).$$

It is well-known (and one easily sees it) that ${}^t : R_F \longrightarrow R_F$ preserves the cuspidal support of irreducible representations.

1.13. Denote the set of all unitarizable classes in $\mathcal{D}_F$ by $\mathcal{D}_F^u$ (in other words, $\mathcal{D}_F^u$ consists of the square-integrable classes). For $\delta \in D^u$ and a positive integer n denote

$$u(\delta, n) = L(\nu^{\frac{n-1}{2}}\delta, \nu^{\frac{n-3}{2}}\delta, \dots, \nu^{-\frac{n-1}{2}}\delta).$$

The following theorem describes irreducible unitarizable representations.

Theorem ([T2]). *Let*

$$\mathcal{B}_F = \{u(\delta, n),\ \ \nu^{\alpha} u(\delta, n) \times \nu^{-\alpha} u(\delta, n)\ \mid\ \delta \in \mathcal{D}_F^u,\ \ n \in \mathbb{N},\ \ 0 < \alpha < 1/2\}.$$

Then

(i) *If* $\sigma_1, \dots, \sigma_k \in \mathcal{B}_F$, *then* $\sigma_1 \times \dots \times \sigma_k$ *is an irreducible unitarizable representation of some general linear group over* F.

(ii) *If* π *is an irreducible unitarizable representation of some general linear group over* F, *then there exist* $\sigma_1, \dots, \sigma_m \in \mathcal{B}_F$, *unique up to a permutation, such that*

$$\pi \cong \sigma_1 \times \dots \times \sigma_m. \quad \square$$

1.14. Let $\rho \in \mathcal{C}_F$. Fix positive integers d and n. Let

$$a(1,d)^{(\rho)} = [\nu^{-(d-1)/2}\rho, \nu^{(d-1)/2}\rho] \in \mathcal{S}_F,$$

$$a(n,d)^{(\rho)} = (a(1,d)^{(\nu^{-(n-1)/2}\rho)}, a(1,d)^{(\nu^{1-(n-1)/2}\rho)}, \dots, a(1,d)^{(\nu^{(n-1)/2}\rho)}) \in M(\mathcal{S}_F).$$

We shall take (formally) $a(0,d)^{(\rho)}$ to be the empty multiset (then $L(a(0,d)^{(\rho)})$ is the one-dimensional representation of the trivial group $GL(0,F)$, which is the identity of R_F). Similarly, we also take $a(n,0)^{(\rho)}$ to be the empty multiset (so again $L(a(n,0)^{(\rho)})$ is the identity in R). Observe that

$$[\nu^{k_1}\rho, \nu^{k_2}\rho] = [k_1, k_2]^{(\rho)} = a(1, k_2 - k_1 + 1)^{(\nu^{(k_1+k_2)/2}\rho)},$$
$$a(1,d)^{(\nu^{\alpha}\rho)} = [-(d-1)/2 + \alpha, (d-1)/2 + \alpha]^{(\rho)}.$$

From 1.7 follows that for a character χ of $F^{\times}$

$$\chi L(a(n,d)^{(\rho)}) \cong L(a(n,d)^{(\chi\rho)}).$$

Further

$$u(\delta(\rho,d), n) = L(a(n,d)^{(\rho)}).$$

1.15. There are two important distinguished bases of R_F, irreducible representations and standard modules $\lambda(d), d \in M(\mathcal{D}_F)$. Theorem 1.13 implies that the following theorem solves the problem of expressing irreducible unitary representations (resp. irreducible unitary characters) in terms of standard modules (resp. standard characters). It is convenient to present it in the following form:

Theorem ([T7]). *Let $n, d \in \mathbb{Z}, n, d \geq 1$. Let W_n be the group of permutations of the set $\{1, 2, \dots, n\}$. Denote $W_n^{(d)} = \{w \in W_n; w(i) + d \geq i$ for all $1 \leq i \leq n\}$. Then we have the following identity in R_F*

$$L(([\nu\rho, \nu^d\rho], \dots, [\nu^n\rho, \nu^{d+n-1}\rho])) = \sum_{w \in W_n^{(d)}} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{n} \delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho]),$$

where sgn(w) denotes the sign of permutation w. □

1.16. From the following theorem one can get all the irreducible subquotients of ends of complementary series. (This is crucial information for determining the topology of the unitary dual.)

Theorem ([T3]). *For positive integers n and d, and $\rho \in \mathcal{C}_F$, we have*

$$\nu^{1/2}L(a(n,d)^{(\rho)}) \times \nu^{-1/2}L(a(n,d)^{(\rho)}) = \\ L(a(n+1,d)^{(\rho)}) \times L(a(n-1,d)^{(\rho)}) + L(a(n,d+1)^{(\rho)}) \times L(a(n,d-1)^{(\rho)}). \quad \square$$

1.17. The following theorem implies that the involution ^{t} carries class of irreducible unitary representations to itself. Moreover, it implies an explicit formula for the involution on irreducible unitary representations.

Theorem ([T2]). *For positive integers n and d, and $\rho \in \mathcal{C}_F$, we have*

$$\left(L(a(n,d)^{(\rho)})\right)^t = L(a(d,n)^{(\rho)}).$$

2. Representations of $GL(n)$ over a division algebra A

In this section, we shall introduce notation and basic results that we shall need in this paper regarding general linear groups over division algebras. Since the situation is very similar to the case of general linear groups over a field, we shall only point out the differences between these two cases. (More details can be found in [T5].)

We shall assume in the sequel that the characteristic of F is 0.

2.1. Let A be a finite dimensional division algebra over F whose center is F. Let

$$\dim_F A = d_A^2.$$

Let $\mathrm{Mat}(n \times n, A)$ be the algebra of all $n \times n$ matrices with entries in A. Then $GL(n, A)$ is the group of invertible matrices with the natural topology. The commutator subgroup is denoted by $SL(n, A)$. We shall denote by

$$\det : GL(n, A) \to GL(1, A)/SL(1, A)$$

the determinant homomorphism, as defined by J. Dieudonné (for $n = 1$ this is just the quotient map). The kernel is $SL(n, A)$.

The reduced norm of $\mathrm{Mat}(n \times n, A)$ will be denoted by $\mathrm{r.n.}_{\mathrm{Mat}(n\times n,A)/F}$. We shall identify characters of $GL(n, A)$ with characters of $F^\times$ using $\mathrm{r.n.}_{\mathrm{Mat}(n\times n,A)/F}$. Let

$$\nu = |\mathrm{r.n.}_{\mathrm{Mat}(n\times n,A)/F}|_F : GL(n, A) \to \mathbb{R}^\times.$$

2.2. Now we shall comment on modifications that one needs to make to 1.3 - 1.12 so that these sections apply also to the case of general linear groups over division algebras. (More details can be found in [T5]). Particularly, small modifications are required for 1.3 - 1.8. We shall first deal with these modifications.

ad 1.3. We define by the same formula, as in 1.3, the multiplication $\times$ of smooth representations of general linear groups over A. Then, as in 1.3, we have $(\pi_1 \times \pi_2) \times \pi_3 \cong \pi_1 \times (\pi_2 \times \pi_3)$.

ad 1.4. In 2.1 we identified characters of $GL(n, A)$ with characters of $F^\times$ using the reduced norm $\mathrm{r.n.}_{\mathrm{Mat}(n\times n,A)/F}$. The character identified with $|\ \ |_F$ was again denoted by ν. Again, for a character χ of $F^\times$, we have $\chi(\pi_1 \times \pi_2) \cong (\chi\pi_1) \times (\chi\pi_2)$.

ad 1.5. Define

$$R_{n,A} = \mathrm{Groth}(GL(n, A))$$

(recall that $R_F = \oplus_{n\geq 0} R_{n,F}$). One defines on R_A the structure of an (associative, commutative) ring in the same way as in 1.5 was done for the field case. Also, characters of $F^\times$ lift to automorphisms of R_A (as in 1.5).

ad 1.6. All essentially square-integrable classes in $\cup_{n\geq 1} GL(n, A)^{\tilde{}}$ are denoted by $\mathcal{D}_A$. One defines $e(\delta)$ and δ^u for $\delta \in \mathcal{D}_A$ as in 1.6.

ad 1.7. For $d \in M(\mathcal{D}_A)$ we define $\lambda(d)$ and $L(d)$ in the same way as in 1.7. The Langlands classification for general linear groups over division algebras have the same expression as in the field case. (The parameters are in $M(\mathcal{D}_A)$.)

Here R_A is a polynomial algebra over $\mathcal{D}_A$.

ad 1.8. Proposition 1.8 holds in the same form for general linear groups over division algebras (Proposition 2.3 of [T5]).

2.3. By [DeKaVi] there exists a bijection

$$\mathrm{JL}_p : \mathcal{D}(GL(p, A)) \to \mathcal{D}(GL(pd_A, F))$$

$$\delta' \leftrightarrow \delta$$

such that characters $\Theta_{\delta'}$ and Θ_δ satisfy

$$(-1)^p \Theta_{\delta'}(g') = (-1)^{pd_A} \Theta_\delta(g)$$

whenever g' and g have same characteristic polynomials, and when this polynomial is separable.

This bijection is uniquely determined by the above character requirement and is called the Jacquet-Langlands correspondence between irreducible essentially square-integrable representations of $GL(p,A)$ and $GL(pd_A,F)$.

This bijection commutes with twisting with characters (see [Ba2]).

Take $\delta' \in \mathcal{D}(GL(p,A))$ cuspidal and let δ' correspond to $\delta \in \mathcal{D}(GL(pd_A,F))$ by the above correspondence. We know that

$$\delta = \delta(\rho, q)$$

for some positive integer q which divides pd_A and for some irreducible cuspidal representation ρ of $GL((pd_A)/q, F)$. Further, it is known that $q|d_A$ and that p is relatively prime to q (pd_A is the lowest common multiple of d_A and $(pd_A)/q$).

We define

$$s_{\delta'} = q,$$

and a character

$$\nu_{\delta'} = \nu^{s_{\delta'}}$$

of $F^{\times}$. As we noted in 2.1, we use $\text{r.n.}_{\text{Mat}(n\times n,A)/F}$ to identify the characters of $F^{\times}$ with characters of $GL(n,A)$. Therefore, we can view $\nu_{\delta'}$ as a character of any $GL(k,A)$. (Note that in this definition, δ' is cuspidal; soon we shall give a definition also in the case that δ' is essentially square-integrable.)

2.4. We shall continue now with modifications that one needs to make to 1.9 - 1.12 in order for these sections to apply also to the case of general linear groups over division algebras.

ad 1.9. We shall denote by $\mathcal{C}_A$ the set of all equivalence classes of irreducible cuspidal representation of all $GL(n,A)$, $n \geq 1$. For $\rho' \in \mathcal{C}_A$ and $k \in \mathbb{Z}, k \geq 0$, the set

$$[\rho', \nu_{\rho'}^k \rho'] = \{\rho', \nu_{\rho'}\rho', \nu_{\rho'}^2\rho', \dots, \nu_{\rho'}^k\rho'\}$$

is called a segment of irreducible cuspidal representations of general linear groups over division algebras. In this case, we shall also write the segment

$$[\nu_{\rho'}^{k_1}\rho, \nu_{\rho'}^{k_2}\rho] = [k_1, k_2]^{(\rho')}$$

($k_1, k_2 \in \mathbb{R}$, $k_2 - k_1 \in \mathbb{Z}$ and $k_2 - k_1 \geq 0$). The set of all such segments will be denoted by $\mathcal{S}_A$. The set of all finite multisets in $\mathcal{S}_A$ will be denoted by $M(\mathcal{S}_F)$ and we shall consider the partial ordering $\leq$ on $M(\mathcal{S}_A)$ introduced in section 4 of [T5] (defined again by linking segments).

ad 1.10. For $\Delta' = [\rho', \nu_{\rho'}^k\rho'] = \{\rho', \nu_{\rho'}\rho', \nu_{\rho'}^2\rho', \dots, \nu_{\rho'}^k\rho'\} \in \mathcal{S}_A$, the representation $\rho' \times \nu_{\rho'}\rho' \times \nu_{\rho'}^2\rho' \times \dots \times \nu_{\rho'}^k\rho'$ has a unique irreducible quotient, denoted by $\delta([\rho', \nu_{\rho'}^k\rho'])$, and it has a unique irreducible subrepresentation, denoted by $\delta([\rho', \nu_{\rho'}^k\rho'])^t$.

The mapping

$$\delta : \mathcal{S}_A \to \mathcal{D}_A, \quad \Delta' \mapsto \delta(\Delta'),$$

is a bijection. If we let

$$\delta(\rho', n) = \delta([-(n-1)/2, (n-1)/2]^{(\rho')}),$$

then we can state the above fact in the following way: $(\rho', n) \mapsto \delta(\rho', n)$ is a bijection of $\mathcal{C}_A \times \mathbb{N}$ onto $\mathcal{D}_A$.

We define $\nu_{\delta(\rho',n)}$ to be $\nu_{\rho'}$, i.e.

$$\nu_{\delta(\rho',n)} = \nu_{\rho'}.$$

We lift $\Delta' \mapsto \delta(\Delta')$ naturally to a bijection $M(\delta) : M(\mathcal{S}_A) \to M(\mathcal{D}_A)$. This gives the Langlands classification for general linear groups over a division algebra A in terms of $M(\mathcal{S}_A)$. For $a \in M(\mathcal{S}_A)$ let $L(a) = L\big(M(\delta)(a)\big)$ and $\lambda(a) = \lambda\big(M(\delta)(a)\big)$ as before.

ad 1.11. Since R_A is a polynomial algebra over $\delta(\Delta'), \Delta' \in \mathcal{S}_A$, the mapping $\delta(\Delta') \mapsto \delta(\Delta')^t, \quad \Delta \in \mathcal{S}_A$, extends uniquely to the ring morphism $^t : R_A \to R_A$, which carries irreducible representations into irreducible ones ([A], [ScSt]). This homomorphism of rings is an involution. Again, for a character χ of $F^\times$, $^t : R_A \to R_A$ and $\chi : R_A \to R_A$ commute.

ad 1.12. One defines the cuspidal support of an irreducible representation in the same way as before (it is an element of $M(\mathcal{C}_A)$). The involution ^{t} preserves the cuspidal support.

2.5. Let $\mathcal{D}_A^u$ be the set of all the unitarizable classes in $\mathcal{D}_A$. Let

$$u(\delta', n) = L(\nu_{\delta'}^{\frac{n-1}{2}}\delta', \nu_{\delta'}^{\frac{n-3}{2}}\delta', \dots, \nu_{\delta'}^{-\frac{n-1}{2}}\delta').$$

for $\delta' \in D_A^u$ and a positive integer n.

Let us first recall a conjecture (U0) from [T5]:

(U0) if π_1 and π_2 are irreducible unitarizable representations of general linear groups over A, then $\pi_1 \times \pi_2$ is irreducible.

Now section 6 of [T5] and [BaRe] imply the following

Proposition. *Assume that (U0) holds. Let*

$$\mathcal{B}_A = \{u(\delta', n), \;\; \nu_{\delta'}^{\alpha} u(\delta', n) \times \nu_{\delta'}^{-\alpha} u(\delta', n) \;\mid\; \delta' \in \mathcal{D}_A^u, \;\; n \in \mathbb{N}, \;\; 0 < \alpha < 1/2\}.$$

Then

(i) *If $\sigma_1, \dots, \sigma_k \in \mathcal{B}_A$, then $\sigma_1 \times \dots \times \sigma_k$ is an irreducible unitarizable representation of some general linear group over A.*

(ii) *If π is an irreducible unitarizable representation of some general linear group over A, then there exist $\sigma_1, \dots, \sigma_m \in \mathcal{B}_A$, unique up to a permutation, such that*

$$\pi \cong \sigma_1 \times \dots \times \sigma_m. \quad \square$$

2.6. For $\rho' \in \mathcal{C}_A$ and positive integers d, n, let

$$a(1,d)^{(\rho')} = [\nu_{\rho'}^{-(d-1)/2}\rho', \nu_{\rho'}^{(d-1)/2}\rho'] \in \mathcal{S}_A,$$

$$a(n,d)^{(\rho')} = (a(1,d)^{(\nu_{\rho'}^{-(n-1)/2}\rho')}, a(1,d)^{(\nu_{\rho'}^{1-(n-1)/2}\rho')}, \dots, a(1,d)^{(\nu_{\rho'}^{(n-1)/2}\rho')}) \in M(\mathcal{S}_A).$$

Further, $a(0,d)^{(\rho')}$ and $a(n,0)^{(\rho')}$ are empty multisets (so $L(a(0,d)^{(\rho')})$ and $L(a(n,0)^{(\rho')})$ are both the identity in R_A). Similarly as before we have

$$[\nu_{\rho'}^{k_1}\rho', \nu_{\rho'}^{k_2}\rho'] = [k_1,k_2]^{(\rho')} = a(1,k_2-k_1+1)^{(\nu_{\rho'}^{(k_1+k_2)/2}\rho')},$$
$$a(1,d)^{(\nu_{\rho'}^{\alpha}\rho')} = [-(d-1)/2+\alpha, (d-1)/2+\alpha]^{(\rho')}.$$

For a character χ of $F^\times$ we have $\chi L(a(n,d)^{(\rho')}) \cong L(a(n,d)^{(\chi\rho')})$. Also

$$u(\delta(\rho',d),n) = L(a(n,d)^{(\rho')}).$$

3. Involution and unitarity, on unitary duals of $GL(n,A)$

3.1 We shall call an irreducible representation π of a general linear group over A rigid if

$$e(\rho')/s_{\rho'} \in (1/2)\mathbb{Z}$$

for all ρ' in the cuspidal support of π.

Lemma. *Assume that (U0) holds. For $d,n \in \mathbb{N}$, $\rho' \in \mathcal{C}_A$ we have*

$$L(a(n,d)^{(\rho')})^t = L(a(d,n)^{(\rho')}).$$

Proof. Since ^{t} commutes with character automorphisms $\chi : R_A \to R_A$, it is enough to prove the above relation for unitary $\rho' \in \mathcal{C}_A$.

The proof goes in several steps.

First we shall prove that $L(a(n,d)^{(\rho')})^t$ is unitarizable. We prove it by induction with respect to n. For $n=1$ we know $L(a(1,d)^{(\rho')})^t = L(a(d,1)^{(\rho')})$, which is unitarizable by Proposition 2.5.

Let $n \geq 1$ and suppose that we have proved the unitarity of the representations $L(a(n,d)^{(\rho')})^t$ for that n. Proposition 1.8 for the division algebra case (see 2.2) and (U0) imply

$$L(a(n+1,d)^{(\rho')}) \times L(a(n-1,d)^{(\rho')}) \leq \nu_{\rho'}^{-1/2}L(a(n,d)^{(\rho')}) \times \nu_{\rho'}^{1/2}L(a(n,d)^{(\rho')}).$$

Applying the involution $^t : R_A \to R_A$ to the above relation, we get

$$L(a(n+1,d)^{(\rho')})^t \times L(a(n-1,d)^{(\rho')})^t \leq \nu_{\rho'}^{-1/2}L(a(n,d)^{(\rho')})^t \times \nu_{\rho'}^{1/2}L(a(n,d)^{(\rho')})^t.$$

First observe that $L(a(n,d)^{(\rho')})^t$ is rigid (since ^{t} preserves the cuspidal support). This fact, the unitarity of $L(a(n,d)^{(\rho')})^t$ and Proposition 2.5 imply that $L(a(n,d)^{(\rho')})^t$ is a product of elements of the form $L(a(n',d')^{(\rho'')})$, where $\rho'' \in \mathcal{C}_A$ are unitarizable. Proposition 2.5 implies that all representations

$$\nu_{\rho''}^{-\alpha} L(a(n',d')^{(\rho'')}) \times \nu_{\rho''}^{\alpha} L(a(n',d')^{(\rho'')}), \quad 0 < \alpha < 1/2,$$

are unitarizable. This implies that the representations

$$\nu_{\rho'}^{-\alpha} L(a(n,d)^{(\rho')})^t \times \nu_{\rho'}^{\alpha} L(a(n,d)^{(\rho')})^t, \quad 0 < \alpha < 1/2,$$

are also unitarizable. Recall that all the irreducible subquotients at the end of complementary series are unitarizable ([M], see also [T2] and [T4]). This implies that all the irreducible subquotients of $\nu_{\rho'}^{-1/2} L(a(n,d)^{(\rho')})^t \times \nu_{\rho'}^{1/2} L(a(n,d)^{(\rho')})^t$ are unitarizable. In particular,

$$L(a(n+1,d)^{(\rho')})^t \times L(a(n-1,d)^{(\rho')})^t$$

is unitarizable (and irreducible, since ^{t} carries irreducible representations to the irreducible ones).

For an irreducible representation π denote by π^+ the Hermitian contragredient of π. Then $\pi \mapsto \pi^+$ lifts to an automorphism of R_A. Observe that ^{t} carries the class of irreducible Hermitian representations to itself, since the automorphisms ^{t} and $\pi \mapsto \pi^+$ of R_A commute (one directly checks this on generators). Therefore, $L(a(n+1,d)^{(\rho')})^t \otimes L(a(n-1,d)^{(\rho')})^t$ is Hermitian. Now (d) in section 3 of [T6] implies that $L(a(n+1,d)^{(\rho')})^t \otimes L(a(n-1,d)^{(\rho')})^t$ is (irreducible) unitarizable, which implies that $L(a(n+1,d)^{(\rho')})^t$ is unitarizable. Therefore, we have proved the inductive step.

So, we have proved that representations $L(a(n,d)^{(\rho')})^t$ for $\rho' \in \mathcal{C}_A$ unitarizable, are unitarizable in general.

Now we are going to get an explicit formula for $L(a(n,d)^{(\rho')})^t$.

First note that $L(a(n,d)^{(\rho')})$ is not induced from a proper parabolic subgroup by an irreducible unitarizable representation (see Proposition 2.5). Therefore, $L(a(n,d)^{(\rho')})^t$ is also not induced in that way. Proposition 2.5 implies that $(L(a(n,d)^{(\rho')}))^t = L(a(n',d')^{(\rho'')})$ for some n', d' and ρ''. Since ^{t} preserves the cuspidal support, one gets directly $\rho' \cong \rho''$ and $\{n,d\} = \{n',d'\}$. This implies the lemma if $n = d$.

It remains to consider the case $n \neq d$. Actually, in this case it is enough to prove

$$L(a(n,d)^{(\rho')})^t \neq L(a(n,d)^{(\rho')}).$$

Since ^{t} is an involution, our previous discussion implies that it is enough to prove the above relation only in the case

$$d < n.$$

We shall prove this by induction with respect to d. For $d = 1$,

$$L(a(n,1)^{(\rho')})^t = L(a(1,n)^{(\rho')}),$$

which is different from $L(a(n,1)^{(\rho')})$ since $n > 1$.

Suppose $d \geq 1$ and that we have proved the claim for $d' \leq d$. Let $d+1 < n$. We have already observed that $L(a(n+1,d)^{(\rho')}) \times L(a(n-1,d)^{(\rho')}) \leq \nu_\rho^{-1/2} L(a(n,d)^{(\rho')}) \times \nu_\rho^{1/2} L(a(n,d)^{(\rho')})$. Applying ^{t} to this relation and using the inductive assumption, we get

$$L(a(n+1,d)^{(\rho')})^t \times L(a(d,n-1)^{(\rho')}) \leq \nu_\rho^{-1/2} L(a(d,n)^{(\rho')}) \times \nu_\rho^{1/2} L(a(d,n)^{(\rho')}).$$

Suppose

$$L(a(n+1,d)^{(\rho')})^t = L(a(n+1,d)^{(\rho')}).$$

Then

$$L(a(n+1,d)^{(\rho')}) \times L(a(d,n-1)^{(\rho')}) \leq \nu_\rho^{-1/2} L(a(d,n)^{(\rho')}) \times \nu_\rho^{1/2} L(a(d,n)^{(\rho')}).$$

Then by the definition of the ordering $\leq$ on $M(\mathcal{S}_A)$, we can not have on the left hand side more segments than on the right hand side (since ordering is generated by linking segments). This implies $n+1+d \leq 2d$, i.e. $n+1 \leq d$ which implies $n < d$. This contradicts $d+1 < n$. Thus $L(a(n+1,d)^{(\rho')})^t \neq L(a(n+1,d)^{(\rho')})$, what we needed to prove. □

3.2. Corollary. *Assume that (U0) holds. Then ^{t} carries the class of irreducible unitary representations to itself.* □

4. On ends of complementary series of $GL(n,A)$; character identities

4.1 The following proposition describes irreducible subquotients in the ends of complementary series. Besides the fact that this is crucial information for determining the topology of the unitary dual, this result (essentially character identity) will be crucial for us in obtaining formulas for (characters of) irreducible unitary representations in terms of (characters of) standard modules.

Proposition. *Assume that (U0) holds. Then for $n, d \in \mathbb{N}, \rho' \in \mathcal{C}_A$ we have in R_A*

$$\nu_{\rho'}^{-1/2} L(a(n,d)^{(\rho')}) \times \nu_{\rho'}^{1/2} L(a(n,d)^{(\rho')}) = \\ L(a(n-1,d)^{(\rho')}) \times L(a(n+1,d)^{(\rho')}) + L(a(n,d-1)^{(\rho')}) \times L(a(n,d+1)^{(\rho')}).$$

Proof. First note that it is enough to prove the above equality for ρ' unitarizable.

Further, Proposition 4.3 of [T5] implies that it is enough to prove the proposition for $n \geq 2$. Applying involution ^{t}, we conclude that it is enough to consider only the case $d \geq 2$.

Let

$$\pi = \nu_{\rho'}^{-1/2} L(a(n,d)^{(\rho')}) \times \nu_{\rho'}^{1/2} L(a(n,d)^{(\rho')}).$$

We know that $L(a(n-1,d)^{(\rho')}) \times L(a(n+1,d)^{(\rho')})$ is a subquotient of multiplicity one in π (see 2.2).

Applying the same argument to $\pi' = \nu_{\rho'}^{-1/2} L(a(d,n)^{(\rho')}) \times \nu_{\rho'}^{1/2} L(a(d,n)^{(\rho')})$, we get that $L(a(d-1,n)^{(\rho')}) \times L(a(d+1,n)^{(\rho')})$ is a subquotient of π' of multiplicity one.

Applying ^{t} to π', we get from Lemma 3.1 that $L(a(n,d-1)^{(\rho')}) \times L(a(n,d+1)^{(\rho')})$ is a subquotient of $\pi = (\pi')^t$ of multiplicity one. Therefore, to complete the proof, it is enough to prove that there are no additional irreducible subquotients besides these two.

Let σ be an irreducible subquotient of π different from the above two subquotients. Since π is an end of complementary series, σ must be unitarizable. Since π is rigid, σ must be rigid. This easily implies that

$$\sigma = L(a(n_1,d_1)^{(\rho')}) \times \ldots \times L(a(n_k,d_k)^{(\rho')})$$

for some n_i's and d_i's. After renumeration, we can (and shall) assume that

$$n_1 + d_1 \geq n_2 + d_2 \geq \cdots \geq n_k + d_k.$$

Look at the cuspidal representation $\nu_{\rho'}^{-(n+d)/2+1-1/2}\rho' = \nu_{\rho'}^{-(n+d)/2+1/2}\rho'$. This is the first representation (from the negative left hand side) in the cuspidal support of π. Then the cuspidal support tells

$$n_1 + d_1 = n + d + 1.$$

(Observe that we must have $n_1 + d_1 > n_2 + d_2$, since the multiplicity of $\nu_{\rho'}^{-(n+d)/2+1/2}\rho'$ in the cuspidal support of π is one.)

The rules for linking segments imply

$$d \leq d_1.$$

(Since $\nu_{\rho'}^{-(n+d)/2+1/2}\rho'$ is the left end of only one segment in π, and there are no segments which are more to the left, the segment starting with $\nu_{\rho'}^{-(n+d)/2+1/2}\rho'$ must be at least of length d.) Applying ^{t}, Lemma 3.1, and repeating the above argument in this situation, we get

$$n \leq n_1.$$

The three above relations imply

$$(n_1, d_1) = (n+1, d) \quad \text{or} \quad (n_1, d_1) = (n, d+1),$$

i.e.

$$L(a(n_1,d_1)^{(\rho')}) = L(a(n+1,d)^{(\rho')}) \quad \text{or} \quad L(a(n_1,d_1)^{(\rho')}) = L(a(n,d+1)^{(\rho')}).$$

Now the first remaining representation in the cuspidal support is $\nu_{\rho'}^{-(n+d)/2+1/2+1}\rho = \nu_{\rho'}^{-(n+d)/2+3/2}\rho$. (Similarly as above, looking at the cuspidal support of π, we can conclude that $n_2 + d_2 > n_3 + d_3$ if $k \geq 3$.) This implies

$$n_2 + d_2 = n + d - 1.$$

Now looking at the rules for linking segments, one gets directly

$$d-1 \leq d_2.$$

(Note that $\nu_{\rho'}^{-(n+d)/2+3/2}\rho'$ must be the beginning of a segment in $a(n_2,d_2)^{(\rho')}$, and the shortest segment that can have this beginning is of length $d-1$, which one gets by intersecting the most left segment in π with the segment in π starting at $\nu_{\rho'}^{-(n+d)/2+1/2}\rho'$).

Repeating the above argument in the case of π^t and using Lemma 3.1 we get

$$n-1 \leq n_2.$$

The three above relations imply

$$(n_2,d_2)=(n-1,d) \quad \text{or} \quad (n_2,d_2)=(n,d-1),$$

i.e.

$$L(a(n_2,d_2)^{(\rho')}) = L(a(n-1,d)^{(\rho')}) \quad \text{or} \quad L(a(n_2,d_2)^{(\rho')}) = L(a(n,d-1)^{(\rho')}).$$

We have now four possibilities for the first two factors of σ. We shall analyze two possibilities. Let

$$\sigma' = L(a(n_3,d_3)^{(\rho')}) \times \ldots \times L(a(n_k,d_k)^{(\rho')})$$

if $k \geq 3$. Otherwise, we take $\sigma'=1$.

Suppose that σ is isomorphic to

$$L(a(n+1,d)^{(\rho')}) \times L(a(n,d-1)^{(\rho')}) \times \sigma' \quad \text{or} \quad L(a(n,d+1)^{(\rho')}) \times L(a(n-1,d)^{(\rho')}) \times \sigma'.$$

The first representation cannot be a subquotient of π since it corresponds to at least $2n+1$ segments, while π is defined by $2n$ segments. For the second representation, observe that

$$\big(L(a(n,d+1)^{(\rho')}) \times L(a(n-1,d)^{(\rho')}) \times \sigma\big)^t = L(a(d+1,n)^{(\rho')}) \times L(a(d,n-1)^{(\rho')}) \times (\sigma')^t$$

is a subquotient of $\pi^t = \nu_{\rho'}^{-1/2}L(a(d,n)^{(\rho')}) \times \nu_{\rho'}^{1/2}L(a(d,n)^{(\rho')})$. This is impossible, for the same reason as in the first case.

Therefore, the only two remaining possibilities for σ are

$$L(a(n-1,d)^{(\rho')}) \times L(a(n+1,d)^{(\rho')}) \times \sigma',$$
$$L(a(n,d-1)^{(\rho')}) \times L(a(n,d+1)^{(\rho')}) \times \sigma'.$$

But since both

$$L(a(n-1,d)^{(\rho')}) \times L(a(n+1,d)^{(\rho')}) \quad \text{and} \quad L(a(n,d-1)^{(\rho')}) \times L(a(n,d+1)^{(\rho')})$$

have in their cuspidal supports $2nd$ representations (counted with multiplicities), which is exactly the number of representations in the cuspidal support of π (counted with multiplicities), we conclude that $\sigma'=1$.

This completes the proof of the lemma. □

5. On characters of irreducible unitary representations of $GL(n, A)$

5.1. The set of non-negative integers is denoted by $\mathbb{Z}_+$. Fix $\rho \in \mathcal{C}_F$ and $\rho' \in \mathcal{C}_A$. Let $R_F(\rho)$ (resp. $R_A(\rho')$) be the subalgebra of R_F (resp. R_A) generated by

$$\left\{\delta([\nu^{k_1}\rho, \nu^{k_2}\rho]) \ \mid \ k_1, k_2 \in (1/2)\mathbb{Z}, k_2 - k_1 \in \mathbb{Z}_+\right\}$$
$$\left(\text{resp. } \ \{\delta([\nu_{\rho'}^{k_1}\rho', \nu_{\rho'}^{k_2}\rho']) \ \mid \ k_1, k_2 \in (1/2)\mathbb{Z}, k_2 - k_1 \in \mathbb{Z}_+\}\right),$$

Then clearly, both algebras are polynomial over the above sets of generators. Define an algebra isomorphism $\Psi_{\rho,\rho'} : R_F(\rho) \longrightarrow R'_A(\rho')$ by

$$\Psi_{\rho,\rho'} : \delta([\nu^{k_1}\rho, \nu^{k_2}\rho]) \mapsto \delta([\nu_{\rho'}^{k_1}\rho', \nu_{\rho'}^{k_2}\rho'])$$

for all $k_1, k_2 \in (1/2)\mathbb{Z}, k_2 - k_1 \in \mathbb{Z}_+$.

Lemma. *If we assume (U0), then*

$$\Psi_{\rho,\rho'} : (L(a(n,d)^{(\nu^k\rho)}) = L(a(n,d)^{(\nu_{\rho'}^k\rho')})$$

for all $n, d \in \mathbb{N}$ and $k \in (1/2)\mathbb{Z}$.

Proof. We shall prove the lemma by induction with respect to n. For $n = 1$ (and all d) the lemma holds by definition of $\Psi_{\rho,\rho'}$ (see 1.14 and 2.6). Fix $n \geq 1$ and assume that the formula of the lemma holds for all $n' \leq n$. Applying $\Psi_{\rho,\rho'}$ to the formula of Theorem 1.15 (with $\nu^k\rho$ instead of ρ in the formula), and using the inductive assumption we get

$$\nu_{\rho'}^{1/2} L(a(n,d)^{(\nu_{\rho'}^k\rho')}) \times \nu_{\rho'}^{-1/2} L(a(n,d)^{(\nu_{\rho'}^k\rho')}) =$$
$$\Psi_{\rho,\rho'}\left(L(a(n+1,d)^{(\nu^k\rho)})\right) \times L(a(n-1,d)^{(\nu_{\rho'}^k\rho')}) + L(a(n,d+1)^{(\nu_{\rho'}^k\rho'}) \times L(a(n,d-1)^{(\nu_{\rho'}^k\rho')}).$$

Now subtracting the above formula from the formula of Proposition 4.1 (with $\nu_{\rho'}^k\rho'$ instead of ρ' in the formula) and using the fact that R_A is an integral domain, one gets $\Psi_{\rho,\rho'}(L(a(n+1,d)^{(\nu^k\rho)}) = L(a(n+1,d)^{(\nu_{\rho'}^k\rho')})$. The proof of the lemma is now complete. $\square$

5.2. As a direct consequence of the above lemma and Theorem 1.15 we get the following

Proposition. *Assume that (U0) holds. Let $\rho' \in \mathcal{C}_A$ and $n, d \in \mathbb{Z}, n, d \geq 1$. Let W_n be the group of permutations of the set $\{1, 2, \dots, n\}$. Denote $W_n^{(d)} = \{w \in W_n; w(i) + d \geq i$ for all $1 \leq i \leq n\}$. Then we have the following identity in R_A*

$$L(([\nu_{\rho'}\rho', \nu_{\rho'}^d\rho'], [\nu_{\rho'}^2\rho', \nu_{\rho'}^{d+1}\rho'], \dots, [\nu_{\rho'}^n\rho', \nu_{\rho'}^{d+n-1}\rho']))$$
$$= \sum_{w \in W_n^{(d)}} (-1)^{\text{sgn}(w)} \prod_{i=1}^{n} \delta([\nu_{\rho'}^i\rho', \nu_{\rho'}^{w(i)+(d-1)}\rho']). \quad \square$$

6. Jacquet-Langlands correspondence

6.1. A.I. Badulescu in [Ba2] studied very natural extensions of Jacquet-Langlands correspondences. We shall recall here some of his considerations (in a slightly different notation).

In section 2.3 we recalled the Jacquet-Langlands correspondences

$$\mathrm{JL}_p : \mathcal{D}(GL(p,A)) \to \mathcal{D}(GL(pd_A,F)), \tag{6-1-1}$$

which are uniquely determined by the requirement that the characters $\Theta_{\delta'}$ and $\Theta_{\mathrm{JL}_p(\delta')}$ satisfy

$$(-1)^p \Theta_{\delta'}(g') = (-1)^{pd_A} \Theta_{\mathrm{JL}_p(\delta')}(g). \tag{6-1-2}$$

whenever g' and g have the same characteristic polynomials, and when this polynomial is separable.

The above correspondences are bijections, so instead of the correspondences JL_p, we could consider their inverses JL_p^{-1}.

The mappings JL_p, $p \geq 1$, define in a natural way an injective mapping

$$\mathrm{JL} : \mathcal{D}_A \to \mathcal{D}_F.$$

Since the algebras R_A and R_F are polynomial over $\mathcal{D}_A$ and $\mathcal{D}_F$ respectively, JL_p can be uniquely extended to a ring homomorphism of R_A into R_F, which will be again denoted by

$$\mathrm{JL} : R_A \to R_F. \tag{6-1-3}$$

Clearly, the extension is also injective.

The homomorphism JL carries $\mathrm{Groth}(GL(p,A))$ to $\mathrm{Groth}(GL(pd_A,F))$, and we shall denote this restriction again by

$$\mathrm{JL}_p : \mathrm{Groth}(GL(p,A)) \to \mathrm{Groth}(GL(pd_A,F)). \tag{6-1-4}$$

Then this extended JL_p again satisfies the relation

$$(-1)^p \Theta_{\pi'}(\pi') = (-1)^{pd_A} \Theta_{\mathrm{JL}_p(\pi')}(g) \tag{6-1-5}$$

for any $\pi \in GL(p,A)\tilde{}$.

Let

$$\mathcal{D}_F^{(d_A)} = \underset{p\geq 1}{\cup} \mathcal{D}(GL(pd_A,F)).$$

Then JL defines a bijection of $\mathcal{D}_A$ onto $\mathcal{D}_F^{(d_A)}$. Denote the inverse mapping by

$$\mathrm{LJ} : \mathcal{D}_F^{(d_A)} \to \mathcal{D}_A.$$

There exists a unique ring homomorphism $R_F \to R_A$ which extends LJ and which sends all the elements from $\mathcal{D}_F \setminus \mathcal{D}_F^{(d_A)}$ to $0 \in R_A$. This extension will be denoted again by

$$\mathrm{LJ} : R_F \to R_A.$$

If $d_A | m$, then we shall denote by LJ_m the restriction

$$\mathrm{LJ}_m : \mathrm{Groth}(GL(m, F)) \to \mathrm{Groth}(GL(m/d_A, A)).$$

Otherwise, we shall take (formally) $\mathrm{LJ}_m = 0$ (as a mapping from $\mathrm{Groth}(GL(m, F))$ into R_A).

Let

$$I_{F,A}$$

be the ideal in R_F generated by $\mathcal{D}_F \setminus \mathcal{D}_F^{(d_A)}$ (clearly, this ideal is graded). This is just the kernel of LJ. Therefore, $R_A \cong R_F/I_{F,A}$.

Further, suppose that $\varphi \in \mathrm{Groth}(GL(m, F))$ is in $I_{F,A}$ and $d_A|m$. Then for regular semisimple $g \in GL(m, F)$ we have

$$\Theta_\varphi(g) = 0,$$

where Θ_φ denotes the formal character of φ. Therefore,

$$(-1)^m \Theta_\varphi(g) = (-1)^{\frac{m}{d_A}} \Theta_{\mathrm{LJ}(\varphi)}(g') \tag{6-1-6}$$

whenever g and g' have the same characteristic polynomials, and when this polynomial is separable. Clearly, $\mathrm{LJ} \circ \mathrm{JL}=\mathrm{id}_{R_A}$.

The correspondence JL: $R_A \to R_F$, which we considered first, does not behave well with respect to irreducibility. Namely, one sees easily (as in the Comments after Theorem 3.1 in [Ba2]) that JL does not in general carry irreducible representations to irreducible ones (up to a sign). A similar situation happens with unitarity. In general even irreducible unitary representations are carried neither to irreducible unitary representations (up to a sign), nor to linear combinations of irreducible unitary representations.

Assuming (U0) to hold, we shall see in the rest of the paper that the correspondence LJ: $R_F \to R_A$ behave well with respect to irreducible unitary representations, i.e., that it carries such representations either again to the irreducible unitary representations (up to a sign) or to 0.

6.2. Let $\rho' \in \mathcal{C}_A$. Suppose that

$$\delta([\rho, \nu^{s_{\rho'}-1}\rho]) \in \mathcal{D}_F \quad \text{corresponds to} \quad \rho'$$

under the Jacquet-Langlands correspondence (here $\rho \in \mathcal{C}_F$). Then

$$\delta([\rho, \nu^{s_{\rho'}k-1}\rho]) \quad \text{corresponds to} \quad \delta([\rho', \nu_{\rho'}^{k-1}\rho']).$$

6.3. Fix an irreducible cuspidal representation ρ of $GL(m,F)$. Let $s'm$ be the smallest common multiple of m and d_A. The fact $s'm|d_Am$ implies

$$s'|d_A.$$

Note that $\delta([\rho,\nu^{s'-1}\rho])$ is an irreducible essentially square-integrable representation of $GL(s'm,F)$. Therefore, it lifts under the Jacquet-Langlands correspondence to an irreducible essentially square-integrable representation ρ' in R_A. A short discussion implies that ρ' is cuspidal, and then $s_{\rho'}=s'$.

Now ρ' is a representation of $GL(p,A)$, where $p=\frac{ms_{\rho'}}{d_A}$. Since $s'm$ is the smallest common multiple of m and d_A, this implies $(p,s_{\rho'})=1$ (if k were the greatest common divisor, then kd_A and km would divide $s'm$). Further, the smallest common multiple of d_A and $\frac{pd_A}{s_{\rho'}}=m$ is $s_{\rho'}m=pd_A$.

6.4. Assuming (U0) we shall compute $\mathrm{LJ}(\pi)$ for irreducible unitary representations of general linear groups over the field. Since LJ is a ring homomorphism, for this it will be enough to compute

$$\mathrm{LJ}(L(a(r,d)^{(\rho)})\in R_A,\quad r,d\in\mathbb{N}.$$

Suppose that we are in the situation of 6.3, and suppose that $s_{\rho'}=1$ (which means that ρ corresponds to ρ' under the Jacquet-Langlands correspondence). Then Theorem 1.15 and Proposition 5.2 directly imply

$$\mathrm{LJ}(L(a(r,d)^{(\rho)}))=L(a(r,d)^{(\rho')}).$$

It remains therefore to consider the case

$$s_{\rho'}\geq 2.$$

We shall assume this in the rest of the paper.

If $r=1$, then we know $\mathrm{LJ}(L(a(1,d)^{(\rho)}))$ by 6.2, so we can assume also $r\geq 2$.

To simplify notation, we shall often denote bellow $s_{\rho'}$ by n, i.e.

$$s_{\rho'}=n.$$

7. Calculation of the Jacquet-Langlands correspondence in the unitary case I, the case of $r\leq d$ and $s_{\rho'}|d$

7.1. In this section, we shall assume $r\leq d$ and $s_{\rho'}|d$ (i.e. $n|d$).

We shall use below the following Lemma 3.1 of [T8]. To follow later calculations more easily, we shall repeat this technical lemma here.

Denote by W_r' the group of permutations of $\{0,1,\dots,r-1\}$. The signature of a permutation w will be denoted by $\mathrm{sgn}(w)$.

Lemma. *Write $r = an + b$, with $a, b \in \mathbb{Z}$ such that $0 \le b \le n-1$.*
(i) Let

$$W'_r(n) = \{w \in W'_r \ ; \ n|(w(i)-i) \text{ for all } 0 \le i \le r-1\}.$$

For $0 \le \ell \le \min(n,r)-1$ denote by

$$W'_r(n;\ell) = \{w \in W'_r \ ; \ w(i) = i \text{ if } n \not| (i-\ell)\}.$$

Then $W'_r(n)$ is a subgroup of W'_r, $W'_r(n;\ell)$ are subgroups of $W'_r(n)$ and $W'_r(n)$ is a direct product of $W'_r(n;\ell)$, $\ell = 0,1,2,\dots,\min(n,r)-1$.
(ii) Let $0 \le \ell \le b-1$ (resp. $b \le \ell \le \min(n,r)-1$). For $w \in W'_{a+1}$ (resp. $w \in W'_a$) define $w^* \in W'_r$ by

$$w^*(j) = \begin{cases} j, & \text{if } \ n \not| \ (i-\ell); \\ \ell + nw(i), & \text{if } \ j = \ell + ni. \end{cases}$$

Then $w \mapsto w^*$ is an isomorphism of W'_{a+1} (resp. W'_a) onto $W'_r(n;\ell)$. Further, sgn(w) = sgn(w^*). □

7.2. Let

$$\Pi = L(([\rho, \nu^{d-1}\rho], [\nu\rho, \nu^d\rho], \dots, [\nu^{r-1}\rho, \nu^{r-1+d-1}\rho])).$$

Now we shall start computation of

$$\begin{aligned} \mathrm{LJ}(\Pi) &= \mathrm{LJ}\left(L(([\rho, \nu^{d-1}\rho], [\nu\rho, \nu^d\rho], \dots, [\nu^{r-1}\rho, \nu^{r-1+d-1}\rho]))\right) \\ &= \sum_{w \in W'_r} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \mathrm{LJ}\left(\delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho])\right) \\ &= \sum_{w \in W'_r(n)} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \mathrm{LJ}\left(\delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho])\right), \end{aligned}$$

since $n|d$ and $r \le d$.

Write $r = an+b, a, b \in \mathbb{Z}, 0 \le b \le n-1$. We shall now use the above lemma to modify the above sum:

$$\mathrm{LJ}(\Pi) = \sum_{w \in W'_r(n)} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \mathrm{LJ}\left(\delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho])\right)$$

$$= \sum_{w_0, w_1, \dots, w_{b-1} \in W'_{a+1}} \quad \sum_{w_b, w_{b+1} \dots, w_{\min(n,r)-1} \in W'_a}$$

$$\left(\prod_{\ell=0}^{b-1} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{a} \mathrm{LJ}(\delta([\nu^{\ell+n\,i}\rho, \nu^{\ell+n\,w_\ell(i)+(d-1)}\rho])) \right.$$

$$\times \left(\prod_{\ell=b}^{\min(n,r)-1} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{a-1} \mathrm{LJ}(\delta([\nu^{\ell+n\,i}\rho, \nu^{\ell+n\,w_\ell(i)+(d-1)}\rho])) \right)$$

$$= \left(\prod_{\ell=0}^{b-1} \sum_{w_\ell \in W'_{a+1}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{a} \mathrm{LJ}(\delta([\nu^{\ell+n\,i}\rho, \nu^{\ell+n\,w_\ell(i)+(d-1)}\rho])) \right)$$

$$\times \left(\prod_{\ell=b}^{\min(n,r)-1} \sum_{w_\ell \in W'_{a}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{a-1} \mathrm{LJ}(\delta([\nu^{\ell+n\,i}\rho, \nu^{\ell+n\,w_\ell(i)+(d-1)}\rho])) \right)$$

(we assume in the sequel that $\delta([\rho, \nu^{n-1}\rho])$ and ρ' correspond under the Jacquet-Langlands correspondence as in 6.2)

$$= \left(\prod_{\ell=0}^{b-1} \sum_{w_\ell \in W'_{a+1}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{a} \delta([\nu^{\ell+n\,i}\rho', \nu^{\ell+n\,i}\nu_{\rho'}^{w_\ell(i)-i+(\frac{d}{n}-1)}\rho']) \right)$$

$$\times \left(\prod_{\ell=b}^{\min(n,r)-1} \sum_{w_\ell \in W'_{a}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{a-1} \delta([\nu^{\ell+n\,i}\rho', \nu^{\ell+n\,i}\nu_{\rho'}^{w_\ell(i)-i+(\frac{d}{n}-1)}\rho']) \right)$$

$$= \left(\prod_{\ell=0}^{b-1} \nu_{\rho'}^{\frac{\ell}{n}} \sum_{w_\ell \in W'_{a+1}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{a} \delta([\nu_{\rho'}^{i}\rho', \nu_{\rho'}^{w_\ell(i)+(\frac{d}{n}-1)}\rho']) \right)$$

$$\times \left(\prod_{\ell=b}^{\min(n,r)-1} \nu_{\rho'}^{\frac{\ell}{n}} \sum_{w_\ell \in W'_{a}} (-1)^{\mathrm{sgn}(w_\ell)} \prod_{i=0}^{a-1} \delta([\nu_{\rho'}^{i}\rho', \nu_{\rho'}^{w_\ell(i)+(\frac{d}{n}-1)}\rho']) \right).$$

Note that $r \leq d$ implies $r - b \leq d$, which implies $an \leq d$, i.e.

$$a \leq d/n.$$

If

$$b \geq 1,$$

then $r - b < d$, which implies $a < \frac{d}{n}$ and further

$$a + 1 \leq d/n$$

(since $n|d$).

Therefore

$$\mathrm{LJ}(\Pi) =$$

$$= \left(\prod_{\ell=0}^{b-1} \nu_{\rho'}^{\frac{\ell}{n}}\; L(([\rho', \nu_{\rho'}^{\frac{d}{n}-1}\rho'], [\nu_{\rho'}\rho', \nu_{\rho'}^{\frac{d}{n}}\rho'], \dots, [\nu_{\rho'}^{a}\rho', \nu_{\rho'}^{a+\frac{d}{n}-1}\rho'])) \right)$$

$$\times\left(\prod_{\ell=b}^{\min(n,r)-1}\nu_{\rho'}^{\frac{\ell}{n}}L(([\rho',\nu_{\rho'}^{\frac{d}{n}-1}\rho'],[\nu_{\rho'}\rho',\nu_{\rho'}^{\frac{d}{n}}\rho'],\dots,[\nu_{\rho'}^{a-1}\rho',\nu_{\rho'}^{a-1+\frac{d}{n}-1}\rho']))\right).$$

7.3. Now suppose that ρ is unitary and that

$$\delta(\rho,s_{\rho'})=\delta(\rho,n)\quad\text{corresponds to}\quad\rho''$$

under the Jacquet-Langlands correspondence. Then ρ'' is unitary.

Further

$$\rho''=\mathrm{LJ}(\delta(\rho,n))=\mathrm{LJ}(\nu^{-\frac{n-1}{2}}\delta([\rho,\nu^{n-1}\rho]))=\nu^{-\frac{n-1}{2}}\mathrm{LJ}(\delta([\rho,\nu^{n-1}\rho]))=\nu_{\rho'}^{-\frac{n-1}{2n}}\rho',$$

i.e.

$$\rho''=\nu_{\rho'}^{-\frac{n-1}{2n}}\rho'$$

(and $\rho'=\nu_{\rho'}^{\frac{n-1}{2n}}\rho''$). Note that

$$\nu_{\rho'}=\nu_{\rho''}.$$

Now we shall compute (for $r\le d$ and $n|d$)

$$\mathrm{LJ}(L(a(r,d)^{(\rho)}))=\mathrm{LJ}(\nu^{-\frac{r+d}{2}+1}L(([\rho,\nu^{d-1}\rho],[\nu\rho,\nu^{d}\rho],\dots,[\nu^{r-1}\rho,\nu^{r-1+d-1}\rho]))$$

$$=\nu^{-\frac{r+d}{2}+1}\left[\left(\prod_{\ell=0}^{b-1}\nu_{\rho'}^{\frac{\ell}{n}}\,\nu_{\rho'}^{\frac{a+1+\frac{d}{n}}{2}-1}L(a(a+1,d/n)^{(\rho')})\right)\right.$$

$$\left.\times\left(\prod_{\ell=b}^{\min(n,r)-1}\nu_{\rho'}^{\frac{\ell}{n}}\,\nu_{\rho'}^{\frac{a+\frac{d}{n}}{2}-1}L(a(a,d/n)^{(\rho')})\right)\right].$$

Since $\nu^{-\frac{r+d}{2}+1}=\nu_{\rho'}^{\frac{-r-d+2}{2n}}=\nu_{\rho'}^{\frac{-an-b-d+2}{2n}}$, we have

$$\mathrm{LJ}(L(a(r,d)^{(\rho)}))=\left(\prod_{\ell=0}^{b-1}\nu_{\rho'}^{\frac{-b+2+2\ell-n}{2n}}L(a(a+1,d/n)^{(\rho')})\right)$$

$$\times\left(\prod_{\ell=b}^{\min(n,r)-1}\nu_{\rho'}^{\frac{-b+2+2\ell-2n}{2n}}L(a(a,d/n)^{(\rho')})\right)$$

$$=\left(\prod_{\ell=0}^{b-1}\nu_{\rho'}^{\frac{-b+2+2\ell-n}{2n}}L(a(a+1,d/n)^{(\nu_{\rho'}^{\frac{n-1}{2n}}\rho'')})\right)$$

$$\times\left(\prod_{\ell=b}^{\min(n,r)-1}\nu_{\rho'}^{\frac{-b+2+2\ell-2n}{2n}}L(a(a,d/n)^{(\nu_{\rho'}^{\frac{n-1}{2n}}\rho'')})\right)$$

$$= \left(\prod_{\ell=0}^{b-1} \nu_{\rho''}^{\frac{-b+1+2\ell}{2n}} L(a(a+1,d/n)^{(\rho'')}) \right)$$

$$\times \left(\prod_{\ell=b}^{\min(n,r)-1} \nu_{\rho''}^{\frac{-b+1+2\ell-n}{2n}} L(a(a,d/n)^{(\rho'')}) \right)$$

$$= \left(\prod_{\ell'=-\frac{b-1}{2}}^{\frac{b-1}{2}} \nu_{\rho''}^{\frac{\ell'}{n}} L(a(a+1,d/n)^{(\rho'')}) \right)$$

$$\times \left(\prod_{\ell=b}^{\min(n,r)-1} \nu_{\rho''}^{\frac{\frac{-b-n+1}{2}+\ell}{n}} L(a(a,d/n)^{(\rho'')}) \right)$$

For an irreducible representation π of $GL(l,A)$, positive integer l and a non-negative integer k, define

$$\text{string}_{\nu_{\rho'}}(k,l,\pi) = (\nu_{\rho'}^{\frac{-(k-1)/2}{l}}\pi) \times (\nu_{\rho'}^{\frac{-(k-1)/2+1}{l}}\pi) \times (\nu_{\rho'}^{\frac{-(k-1)/2+2}{l}}\pi) \times \cdots \times (\nu_{\rho'}^{\frac{(k-1)/2}{l}}\pi).$$

If $k=0$, we take the string to be the identity of R_A (i.e., the trivial representation of the trivial group $GL(0,A)$).

From this directly follows the

Proposition. *Suppose that (U0) holds. Assume that $\delta(\rho,n)$ corresponds to $\rho'' \in \mathcal{C}_A$ under the Jacquet-Langlands correspondence. Let*

$$r \leq d, \quad n|d, \quad 1 \leq n.$$

Write

$$r = an+b, \quad a,b \in \mathbb{Z}, \quad 0 \leq b \leq n-1.$$

Then

$$LJ(L(a(r,d)^{(\rho)})) = \text{string}_{\nu_{\rho''}}(b,n,L(a(a+1,d/n)^{(\rho'')}))$$
$$\times \text{string}_{\nu_{\rho''}}(\min(n,r)-b,n,L(a(a,d/n)^{(\rho'')})). \quad \square$$

Remark. (i) Our definition of $\text{string}_{\nu_{\rho'}}(k,l,\pi)$ is more general that we need for the present paper. In this paper, whenever we use $\text{string}_{\nu_{\rho'}}(k,l,\pi)$, it will be $l = s_{\rho'} = n$. Then $\text{string}_{\nu_{\rho'}}(k,n,\pi)$, which equals $\text{string}_{\nu}(k,1,\pi)$ in this case, will be denoted simply by

$$\text{string}(k,\pi).$$

In this case $\text{string}(k,\pi)$ has the following simpler form:

$$\text{string}(k,\pi) = (\nu^{-(k-1)/2}\pi) \times (\nu^{-(k-1)/2+1}\pi) \times (\nu^{-(k-1)/2+2}\pi) \times \cdots \times (\nu^{(k-1)/2}\pi).$$

Note that neither $\nu_{\rho'}$ nor $n = s_{\rho'}$ show up on the right hand side.

We shall use both notation, $\text{string}(k,\pi)$ and $\text{string}_{\nu_{\rho'}}(k,n,\pi)$. The later will be convenient to us when we study unitarity.

Now the above proposition can be written in the following form:

Proposition′. *With the same assumptions as in the above proposition, we have*

$$\mathrm{LJ}(L(a(r,d)^{(\rho)})) = \mathrm{string}(b, L(a(a+1,d/n)^{(\rho'')}))$$

$$\times \mathrm{string}(min(n,r)-b, L(a(a,d/n)^{(\rho'')})). \quad \square$$

We can rewrite the above formula as

$$\mathrm{LJ}(L(a(r,d)^{(\rho)})) = \mathrm{string}(r-[r/n]n, L(a([r/n]+1,d/n)^{(\rho'')}))$$

$$\times \mathrm{string}(\min(n,r)-r+[r/n]n, L(a([r/n],d/n)^{(\rho'')})).$$

where $[x]$ denotes the greatest integer which does not exceed x.

8. Unitarity of the Jacquet-Langlands correspondence of irreducible unitary representations I

8.1. We shall now show that $\mathrm{LJ}(L(a(r,d)^{(\rho)}))$, which we have computed in the previous section, is irreducible and unitary if $\rho \in \mathcal{C}_F$ is unitary. (After this, we shall show that $\mathrm{LJ}(\nu^\beta(L(a(r,d)^{(\rho)})) \times \nu^{-\beta}(L(a(r,d)^{(\rho)})))$ is unitary for $0 < \beta < 1/2$.)

If $b = 0$, then $\mathrm{string}_{\nu_{\rho''}}(b, n, L(a(a+1,d/n)^{(\rho'')})) = 1$. So $\mathrm{string}_{\nu_{\rho''}}(b, n, L(a(a+1,d/n)^{(\rho'')}))$ is irreducible unitary. Suppose $b \geq 1$. Then $0 \leq (b-1)/(2n) < n/(2n) = 1/2$. Therefore, $\mathrm{string}_{\nu_{\rho''}}(b, n, L(a(a+1,d/n)^{(\rho'')}))$ is again irreducible unitary.

If $r < n$, then $\mathrm{string}_{\nu_{\rho''}}(\min(n,r)-b, n, L(a(a,d/n)^{(\rho'')})) = 1$, and $\mathrm{string}_{\nu_{\rho''}}(\min(n,r)-b, n, L(a(a,d/n)^{(\rho'')}))$ is irreducible unitary. Suppose $r \geq n$. Then

$$\mathrm{string}_{\nu_{\rho''}}(\min(n,r)-b, n, L(a(a,d/n)^{(\rho'')})) = \mathrm{string}_{\nu_{\rho''}}(n-b, n, L(a(a,d/n)^{(\rho'')})).$$

Since now $0 \leq (n-b-1)/(2n) \leq (n-1)/(2n) < 1/2$, this implies that $\mathrm{string}_{\nu_{\rho''}}(\min(n,r)-b, n, L(a(a,d/n)^{(\rho'')})$ is irreducible unitary. Therefore, $\mathrm{LJ}(L(a(r,d)^{(\rho)}))$ is irreducible unitary.

For $0 < \beta < 1/2$ let

$$\pi(L(a(r,d)^{(\rho)}), \beta) = \nu^\beta L(a(r,d)^{(\rho)}) \times \nu^{-\beta} L(a(r,d)^{(\rho)}).$$

Now we shall show that

$$\mathrm{LJ}(\pi(L(a(r,d)^{(\rho)}), \beta))$$

is irreducible unitary. Observe

$$\mathrm{LJ}(\pi(L(a(r,d)^{(\rho)}), \beta))$$

$$= \nu^\beta \mathrm{LJ}(L(a(r,d)^{(\rho)}) \times \nu^{-\beta} \mathrm{LJ}(L(a(r,d)^{(\rho)})$$

$$= \nu_{\rho''}^{\frac{\beta}{n}} \operatorname{string}_{\nu_{\rho''}}(b, n, L(a(a+1, d/n)^{(\rho'')})) \times \nu_{\rho''}^{-\frac{\beta}{n}} \operatorname{string}_{\nu_{\rho''}}(b, n, L(a(a+1, d/n)^{(\rho'')}))$$
$$\times \nu_{\rho''}^{\frac{\beta}{n}} \operatorname{string}_{\nu_{\rho''}}(\min(n, r) - b, n, L(a(a, d/n)^{(\rho'')})$$
$$\times \nu_{\rho''}^{-\frac{\beta}{n}} \operatorname{string}_{\nu_{\rho''}}(\min(n, r) - b, n, L(a(a, d/n)^{(\rho'')}).$$

This implies that it is enough to show unitarity (and irreducibility, which follows from unitarity) of

$$\nu_{\rho''}^{\frac{\beta}{n}} \operatorname{string}_{\nu_{\rho''}}(b, n, L(a(a+1, d/n)^{(\rho'')})) \times \nu_{\rho''}^{-\frac{\beta}{n}} \operatorname{string}_{\nu_{\rho''}}(b, n, L(a(a+1, d/n)^{(\rho'')}))$$

for $b \geq 1$, and of

$$\nu_{\rho''}^{\frac{\beta}{n}} \operatorname{string}_{\nu_{\rho''}}(n - b, n, L(a(a, d/n)^{(\rho'')}) \times \nu_{\rho''}^{-\frac{\beta}{n}} \operatorname{string}_{\nu_{\rho''}}(n - b, n, L(a(a, d/n)^{(\rho'')})$$

for $r \geq n$. For the first representation, we need to show

$$0 \leq (b-1)/(2n) + \beta/n < 1/2 \qquad \text{if} \quad b \geq 1,$$

and for the second,

$$0 \leq (n-b-1)/(2n) + \beta/n < 1/2 \quad \text{if} \ \ r \geq n.$$

This obviously holds since $\beta < 1/2$.

9. Calculation of the Jacquet-Langlands correspondence in the unitary case II, the case of $r \leq d$ and $s_{\rho'}$ not dividing d

9.1. In this section we shall assume that $r \leq d$ and $s_{\rho'} \not| d$ (i.e. $n \not| d$).

If $n \not| rd$, then one sees directly that

$$\operatorname{LJ}\left(L(([\rho, \nu^{d-1}\rho], [\nu\rho, \nu^{d}\rho], \ldots, [\nu^{r-1}\rho, \nu^{r-1+d-1}\rho]))\right) = 0.$$

Therefore, we need only to consider the case $n|rd$. Soon we shall see that a stronger assumption needs to be imposed to get a non-zero result.

Write

$$r = an + b, \quad a, b \in \mathbb{Z},\ 0 \leq b \leq n-1.$$

Now $n|rd$ implies $n|bd$.

We continue to assume

$$n \geq 2.$$

If $r = 1$, then $\operatorname{LJ}(L(([\rho, \nu^{d-1}\rho]))) = 0$ since $n \not| d$. Therefore, we shall assume in the sequel

$$r \geq 2.$$

9.2. The following lemma is a modification of Lemma 4.1 of [T8].

Lemma. *Suppose $n \not| d$. If the set*

$$X_r(n,d) = \{w \in W_r';\ n|(d + w(i) - i) \text{ for all } 0 \le i \le r-1\}$$
$$= \{w \in W_r';\ n|(d + i - w^{-1}(i)) \text{ for all } 0 \le i \le r-1\}$$

is non-empty, then $n|r$.

Proof. Suppose $X_r(n,d) \neq \emptyset$. Clearly, the identity is not in $X_r(n,d)$ since $n \not| d$.

Take some $w \in X_r(n,d)$. Note that for $0 \le i \le r-2$, $n|(d + w(i) - i)$ and $n|(d + w(i+1) - i - 1)$ imply $n|(w(i) - w(i+1) + 1)$ Suppose $w(i) - w(i+1) + 1 = 0$ for all i as above. This implies $w(1) = w(0) + 1$, $w(2) = w(0) + 2, \dots, w(r-2) = w(0) + r - 2$, which implies $w(0) = 1$ (since w cannot be identity). This implies $w(r-1) = 0$.

Since $w \in X_r(n,d)$, we get $n|(d+1)$ and $n|(d + w(r-1) - (r-1)) = (d+1-r)$. These two relations imply $n|r$.

Therefore it remains to consider the case when

$$w(i) - w(i+1) + 1 \neq 0$$

for some $0 \le i \le r-2$. If the above number is negative, then $w(i) - w(i+1) + 1 \le -n$, which implies $w(i) + n + 1 \le w(i+1)$. This implies $n + 1 \le r - 1$. If the above number is positive, then $n \le w(i) - w(i+1) + 1$, which implies $n \le r$. Thus we have proved (up to now) that

$$n \le r.$$

We have written $r = an + b$ $a, b \in \mathbb{Z}$, $0 \le b \le n-1$. We know $a \ge 1$.

Write

$$d = cn + d', \quad c, d' \in \mathbb{Z},$$
$$1 \le d' \le n - 1 \ (\le r - 1)$$

(since $n \not| d$).

Since elements in $X_r(n,d) \subseteq W_r'$ are bijections, for each $i \in \{0, 1, \dots, n-1\}$ it must hold that

$$\begin{aligned}
&\text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, r-1]) \\
&\quad = \text{card}(\{i - d + kn; k \in \mathbb{Z}\} \cap [0, r-1]) \\
&\quad = \text{card}(\{i - d' + kn; k \in \mathbb{Z}\} \cap [0, r-1]) \\
&\quad = \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d', d' + r - 1]).
\end{aligned}$$

From the above relation we get

$$\begin{aligned}
&\text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d'-1]) + \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d', r-1]) \\
&\quad = \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d', r-1]) + \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [r, d' + r - 1]).
\end{aligned}$$

Thus for each $i \in \{0, 1, \dots, n-1\}$ we have

$$\text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d'-1]) = \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [r, d' + r - 1]).$$

The last relation can be written as

$$\mathrm{card}(\{i+kn; k\in\mathbb{Z}\}\cap[0,d'-1]) = \mathrm{card}(\{i+kn; k\in\mathbb{Z}\}\cap[an+b, an+b+d'-1]).$$

Therefore, for each $i \in \{0,1,\dots,n-1\}$

$$\mathrm{card}(\{i+kn; k\in\mathbb{Z}\}\cap[0,d'-1]) = \mathrm{card}(\{i+kn; k\in\mathbb{Z}\}\cap[b, b+d'-1]),$$

i.e.,
$$\mathrm{card}(\{i\}\cap[0,d'-1]) = \mathrm{card}(\{i+kn; k\in\mathbb{Z}\}\cap[b, b+d'-1]).$$

Suppose $b \geq 1$. The above relation for $i = d'-1$ implies $b \leq d'-1$, and further $b < d'$. The case $i = d'$ implies $d' < b$. This is a contradiction. The proof is now complete. □

9.3. We shall also need the following lemma. It is a slight modification of Lemma 4.2 of [T8]. The proof is almost the same. Therefore, we omit it here.

Lemma. *Suppose $r = an$ ($a \in \mathbb{Z}$, $a \geq 1$) and $n \not| d$. Then:*
(i) $W'_r(n)X_r(n,d)W'_r(n) = X_r(n,d)$.
(ii) $X_r(n,d)$ normalizes $W'_r(n)$.
(iii) For any $w \in X_r(n,d)$ we have $X_r(n,d) = wW'_r(n) = W'_r(n)\,w$.
(iv) Each $i \in \{0,1,2,\dots,r-1\}$ write $i = s(i)n + t(i)$ where $s(i), t(i) \in \mathbb{Z}$ and $0 \leq t(i) \leq n-1$. Let $d = cn + d'$, where $c, d' \in \mathbb{Z}, 1 \leq d' \leq n-1$ ($\leq r-1$) ($d' \neq 0$ since $n \not| d$). Define $w_{(n,d)} \in W'_r$ by

$$w_{(n,d)}(i) = \begin{cases} i + (n-d'), & \text{if} \quad t(i) \leq d'-1; \\ i - d', & \text{if} \quad t(i) \geq d'. \end{cases}$$

Then $w_{(n,d)} \in X_r(n,d)$ and sgn $(w_{(n,d)}) = (-1)^{\frac{r}{n}(n-d')d'} = (-1)^{\frac{r}{n}(n-d)d}$.

9.4. As before, let

$$\Pi = L(([\rho, \nu^{d-1}\rho], [\nu\rho, \nu^{d}\rho], \dots, [\nu^{r-1}\rho, \nu^{r-1+d-1}\rho])).$$

Recall that we assume in this section $r \leq d$ and $s_{\rho'} = n \not| d$. To get a non-trivial LJ(Π), we have seen that one needs to consider only the case $s_{\rho'} = n | rd$. Lemma 9.2 gives further reduction to the case $s_{\rho'} = n | r$ (see the calculation bellow). This is the reason why we shall assume it in the sequel.

Now we shall compute in this case

$$\mathrm{LJ}(\Pi) = \mathrm{LJ}\left(L(([\rho, \nu^{d-1}\rho], [\nu\rho, \nu^{d}\rho], \dots, [\nu^{r-1}\rho, \nu^{r-1+d-1}\rho]))\right)$$
$$= \sum_{w\in W'_r} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \mathrm{LJ}\left(\delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho])\right)$$
$$= \sum_{w\in X_r(n,d)} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \mathrm{LJ}\left(\delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho])\right)$$

$$= \sum_{w \in w_{(n,d)} W_r'(n)} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \mathrm{LJ}\left(\delta([\nu^i\rho, \nu^{w(i)+(d-1)}\rho])\right)$$

$$= (-1)^{\mathrm{sgn}(w_{(n,d)})} \sum_{w \in W_r'(n)} (-1)^{\mathrm{sgn}(w)} \prod_{i=0}^{r-1} \mathrm{LJ}\left(\delta([\nu^i\rho, \nu^{w_{(n,d)}w(i)+(d-1)}\rho])\right)$$

$$= (-1)^{a(n-d)d} \sum_{w \in W_r'(n)} (-1)^{\mathrm{sgn}(w)} \prod_{\ell=0}^{n-1} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{w_{(n,d)}w(\ell+nj)+(d-1)}\rho])\right)\right)$$

$$= (-1)^{a(n-d)d} \sum_{w_0' \in W_r'(n;0),\dots,w_{n-1}' \in W_r'(n;n-1)} \prod_{\ell=0}^{n-1} (-1)^{\mathrm{sgn}(w_\ell')} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{w_{(n,d)}w_\ell'(\ell+nj)+(d-1)}\rho])\right)\right)$$

$$= (-1)^{a(n-d)d} \sum_{w_0,\dots,w_{n-1} \in W_a'} \prod_{\ell=0}^{n-1} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{w_{(n,d)}(\ell+nw_\ell(j))+(d-1)}\rho])\right)\right)$$

$$= (-1)^{a(n-d)d} \sum_{w_0,\dots,w_{n-1} \in W_a'} \prod_{\ell=0}^{d'-1} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{w_{(n,d)}(\ell+nw_\ell(j))+(d-1)}\rho])\right)\right)$$
$$\times \prod_{\ell=d'}^{n-1} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{w_{(n,d)}(\ell+nw_\ell(j))+(d-1)}\rho])\right)\right)$$

$$= (-1)^{a(n-d)d} \sum_{w_0,\dots,w_{n-1} \in W_a'} \prod_{\ell=0}^{d'-1} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{\ell+nw_\ell(j)+(n-d')+(d-1)}\rho])\right)\right)$$
$$\times \prod_{\ell=d'}^{n-1} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{\ell+nw_\ell(j)-d'+(d-1)}\rho])\right)\right)$$

$$= (-1)^{a(n-d)d} \prod_{\ell=0}^{d'-1} \sum_{w_\ell \in W_a'} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{\ell+nw_\ell(j)+(n-d')+(d-1)}\rho]) \right) \right)$$
$$\times \prod_{\ell=d'}^{n-1} \sum_{w_\ell \in W_a'} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \mathrm{LJ}\left(\delta([\nu^{\ell+nj}\rho, \nu^{\ell+nw_\ell(j)-d'+(d-1)}\rho]) \right) \right)$$

(we assume in the sequel that $\delta([\rho, \nu^{n-1}\rho])$ corresponds to ρ' under the Jacquet-Langlands correspondence as in 6.2)

$$= (-1)^{a(n-d)d} \prod_{\ell=0}^{d'-1} \sum_{w_\ell \in W_a'} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \delta([\nu^{\ell+nj}\rho', \nu^{\ell+nj}\nu_{\rho'}^{w_\ell(j)-j+c}\rho']) \right)$$
$$\times \prod_{\ell=d'}^{n-1} \sum_{w_\ell \in W_a'} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \delta([\nu^{\ell+nj}\rho', \nu^{\ell+nj}\nu_{\rho'}^{w_\ell(j)-j+c-1}\rho']) \right)$$

$$= (-1)^{a(n-d)d} \prod_{\ell=0}^{d'-1} \nu_{\rho'}^{\frac{\ell}{n}} \sum_{w_\ell \in W_a'} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \delta([\nu_{\rho'}^{j}\rho', \nu_{\rho'}^{w_\ell(j)+c}\rho']) \right)$$
$$\times \prod_{\ell=d'}^{n-1} \nu_{\rho'}^{\frac{\ell}{n}} \sum_{w_\ell \in W_a'} (-1)^{\mathrm{sgn}(w_\ell)} \left(\prod_{j=0}^{a-1} \delta([\nu_{\rho'}^{j}\rho', \nu_{\rho'}^{w_\ell(j)+c-1}\rho']) \right)$$

$$= (-1)^{a(n-d)d} \prod_{\ell=0}^{d'-1} \nu_{\rho'}^{\frac{\ell}{n}} L(([\rho', \nu_{\rho'}^{c}\rho'], \dots, [\nu_{\rho'}^{a-1}\rho', \nu_{\rho'}^{a-1+c}\rho']))$$
$$\times \prod_{\ell=d'}^{n-1} \nu_{\rho'}^{\frac{\ell}{n}} L(([\rho', \nu_{\rho'}^{c-1}\rho'], \dots, [\nu_{\rho'}^{a-1}\rho', \nu_{\rho'}^{a-1+c-1}\rho']))$$

since $a \leq c$, and then clearly $a < c+1$. (Note that we assume $r \leq d$, i.e. $an \leq cn + d'$, which implies $an < cn$, i.e. $a < c$; therefore in particular $a \leq c$.)

We have proved

$$\mathrm{LJ}(\Pi) = \mathrm{LJ}\left(L(([\rho, \nu^{d-1}\rho], \dots, [\nu^{r-1}\rho, \nu^{r-1+d-1}\rho]))\right)$$
$$= (-1)^{a(n-d)d} \left(\prod_{\ell=0}^{d'-1} \nu_{\rho'}^{\frac{\ell}{n}} L(([\rho', \nu_{\rho'}^{c}\rho'], \dots, [\nu_{\rho'}^{a-1}\rho', \nu_{\rho'}^{a-1+c}\rho'])) \right)$$
$$\times \left(\prod_{\ell=d'}^{n-1} \nu_{\rho'}^{\frac{\ell}{n}} L(([\rho', \nu_{\rho'}^{c-1}\rho'], \dots, [\nu_{\rho'}^{a-1}\rho', \nu_{\rho'}^{a-1+c-1}\rho'])) \right).$$

9.5 Suppose now that ρ is unitary and that ρ'' corresponds to $\delta(\rho, s_{\rho'}) = \delta(\rho, n)$ under the Jacquet-Langlands correspondence (as at the beginning of 7.3). Then ρ'' is unitary and $\rho'' = \mathrm{LJ}(\delta(\rho, n)) = \mathrm{LJ}(\nu^{-\frac{n-1}{2}}\delta([\rho, \nu^{n-1}\rho])) = \nu_{\rho'}^{-\frac{n-1}{2n}}\rho'$, i.e.

$$\rho'' = \nu_{\rho'}^{-\frac{n-1}{2n}}\rho', \quad \rho' = \nu_{\rho'}^{\frac{n-1}{2n}}\rho'', \quad \nu_{\rho'} = \nu_{\rho''}.$$

Now (for $n \not| d$)

$$\mathrm{LJ}(L(a(r,d)^{(\rho)})) = \mathrm{LJ}(\nu^{-\frac{r+d}{2}+1}L(([\rho, \nu^{d-1}\rho], \dots, [\nu^{r-1}\rho, \nu^{r-1+d-1}\rho]))$$

$$= (-1)^{a(n-d)d}\nu^{-\frac{r+d}{2}+1}\Bigg[\prod_{\ell=0}^{d'-1}\nu_{\rho'}^{\frac{\ell}{n}}L(([\rho', \nu_{\rho'}^{c}\rho'], \dots, [\nu_{\rho'}^{a-1}\rho', \nu_{\rho'}^{a-1+c}\rho']))$$

$$\times \prod_{\ell=d'}^{n-1}\nu_{\rho'}^{\frac{\ell}{n}}L(([\rho', \nu_{\rho'}^{c-1}\rho'], \dots, [\nu_{\rho'}^{a-1}\rho', \nu_{\rho'}^{a-1+c-1}\rho']))\Bigg].$$

$$= (-1)^{a(n-d)d}\nu_{\rho'}^{-\frac{a+c+\frac{d'}{n}}{2}+\frac{1}{n}}\Bigg[\Bigg(\prod_{\ell=0}^{d'-1}\nu_{\rho'}^{\frac{\ell}{n}}\nu_{\rho'}^{\frac{a+c+1}{2}-1}L(a(a,c+1)^{(\rho')})\Bigg)$$

$$\times\Bigg(\prod_{\ell=d'}^{n-1}\nu_{\rho'}^{\frac{\ell}{n}}\nu_{\rho'}^{\frac{a+c}{2}-1}L(a(a,c)^{(\rho')})\Bigg)\Bigg].$$

$$= (-1)^{a(n-d)d}\Bigg(\prod_{\ell=0}^{d'-1}\nu_{\rho'}^{\frac{2\ell-n-d'+2}{2n}}L(a(a,c+1)^{(\rho')})\Bigg)$$

$$\times\Bigg(\prod_{\ell=d'}^{n-1}\nu_{\rho'}^{\frac{2\ell-2n-d'+2}{2n}}L(a(a,c)^{(\rho')})\Bigg).$$

$$= (-1)^{a(n-d)d}\Bigg(\prod_{\ell=0}^{d'-1}\nu_{\rho'}^{\frac{2\ell-n-d'+2}{2n}}L(a(a,c+1)^{(\nu_{\rho'}^{\frac{n-1}{2n}}\rho'')})\Bigg)$$

$$\times\Bigg(\prod_{\ell=d'}^{n-1}\nu_{\rho'}^{\frac{2\ell-2n-d'+2}{2n}}L(a(a,c)^{(\nu_{\rho'}^{\frac{n-1}{2n}}\rho'')})\Bigg).$$

$$= (-1)^{a(n-d)d}\Bigg(\prod_{\ell=0}^{d'-1}\nu_{\rho''}^{\frac{2\ell-d'+1}{2n}}L(a(a,c+1)^{(\rho'')})\Bigg)$$

$$\times\Bigg(\prod_{\ell=d'}^{n-1}\nu_{\rho''}^{\frac{2\ell-n-d'+1}{2n}}L(a(a,c)^{(\rho'')})\Bigg).$$

$$= (-1)^{a(n-d)d}\Big(\prod_{\ell'=-\frac{d'-1}{2}}^{\frac{d'-1}{2}} \nu_{\rho''}^{\frac{\ell}{n}} L(a(a,c+1)^{(\rho'')})\Big)$$
$$\times\Big(\prod_{\ell=d'}^{n-1} \nu_{\rho''}^{\frac{\frac{-n-d'+1}{2}+\ell}{n}} L(a(a,c)^{(\rho'')})\Big).$$

Therefore, we have proved

Proposition. *Suppose that (U0) holds. Assume* $n \not| d$, $n|r$ *and* $r \leq d$. *Write*

$$d = cn + d', \quad c, d \in \mathbb{Z}, \quad 1 \leq d' \leq n-1.$$

Then

$$\mathrm{LJ}(L(a(r,d)^{(\rho)})) = (-1)^{a(n-d)d}\mathrm{string}_{\nu_{\rho''}}(d', n, L(a(r/n, c+1)^{(\rho'')}))$$
$$\times\mathrm{string}_{\nu_{\rho''}}(n-d', n, L(a(r/n, c)^{(\rho'')}))$$
$$= (-1)^{a(n-d)d}\mathrm{string}(d', L(a(r/n, c+1)^{(\rho'')}))$$
$$\times\mathrm{string}(n-d', L(a(r/n, c)^{(\rho'')})). \quad \square$$

This can be also written as

$$= (-1)^{a(n-d)d}\mathrm{string}(d-[d/n]n, L(a(r/n, [d/n]+1)^{(\rho'')}))$$
$$\times\mathrm{string}(([d/n]+1)n-d, L(a(r/n, [d/n])^{(\rho'')})),$$

where $[x]$ denotes the greatest integer which does not exceed x.

10. Unitarity of Jacquet-Langlands correspondence of irreducible unitary representations II

10.1. Under the assumptions of the previous section, we shall first show in this section that $\mathrm{LJ}(L(a(r,d)^{(\rho)}))$ is unitary (we assume ρ to be unitary). Then we shall show that

$$\mathrm{LJ}(\nu^{\beta}(L(a(r,d)^{(\rho)})) \times \nu^{-\beta}(L(a(r,d)^{(\rho)})))$$

is unitary for $0 < \beta < 1/2$ (under the same assumptions).

For the unitarity of $\mathrm{LJ}(L(a(r,d)^{(\rho)}))$, it is enough to show

$$0 \leq (d'-1)/(2n) < 1/2$$

and

$$0 \leq (n-d'-1)/(2n) < 1/2.$$

Both obviously hold (recall $1 \leq d' \leq n-1$).

For the complementary series, we need to see that for $0 < \beta < 1/2$

$$0 \leq (d'-1)/(2n) + \beta/n < 1/2$$

and

$$0 \leq (n-d'-1)/(2n) + \beta/n < 1/2$$

(see section 9). This holds since $1 \leq d' \leq n-1$ and $\beta < 1/2$.

11. Jacquet-Langlands correspondence in the unitary case, the remaining cases

11.1. It remains to compute $\mathrm{LJ}(L(a(d,r)^{(\rho)}))$ in the case $d \geq r$. Since by Theorem 3.17 of [Ba2], LJ and ^{t} commute up to a sign, we have

$$\mathrm{LJ}(L(a(d,r)^{(\rho)})) = \mathrm{LJ}(L(a(r,d)^{(\rho)})^t) = \pm \mathrm{LJ}(L(a(r,d)^{(\rho)}))^t.$$

Therefore, we can apply our previous calculations. We have two cases.

If

$$r \leq d, \quad n|d, \quad r = an + b, \quad a, b \in \mathbb{Z}, \quad 0 \leq b \leq n-1, \quad 1 \leq n,$$

then

$$\mathrm{LJ}(L(a(d,r)^{(\rho)})) = \pm \mathrm{LJ}(L(a(r,d)^{(\rho)}))^t = \pm \mathrm{string}(b, L(a(d/n, a+1)^{(\rho'')}))$$
$$\times \mathrm{string}(\min(n,r) - b, L(a(d/n, a)^{(\rho'')})).$$

If

$$n \not| d, \quad n|r, \quad r \leq d, \quad d = cn + d', \quad c, d \in \mathbb{Z}, \quad 1 \leq d' \leq n-1,$$

then

$$\mathrm{LJ}(L(a(d,r)^{(\rho)})) = \pm \mathrm{LJ}(L(a(r,d)^{(\rho)}))^t = \pm(-1)^{a(n-d)d} \mathrm{string}(d', L(a(c+1, r/n)^{(\rho'')}))$$
$$\times \mathrm{string}(n - d', L(a(c, r/n)^{(\rho'')})).$$

Note that here unitarity is also preserved by sections 8 and 10, since the involution preserves unitarity. (We assume (U0) to hold as before.)

One can compute the exact sign in the above two formulas using Theorem 3.17 of [Ba2]. (Note that our involutions differ up to a sign from the ones used in [Ba2].)

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