AN EXERCISE ON UNITARY REPRESENTATIONS IN THE CASE OF COMPLEX CLASSICAL GROUPS

Marko Tadić

Introduction

Generic irreducible unitary representations of classical groups have been classified in [LMT]. The purpose of this paper is to present that classification in a very special case, in the case of classical complex groups. In this case the theorem and the proofs are technically very simple, but they still contain main ideas used in the general case.

Further, the same ideas show up in the proof of the exhaustion in the external approach to the classification of spherical unitary duals of classical groups.

Now we shall describe the result that we shall prove here (as we already mentioned, this is a special case of [LMT]). We shall denote by $(\mathbb{C}^\times)\hat{}$ the group of all unitary characters of $\mathbb{C}^\times$. The character $z \mapsto z\bar{z} = |z|^2$ of $\mathbb{C}^\times$ is denoted by ν. We fix a series of symplectic or the series of odd-orthogonal groups. Denote by S_q the group of rank q from that series. The minimal parabolic subgroup in S_q is denoted by P_{min} (see the second section for more details regarding notation). Then we have the following classification theorems. Before we state them, let us note that these results can be stated uniformly, as one theorem (see Theorem 5.2 of the paper). Moreover, they are special case of even a more general theorem, which addresses all the classical groups and all the local fields in the same time (see [LMT]).

Theorem ($SO(2n+1,\mathbb{C})$). *(i) Take real numbers $0 < a_1, \dots, a_m, b_1, \dots, b_l < 1/2$, and characters $\varphi_1, \dots, \varphi_l, \chi_1, \dots, \chi_r \in (\mathbb{C}^\times)\hat{}$ such that $\varphi_1, \dots, \varphi_l$ are all non-trivial (possibilities $m = 0$ or $l = 0$ or $r = 0$ are not excluded). Denote $n = m + l + r$. Then*

$$\pi = Ind_{P_{min}}^{SO(2n+1,\mathbb{C})}(\nu^{a_1} \otimes \dots \otimes \nu^{a_m} \otimes \nu^{b_1}\varphi_1 \otimes \nu^{b_1}\bar{\varphi}_1 \otimes \dots \otimes \nu^{b_l}\varphi_l \otimes \nu^{b_l}\bar{\varphi}_l \otimes \chi_1 \otimes \dots \otimes \chi_r)$$

is irreducible (and generic) representation of $SO(2n+1,\mathbb{C})$. This representation is unitarizable.

(ii) Each irreducible principal series (resp. generic) representation of $SO(2n+1,\mathbb{C})$ which is unitarizable, is equivalent to some representation π from (i).

1991 *Mathematics Subject Classification.* Primary.

The author was partly supported by Croatian Ministry of Science, Education and Sports grant # 037-0372794-2804.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Theorem ($Sp(2n,\mathbb{C})$). *(i) Take $0 < \alpha_1 \leq \cdots \leq \alpha_k \leq 1/2 < \beta_1 < \cdots < \beta_\ell < 1$, $0 < b_1, \dots, b_l < 1/2$ and characters $\varphi_1, \dots, \varphi_l, \chi_1, \dots, \chi_r \in (\mathbb{C}^\times)\hat{}$ such that $\varphi_1, \dots, \varphi_l$ are all non-trivial (possibilities $k = 0$ or $\ell = 0$ or $l = 0$ or $r = 0$ are not excluded). Denote $n = k + \ell + l + r$. Suppose that holds:*

(a) *$\alpha_i + \beta_j \neq 1$ for $1 \leq i \leq k$, $1 \leq j \leq \ell$ and $\alpha_{k-1} + \alpha_k < 1$ if $k \geq 2$;*
(b) *$card\big\{i \in \{1, 2, \dots, k\}; 1 - \alpha_i < \beta_1\big\}$ is even.*
(c) *$card\big\{i \in \{1, 2, \dots, k\}; \beta_j < 1 - \alpha_i < \beta_{j+1}\big\}$ is odd if $j \in \{1, 2, \dots, \ell - 1\}$.*

Then the representation π defined as

$$Ind_{P_{min}}^{Sp(2n,\mathbb{C})}(\nu^{\alpha_1}\otimes\dots\otimes\nu^{\alpha_k}\otimes\nu^{\beta_1}\otimes\dots\otimes\nu^{\beta_\ell}\otimes\nu^{b_1}\varphi_1\otimes\nu^{b_1}\bar{\varphi}_1\otimes\dots\otimes\nu^{b_l}\varphi_l\otimes\nu^{b_l}\bar{\varphi}_l\otimes\chi_1\otimes\dots\otimes\chi_r)$$

is irreducible (and generic) representation of $Sp(2n,\mathbb{C})$. This representation is unitarizable.

(ii) Each irreducible principal series (resp. generic) representation of $Sp(2n,\mathbb{C})$ which is unitarizable, is equivalent to some representation π from (i).

In the paper all equivalences among representations π in the theorem are explained.

Following G. Muić's suggestion, we prepared this paper, which is based on an older manuscript. We thank him for the suggestion. The referee has found a number of typos in the previous version of the paper. He also gave a number of useful suggestions, which helped a lot to improve the readability and the style of the paper . We are thankful to him for that.

At the University of Minnesota in 2003, we gave a series of three talks explaining (among others things) the material presented here. We are thankful to D. Jiang and University of Minnesota for the hospitality.

In the first section we recall the basic simple constructions of irreducible unitary representations. The notation that we shall use in this paper from representation theory of general linear groups, is introduced in the second section, while the third sections does the same for classical groups. The fourth section recalls the very old, simple and well known representation theory of complex rank one groups (essentially $SL(2,\mathbb{C})$). The fifth section recalls the classification theorem from [LMT] in the complex case. A lemma giving upper bounds for complementary series is in the sixth section. The (very short) seventh section gives the proof of the classification theorem in the case of odd-orthogonal groups, while the eighth section brings the proof in the symplectic case.

1. Simple constructions of irreducible unitary representations

In this paper we shall deal with representations of (connected) classical complex groups. For such a group G we shall fix a maximal compact subgroup K of G. The complexified Lie algebra of G, viewed as a real Lie group, will be denoted by $\mathfrak{g}$. A $(\mathfrak{g}, K)$-module will be called simply a representation of G in this paper. Such a representation is called unitarizable (resp. Hermitian) if on the representation space there exists a positive definite (resp. non-degenerate) K-invariant Hermitian form which is skew-symmetric for the action of $\mathfrak{g}$. Contragredient (resp. Hermitian contragredient) of a representation π will be denoted by $\tilde{\pi}$ (resp. π^+). Complex conjugate will be denoted by $\bar{\pi}$.

We shall denote by $\tilde{G}$ the set of all equivalence classes of irreducible representations of G, and by $\hat{G}$ the subset of unitarizable classes. Then $\hat{G}$ is in a natural bijection with the unitary dual of G, i.e. with the set of equivalence classes of (topologically) irreducible unitary representations of G. The set $\tilde{G}$ is called nonunitary (or admissible) dual of G.

We shall list here a simple and well known constructions of irreducible unitary representations of reductive groups. Let $P = MN$ be a parabolic subgroup of G and σ a representation of M.

(UI) Unitary parabolic induction: If σ unitarizable, then parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ is unitarizable.

(UR) Unitary parabolic reduction: If σ is a Hermitian representation, such that parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ is irreducible and unitarizable, then σ is (irreducible) unitarizable representation.

(D) Deformation (or complementary series): Suppose that X is a connected set of characters of M. Suppose that each representation $\mathrm{Ind}_P^G(\chi\sigma)$ is Hermitian and irreducible for $\chi \in X$. If there exists $\chi_0 \in X$ such that $\mathrm{Ind}_P^G(\chi_0\sigma)$ is unitarizable, then all $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in X$ are unitarizable.

(ED) Ends of deformations: Let Y be a set of characters of M and X a dense subset of Y. Suppose that X satisfies the conditions of (D). Then each irreducible subquotient of each $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in Y$ is unitarizable.

2. $GL(n, \mathbb{C})$

The standard absolute value on $\mathbb{C}$ is denoted by $|\ \ |$. We shall denote by $|\ \ |_{\mathbb{C}}$ the square of the standard absolute value on $\mathbb{C}$, i.e. $|z|_{\mathbb{C}} = z\bar{z}$ (observe that the standard absolute value is without index $\mathbb{C}$) . We denote

$$\nu : \mathbb{C}^\times \to \mathbb{R}^\times, \quad z \mapsto |z|_{\mathbb{C}}.$$

In each $GL(n, \mathbb{C})$ we fix the maximal compact subgroup $K = U(n, \mathbb{C})$ consisting of all unitary matrices. Further, we fix the minimal parabolic subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of all upper triangular matrices in $\mathrm{GL}(n, \mathbb{C})$. For representations π_i of $\mathrm{GL}(n_i, \mathbb{C})$, $i = 1, 2$, denote by $\pi_1 \times \pi_2$ the representation of $\mathrm{GL}(n_1 + n_2, \mathbb{C})$ parabolically induced by $\pi_1 \otimes \pi_2$ from the standard parabolic subgroup having Levi factor naturally isomorphic to $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C})$. If π_i's have finite length, then $\pi_1 \times \pi_2$ is also of finite length. Then

(2-1) $\qquad \pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ have the same composition series

(this follows from the fact concerning parabolic induction from associate parabolic subgroups and representations). In particular:

(2-2) $\qquad$ if $\pi_1 \times \pi_2$ is irreducible, then $\pi_1 \times \pi_2 \cong \pi_2 \times \pi_1$.

A consequence of a general simple fact about induction in stages is

$$\pi_1 \times (\pi_2 \times \pi_3) \cong (\pi_1 \times \pi_2) \times \pi_3. \tag{2-3}$$

Using the determinant homomorphism, we identify characters of $GL(n, \mathbb{C})$ and $\mathbb{C}^\times$. Then for a character χ of $\mathbb{C}^\times$ holds

$$\chi(\pi_1 \times \pi_2) \cong (\chi\pi_1) \times (\chi\pi_2). \tag{2-4}$$

Clearly,

$$(\pi_1 \times \pi_2)\tilde{} \cong \tilde{\pi}_1 \times \tilde{\pi}_2. \tag{2-5}$$

3. Complex classical groups

Denote

$$J_n = \begin{bmatrix} 00 & \dots & 01 \\ 00 & \dots & 10 \\ \vdots & & \\ 10 & \dots & 0 \end{bmatrix} \in GL(n,\mathbb{C}).$$

Further, the identity matrix in $GL(n,\mathbb{C})$ is denoted by I_n. For $g \in GL(2n,\mathbb{C})$ let

$$^{\times}g = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^{t}g \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix},$$

where ${}^t g$ denotes the transposed matrix of g. Then ${}^{\times}(g_1 g_2) = {}^{\times}g_2\, {}^{\times}g_1$. Symplectic group is defined as

$$Sp(2n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C});\ {}^{\times}g\, g = I_{2n}\}.$$

We take $\mathrm{Sp}(0,\mathbb{C})$ to be the trivial group. We take formally that the (trivial) element of this group is 0×0 matrix.

By ${}^{\tau}g$ we shall denote the transposed matrix of $g \in GL(n,\mathbb{C})$ with respect to the second diagonal. Then we define odd (special) orthogonal group as

$$SO(2n+1,\mathbb{C}) = \{g \in SL(2n+1,\mathbb{C});\ {}^{\tau}g\, g = I_{2n+1}\}.$$

We shall fix one series of classical groups, either symplectic or odd orthogonal. The group of rank n will be denoted by S_n. We fix the minimal the parabolic subgroup $P_{\min}$ in S_n consisting of all upper triangular matrices in S_n. Fix maximal compact subgroup in S_n consisting of unitary matrices in S_n. Sometimes we shall write $\rtimes_{Sp}$ or $\rtimes_{SO}$ to indicate with which series of groups we are working.

Let τ be a representation of S_n and let π be a representation of $\mathrm{GL}(m,\mathbb{C})$. We denote by $M_{(n)}$ the Levi subgroup in S_{n+m} consisting of matrices

$$\begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & {}^{\tau}g^{-1} \end{bmatrix}$$

where $g \in GL(m,\mathbb{C})$ and $h \in S_n$. Denote by $\pi \rtimes \sigma$ the representation of S_{n+m} parabolically induced from $M_{(n)}P_{min}$ by $\pi \otimes \sigma$. Here $\pi \otimes \sigma$ maps the above matrix into $\pi(g) \otimes \sigma(h)$.

Suppose that π, π_1 and π_2 are admissible representations of $\mathrm{GL}(m,\mathbb{C})$, $\mathrm{GL}(m_1,\mathbb{C})$ and $\mathrm{GL}(m_2,\mathbb{C})$ respectively. Let σ be a representation of S_n. Then

(3-1) $$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma$$

and

(3-2) $$(\pi \rtimes \sigma)^{\sim} \cong \tilde{\pi} \rtimes \tilde{\sigma}$$

(this corresponds to (2-3) and (2-5) respectively). Further

(3-3) $\pi \rtimes \sigma$ and $\tilde{\pi} \rtimes \sigma$ have the same composition series.

In particular,

(3-4) if $\pi \rtimes \sigma$ is irreducible, then $\pi \rtimes \sigma \cong \tilde{\pi} \rtimes \sigma$

(this corresponds to (2-1) and (2-2) respectively).

4. Rank one groups

The trivial (one dimensional) representation of a group G will be denoted by 1_G. In the case of trivial group, this representation will be also denoted simply by 1. Recall that $|\ \ |$ denotes the standard absolute value on $\mathbb{C}$, while $|\ \ |_{\mathbb{C}}$ denotes the square of the standard absolute value, i.e. $|\ \ |_{\mathbb{C}} = |\ \ |^2$.

4.1. Characters of $\mathbb{C}^\times$: Observe that $(\mathbb{C}^\times)\hat{}$ is just the set of unitary characters of $\mathbb{C}^\times$, and $(\mathbb{C}^\times)\tilde{}$ is the set of all characters of $\mathbb{C}^\times$. Let χ be a character of $\mathbb{C}^\times$. Then we can find unique $\chi^u \in (\mathbb{C}^\times)\hat{}$ and $e(\chi) \in \mathbb{R}$ such that

$$\chi(z) = \nu^{e(\chi)}\chi^u(z) = |z|_{\mathbb{C}}^{e(\chi)}\chi^u(z).$$

This defines χ^u and $e(\chi)$.

For $x, y \in \mathbb{C}$ satisfying $x - y \in \mathbb{Z}$ and $z \in \mathbb{C}^\times$ set

$$\gamma(x,y)(z) = (z/|z|)^{x-y}|z|^{x+y}.$$

Then $\gamma(x,y)$ is a character of $\mathbb{C}^\times$ and $e(\gamma(x,y)) = 1/2\operatorname{Re}(x+y)$. Observe $\gamma(x,y)\gamma(x',y') = \gamma(x+x', y+y')$. Further

$$\gamma(x,y)\tilde{} = \gamma(-x,-y), \quad \gamma(x,y)^- = \gamma(\bar{y},\bar{x}), \quad \gamma(x,y)^+ = \gamma(-\bar{y},-\bar{x}).$$

Also $\gamma(x,x)(z) = |z|_{\mathbb{C}}^x$.

4.2. Representations of $GL(2,\mathbb{C})$: The representation $\gamma(x_1,y_1) \times \gamma(x_2,y_2)$ reduces if and only if

$$x_1 - x_2 \in \mathbb{Z} \quad \text{and} \quad (x_1 - x_2)(y_1 - y_2) > 0.$$

From this follows that if $\gamma(x_1,y_1)\times\gamma(x_2,y_2)$ reduces, then $(x_1+y_1)/2-(x_2+y_2)/2 \in (1/2)\mathbb{Z}\backslash\{0,\pm 1/2\}$. For the same reason, if $x_1 - y_1 \neq x_2 - y_2$, then reducibility of $\gamma(x_1,y_1) \times \gamma(x_2,y_2)$ implies $(x_1+y_1)/2 - (x_2+y_2)/2 \in (1/2)\mathbb{Z}\backslash\{0,\pm 1/2,\pm 1\}$.

Therefore, if $\chi_1\times\chi_2$ reduces ($\chi_1,\ \chi_2 \in (\mathbb{C}^\times)\tilde{}$), then $e(\chi_1)-e(\chi_2) \in (1/2)\mathbb{Z}\backslash\{0,\pm 1/2\}$. If additionally $\chi_1^u \neq \chi_2^u$, then $e(\chi_1) - e(\chi_2) \in (1/2)\mathbb{Z}\backslash\{0,\pm 1/2,\pm 1\}$.

Observe that if $\nu^x \times \nu^y = \gamma(x,x) \times \gamma(y,y)$ reduces for some $x, y \in \mathbb{R}$, then $x - y \in \mathbb{Z}$.

If we have reducibility of $\gamma(x_1,y_1)\times\gamma(x_2,y_2)$, then the composition series of this representation consists of

$$L(\gamma(x_1,y_1),\gamma(x_2,y_2)) \quad \text{and} \quad \gamma(x_1,y_2)\times\gamma(x_2,y_1).$$

Unitary dual of $GL(2,\mathbb{C})$ consists of the trivial representation, the unitary principal series are $\chi_1 \times \chi_2$, $\chi_1,\ \chi_2 \in (\mathbb{C}^\times)\hat{}$, and complementary series $\nu^\alpha\chi \times \nu^{-\alpha}\chi$, where $\chi \in (\mathbb{C}^\times)\hat{}$ and $0 < \alpha < 1/2$.

4.3. Representations of $SL(2,\mathbb{C})$: Restricting $\gamma(x,y) \times \gamma(0,0)$ to $SL(2,\mathbb{C})$ we get $\gamma(x,y) \rtimes_{Sp} 1$. Therefore $\gamma(x,y) \rtimes_{Sp} 1$ reduces if and only if

$$x \in \mathbb{Z} \quad \text{and} \quad xy > 0.$$

Further, if $\gamma(x,y) \rtimes_{Sp} 1$ reduces, then $(x+y)/2 \in (1/2)\mathbb{Z}\backslash\{0 \pm 1/2\}$.

In other words, if $\chi \rtimes_{Sp} 1$ reduces ($\chi \in (\mathbb{C}^\times)\tilde{}$), then $e(\chi) \in (1/2)\mathbb{Z}\backslash\{0,\pm 1/2\}$. Further, if $\nu^x \rtimes_{Sp} 1 = \gamma(x,x) \rtimes_{Sp} 1$ reduces for some $x \in \mathbb{R}$, then $x \in \mathbb{Z}$.

In the case of reducibility of $\gamma(x,y) \rtimes_{Sp} 1$, composition series consists of

$$L(\gamma(x,y),1) \quad \text{and} \quad \gamma(x,-y) \rtimes_{Sp} 1.$$

Unitary dual of $SL(2,\mathbb{C})$ consists of the trivial representation, the unitary principal series and complementary series $\nu^\alpha 1_{\mathbb{C}^\times} \rtimes_{Sp} 1$, where $0 < \alpha < 1$.

4.4. Representations of $SO(3,\mathbb{C})$: Consider the epimorphism $SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$ which comes from the adjoint action on the Lie algebra. Using this epimorphism, the representation $\gamma(x,y) \rtimes_{SO} 1$ pulls back to $\gamma(2x,2y) \rtimes_{Sp} 1$. Therefore, $\gamma(x,y) \rtimes_{SO} 1$ reduces if and only if

$$x \in (1/2)\mathbb{Z} \quad \text{and} \quad xy > 0.$$

From this follows: if $\gamma(x,y) \rtimes_{SO} 1$ reduces, then $x+y \in (1/2)\mathbb{Z}\backslash\{0,\pm 1/2\}$.

Thus, if $\chi \rtimes_{SO} 1$ reduces ($\chi \in (\mathbb{C}^\times)\tilde{}$), then $2e(\chi) \in (1/2)\mathbb{Z}\backslash\{0,\pm 1/2\}$. Also, if $\nu^x \rtimes_{SO} 1 = \gamma(x,x) \rtimes_{SO} 1$ reduces for some $x \in \mathbb{R}$, then $x \in (1/2)\mathbb{Z}$.

If we have reducibility of $\gamma(x,y) \rtimes_{SO} 1$, composition series again consists of

$$L(\gamma(x,y),1) \text{ and } \gamma(x,-y) \rtimes_{SO} 1.$$

Unitary dual of $SO(3,\mathbb{C})$ consists of the trivial representation, the unitary principal series and complementary series $\nu^\alpha 1_{\mathbb{C}^\times} \rtimes_{SO} 1$, where $0 < \alpha < 1/2$.

4.5. Observe that for $\alpha \in \mathbb{Z}$ if $\nu^\alpha 1_{\mathbb{C}^\times} \rtimes 1_{S_0}$ reduces, then it has a tempered subquotient. For $\alpha = 0$, we have irreducibility, but the whole induced representation is tempered.

5. Unitarizable irreducible principal series representations

All irreducible tempered representations of semisimple complex groups are fully induced by unitary characters. We fix a series S_n of classical groups, symplectic or odd orthogonal.

The set of equivalence classes of irreducible tempered representations of groups S_n. $n \in \mathbb{Z}_{\geq 0}$, is denoted by T. Observe that T consists of all classes $\chi_1 \times \ldots \times \chi_n \rtimes 1$, $\chi_i \in (\mathbb{C}^\times)\hat{}$, $n \in \mathbb{Z}_{\geq 0}$. If we replace some of χ_i by χ_i^{-1}, or change the order of $\chi_1, \ldots, \chi_n$ in $\chi_1 \times \ldots \times \chi_n \rtimes 1$, we get the same class. These are the only equivalence among representations $\chi_1 \times \ldots \times \chi_n \rtimes 1$.

Denote $(\mathbb{C}^\times)\tilde{}_+ = \{\chi \in (\mathbb{C}^\times)\tilde{}; e(\chi) > 0\}$. A finite multiset in $(\mathbb{C}^\times)\tilde{}_+$ is defined to be an unordered n-tuple of characters in $(\mathbb{C}^\times)\tilde{}_+$, $n \in \mathbb{Z}_{\geq 0}$. The set of all finite multisets in $(\mathbb{C}^\times)\tilde{}_+$ will be denoted by $M((\mathbb{C}^\times)\tilde{}_+)$. For $t = (d,\tau) = ((\chi_1,\ldots,\chi_n),\tau) \in M((\mathbb{C}^\times)\tilde{}_+) \times T$ take a permutation p of order n such that $e(\chi_{p(1)}) \geq e(\chi_{p(2)}) \geq \cdots \geq e(\chi_{p(n)})$. Denote

$$\lambda(t) = \chi_{p(1)} \times \chi_{p(2)} \times \ldots \times \chi_{p(n)} \rtimes \tau.$$

Then $\lambda(t)$ has a unique irreducible quotient, which will be denoted by $L(t)$. In this way we get parameterization of admissible duals of all groups S_n by the set $M((\mathbb{C}^\times)\tilde{}_+) \times T$. This is Langlands classification (of admissible duals of these groups). The representation $\lambda(t)$ is called standard module. The formula for Hermitian contragredient is simply

$$L(((\chi_1,\ldots,\chi_n),\tau))^+ \cong L(((\bar{\chi}_1,\ldots,\bar{\chi}_n),\tau)) \tag{5-1}$$

By Vogan's result [V], representation $L(t)$ is generic if and only if $\lambda(t)$ is irreducible, i.e. if and only if $\lambda(t) = L(t)$.

As we already mentioned in the introduction, we shall present here the proof of classification of irreducible unitarizable generic representations of classical groups S_n. By Vogan's result, it is the same as classifying unitarizable representations among irreducible principal series representations, since by Vogan's result, irreducible generic representations of a classical group S_n are exactly principal series representations $\chi_1 \times \ldots \times \chi_n \rtimes 1$, $\chi_i \in (\mathbb{C}^\times)\check{}$, which are irreducible (use (2-2), (2-3), (3-1), (3-4) and the following proposition). Because of this, for us is important the following

5.1. Proposition. *Let $\chi_1, \ldots, \chi_k \in (\mathbb{C}^\times)\check{}$. Then $\chi_1 \times \chi_2 \times \cdots \times \chi_k \rtimes 1$ is irreducible if and only if all the representations $\chi_i \times \chi_j$, $\chi_i \times \tilde{\chi}_j$, $1 \leq i < j \leq k$, and $\chi_i \rtimes 1$, $1 \leq i \leq k$, are irreducible.*

Proof. If some of the representations $\chi_i \times \chi_j$, $\chi_i \times \tilde{\chi}_j$ or $\chi_i \rtimes 1$ is reducible, then (3-1), (2-3), (2-1) (3-3) and (2-5) imply that $\chi_1 \times \chi_2 \times \cdots \times \chi_k \rtimes 1$ is reducible.

For the other implication, suppose that these representations are irreducible. To prove irreducibility of $\chi_1 \times \chi_2 \times \cdots \times \chi_k \rtimes 1$, using (3-1), (2-3), (2-2) and (3-4) we can easily reduce the proof to the case $e(\chi_1) \geq \cdots \geq e(\chi_k) \geq 0$. Recall that by Langlands classification, the space of intertwining operators $\chi_1 \times \chi_2 \times \cdots \times \chi_k \rtimes 1 \to \chi_1^{-1} \times \chi_2^{-1} \times \cdots \times \chi_k^{-1} \rtimes 1$ is one dimensional, and the image of non-zero intertwining is (irreducible) Langlands quotient. To be consistent with Langlands classification as we have described it above, take minimal $1 \leq i \leq k$ such that $e(\chi_i) > 0$ if such i exists, and take $i = 0$ otherwise. Denote $\tau = \chi_{i+1} \times \ldots \times \chi_k \rtimes 1$. Then the space of intertwining operators $\chi_1 \times \cdots \times \chi_i \rtimes \tau \to \chi_1^{-1} \times \cdots \times \chi_i^{-1} \rtimes \tau$ is one dimensional. But $\chi_1 \times \cdots \times \chi_i \rtimes \tau \cong \chi_1 \times \chi_2 \times \cdots \times \chi_k \rtimes 1$ and $\chi_1^{-1} \times \cdots \times \chi_i^{-1} \rtimes \tau \cong \chi_1^{-1} \times \chi_2^{-1} \times \cdots \times \chi_k^{-1} \rtimes 1$ by (2-2), (2-3), (3-1) and (3-4). Observe that (3-1), (2-3), (2-2) and (3-4) imply that $\chi_1 \times \chi_2 \times \cdots \times \chi_k \rtimes 1 \cong \chi_1^{-1} \times \chi_2^{-1} \times \cdots \times \chi_k^{-1} \rtimes 1$. This implies the irreducibility of $\chi_1 \times \chi_2 \times \cdots \times \chi_k \rtimes 1$. □

Now we have a special case of classification theorem of [LMT][1] (one can find in the introduction of the paper formulation of the theorem separately for symplectic and odd-orthogonal groups):

5.2. Theorem. *(i) Take $\varphi_1, \ldots, \varphi_l \in (\mathbb{C}^\times)\hat{}\backslash\{1_{\mathbb{C}^\times}\}$, $0 < a_1, \ldots, a_m < 1$, $0 < b_1, \ldots, b_l < 1/2$ and $\tau \in T$. Suppose that holds:*

If $\nu^{1/2} \rtimes 1$ reduces, then all $a_i < 1/2$.

If $\nu^{1/2} \rtimes 1$ does not reduce, write the numbers $a_1, \ldots, a_m$ as a non-decreasing sequence $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell$ where $\alpha_k \leq 1/2 < \beta_1$ (possibilities $k = 0$ or $\ell = 0$ are not excluded). Assume

$$\beta_1 < \beta_2 < \cdots < \beta_\ell. \tag{5-2}$$

Further assume that hold

(a) *$\alpha_i + \beta_j \neq 1$ for $1 \leq i \leq k$, $1 \leq j \leq \ell$ and $\alpha_{k-1} + \alpha_k < 1$ if $k \geq 2$;*
(b) *$card\big\{i \in \{1, 2, \ldots, k\}; 1 - \alpha_i < \beta_1\big\}$ is even.*
(c) *$card\big\{i \in \{1, 2, \ldots, k\}; \beta_j < 1 - \alpha_i < \beta_{j+1}\big\}$ is odd if $j \in \{1, 2, \ldots, \ell - 1\}$.*

[1]The classification theorem in [LMT] holds over all locally compact non-discrete fields. In that classification theorem shows up also reducibility at 0, which does not show up in the complex case

Then the representation

$$\pi = \nu^{a_1} \times \cdots \times \nu^{a_m} \times (\nu^{b_1}\varphi_1 \times \nu^{b_1}\bar{\varphi}_1) \times \ldots \times (\nu^{b_l}\varphi_l \times \nu^{b_l}\bar{\varphi}_l) \rtimes \tau \tag{5-3}$$

is irreducible (and generic) representation of some S_n. This representation is unitarizable.

(ii) Each irreducible principal series (resp. generic) representation of S_q which is unitarizable, is equivalent to some representation π from (i).

Recall that τ is equivalent to $\chi_1 \times \ldots \times \chi_n \rtimes 1$ for some $\chi_i \in (\mathbb{C}^\times)\hat{}$.

6. Lemma

Observe that for $\chi \in (\mathbb{C}^\times)\hat{}$, $\chi = \tilde{\chi}$ if and only if $\chi = 1_{\mathbb{C}^\times}$.

6.1. Lemma. *Suppose that $\pi = \mu_1 \times \ldots \times \mu_s \rtimes 1$, $\mu_i \in (\mathbb{C}^\times)\tilde{}$, is an irreducible unitarizable representation. Let $1 \leq i \leq s$. Then:*

1. *If $\mu_i^u \neq 1_{\mathbb{C}^\times}$, then $|e(\mu_i)|_{\mathbb{C}} < 1/2$.*
2. *If $\mu_i^u = 1_{\mathbb{C}^\times}$, then $\mu_i \rtimes 1$ is unitarizable.*

Proof. Observe that by Proposition 5.1, all $\mu_i \times \mu_j$, $\mu_i \times \tilde{\mu}_j$ and $\mu_i \rtimes 1$ are irreducible.

Using relations of the second and third section, we reduce the lemma to the case when all $e(\mu_j) \geq 0$. Further, we need to consider only the case $e(\mu_i) > 0$.

(1) Suppose $\mu_i^u \neq \tilde{\mu}_i^u$ (i.e $\mu_i^u \neq 1_{\mathbb{C}^\times}$). Relations of the second and third section imply that after renumeration we can assume that $i = 1$. Since $\pi = \mu_1 \times \ldots \times \mu_s \rtimes 1$ is Hermitian, by (5-1) there exists $j \neq 1$ such that $\mu_j = \bar{\mu}_1$. Now using relations of sections two and three, we can take $j = 2$. This implies $\tilde{\mu}_2 = \mu_1^+$. Now the relations of the second and the third section imply

$$\pi \cong \mu_1 \times \tilde{\mu}_2 \times \mu_3 \times \ldots \times \mu_s \rtimes 1 = (\mu_1 \times \mu_1^+) \times \mu_3 \times \ldots \times \mu_s \rtimes 1$$

Since $\big(\mu_1 \times \mu_1^+\big) \otimes \big(\mu_3 \times \ldots \times \mu_s \rtimes 1\big)$ is Hermitian (and irreducible), unitary parabolic reduction (UR) implies that $\big(\mu_1 \times \mu_1^+\big) \otimes \big(\mu_3 \times \ldots \times \mu_s \rtimes 1\big)$ is unitarizable. From this directly follows that $\mu_1 \times \mu_1^+ = \nu^{e(\mu_1)}\mu_1^u \times \nu^{-e(\mu_1)}\mu_1^u$ is unitarizable. Now the description of the unitary dual of $GL(2,\mathbb{C})$ in 4.2 implies $e(\mu_1) < 1/2$. This ends the proof of (1).

(2) Suppose now $\mu_i^u = 1_{\mathbb{C}^\times}$. If $e(\mu_i) < 1/2$, then from 4.3 and 4.4 we know that $\mu_i \rtimes 1$ is unitarizable, and thus (2) holds. Therefore, it remains to consider the case $e(\mu_i) \geq 1/2$, and we shall assume this in the sequel. Relations of the second and third section imply that after renumeration we can assume that $i = 1$. Further, using reduction as in the proof of (1), we can suppose that $\mu_j^u = \tilde{\mu}_j^u$ for all j (i.e. $\mu_j^u = 1_{\mathbb{C}^\times}$). We need only to consider the case $s \geq 2$.

Using deformation (D) and the fact that the reducibility happens on a closed set (see Proposition 5.1 and 4.2 - 4.4), twisting μ_j, $j \geq 2$, by ν^{ε_j} for small enough (by real absolute value) real numbers ε_j, we can assume that $e(\mu_u) \pm e(\mu_v) \notin \mathbb{Q}$ for all $u \neq v$, and $e(\mu_u) \notin (1/2)\mathbb{Z}$ for all $u \geq 2$

We can write $\mu_1 = \gamma(a,a)$, where $a = e(\mu_1)$. Since $a \geq 1/2$, we could deform a to the case $a > 1/2$ (and not in $\mathbb{Q}$) in a way that $\mu_1 \times \ldots \times \mu_s \rtimes 1$ stays irreducible and unitarizable. Take $k \in \mathbb{Z}_{>0}$ such that

$$|a - k| < 1/2.$$

Therefore, the representation $\gamma(a-k,a-k)\times\gamma(-(a-k),-(a-k))$ is an irreducible unitarizable (complementary series) representation of $GL(2,\mathbb{C})$. Therefore

$$\text{(6-1)} \qquad \gamma(a-k,a-k)\times\gamma(-(a-k),-(a-k))\times\mu_1\times\ldots\times\mu_s\rtimes 1$$

is unitarizable. By 4.2 (and relations of the second and the third section), one subquotient of the above representation is

$$\text{(6-2)} \qquad \gamma(a-k,a)\times\gamma(a,a-k)\times\gamma(-(a-k),-(a-k))\times\mu_2\times\ldots\times\mu_s\rtimes 1$$

Therefore, this subquotient is unitarizable. Note that by the relations of the second and the third section, the representation (6-2) is isomorphic to

$$\big[\gamma(a-k,a)\times\gamma(-\bar{a},-\overline{(a-k)})\big]\times\big[\gamma(-(a-k),-(a-k))\times\mu_2\times\ldots\times\mu_s\rtimes 1\big].$$

Using parabolic reduction we conclude that both representations

$$\text{(6-3)} \quad \gamma(a-k,a)\times\gamma(-\bar{a},-\overline{(a-k)}) \text{ and } \gamma(-(a-k),-(a-k))\times\mu_2\times\ldots\times\mu_s\rtimes 1$$

are unitarizable.

From the unitary dual of $GL(2,\mathbb{C})$ we know that $|e(\gamma(a-k,a))|=|a-k/2|<1/2$.

Suppose that $e(\mu_1)\geq 1$. First, we can deform it to $e(\mu_1)=a>1$ so that π stays irreducible and unitarizable. Now from the inequalities $|a-k|,|a-k/2|<1/2$ we get $k/2=|(k-a)+(a-k/2)|<1$. This implies $k=1$. From this and $|a-k/2|<1/2$ we get $a-1/2<1/2$, which implies $a<1$. This contradiction shows that $|e(\mu_1)|<1$. This implies that (2) holds for symplectic groups, since complementary series for $Sp(2,\mathbb{C})=SL(2,\mathbb{C})$ end at 1 (see (4.3)).

It remains to prove (2) for odd-orthogonal groups. Recall that we have started the proof with assumption $e(\mu_1)\geq 1/2$ and got that $\gamma(-(a-k),-(a-k))\times\mu_2\times\ldots\times\mu_s\rtimes 1$ is unitarizable (see (6-3)), where $|a-k|<1/2$. Repeating this reductions and using relations of the second and third section, we can suppose that that $\mu_1\times\ldots\times\mu_s\rtimes 1$ is unitarizable, where $e(\mu_1)>1/2$ and $0<e(\mu_j)<1/2$ for $j\geq 2$.

Denote $b=|a-k|$. Suppose $e(\mu_2)<b$. Then $\nu^\alpha\times\mu_1\times\mu_3\times\ldots\times\mu_s\rtimes 1$, $0\leq\alpha\leq e(\mu_2)$ is a continuous family of irreducible Hermitian representations (use Proposition 5.1 and 4.2 - 4.4 to see this), and for $\alpha=e(\mu_2)$ we have unitarizability. Therefore, we have unitarizability for $\alpha=0$. Now using unitary parabolic reduction (UR) we get that $\mu_1\times\mu_3\times\ldots\times\mu_s\rtimes 1$ is unitarizable.

Consider now the case $e(\mu_2)>b$. Then $\mu_1\times\mu_3\times\ldots\times\mu_s\times\nu^\alpha\rtimes 1$, $e(\mu_2)\leq\alpha<1/2$ is a continuous family of irreducible Hermitian representations (again use Proposition 5.1, 4.2 and 4.4 to see this), and for $\alpha=e(\mu_2)$ we have unitarizability. So, all the representations in the family are unitarizable. For $\alpha=1/2$, one irreducible subquotient is $\mu_1\times\mu_3\times\ldots\times\mu_s\times\mu\rtimes 1$ for some unitary character μ of $\mathbb{C}^\times$ (to be precise, for $\mu=\gamma(1/2,-1/2)$). Using (ED) we conclude that the last representation is unitarizable. Now using using the relations of the second and third section we get $\mu_1\times\mu_3\times\ldots\times\mu_s\times\mu\rtimes 1\cong\mu\times\mu_1\times\mu_3\times\ldots\times\mu_s\rtimes 1$. Using parabolic reduction (R) we get that $\mu_1\times\mu_3\times\ldots\times\mu_s\rtimes 1$ is unitarizable.

Applying above reductions $s-1$ times, we get (2) for odd-orthogonal groups. $\square$

7. Proof of Theorem 5.2 for odd-orthogonal groups

Observe that in this case $\nu^{1/2} \rtimes 1$ reduces.

Proof. Irreducibility follows directly from Proposition 5.1 and rank one reducibility 4-2 - 4.4.

To prove the unitarizability of π, consider the family

$$\nu^{x_1} \times \cdots \times \nu^{x_m} \times (\nu^{y_1}\varphi_1 \times \nu^{y_1}\bar{\varphi}_1) \times \ldots \times (\nu^{y_l}\varphi_l \times \nu^{y_l}\bar{\varphi}_l) \rtimes \tau$$

where $0 \leq x_i \leq a_i$ and $0 \leq y_j \leq b_j$. Now Proposition 5.1, 4.2 and 4.4 imply directly that this is a continuous family of irreducible Hermitian representations, which contains π. Since this family contains unitarizable representation (for all $x_i = y_j = 0$), by (D) all these representations are unitarizable. Thus, π is unitarizable.

Recall that by Vogan's result [V], irreducible generic representations of a classical group S_n are exactly principal series representations $\chi_1 \times \ldots \times \chi_n \rtimes 1$, $\chi_i \in (\mathbb{C}^\times)\tilde{}$, which are irreducible. Using this, exhaustion follows now directly from the last lemma. $\square$

8. Proof of Theorem 5.2 for symplectic groups

Proof. Irreducibility of the representations π from the theorem follows from Proposition 5.1, conditions on exponents a_i and b_j in (i) ($0 < a_i < 1$, $0 < b_j < 1/2$ and (a)), and rank one reducibility facts 4.2, 4.3.

Now we shall prove unitarizability of the representations π in the theorem by induction on $m+2l$. If $m+2l = 1$, then $m = 1$ and $l = 0$. In this case we obviously have unitarizability (see 4.3). Therefore, it remains to consider the case $m+2l \geq 2$.

First suppose that $l \geq 1$. Then $\pi \cong (\nu^{b_1}\varphi_1 \times \nu^{b_1}\bar{\varphi}_1) \rtimes \pi' \cong (\nu^{b_1}\varphi_1 \times \nu^{-b_1}\varphi_1) \rtimes \pi'$, where π' satisfy conditions of (i) of the theorem. Since π' is unitarizable by the inductive assumption, and $0 < b_1 < 1/2$, 4.2 implies that π is unitarizable.

Therefore, we need to consider the case when $l = 0$ and $m = k + \ell \geq 2$. Suppose first that $\ell = 0$. Then we get unitarizability of π in the same way as in the case of odd-orthogonal groups (by deformations).

Therefore, we need to consider the case $\ell \geq 1$ and $m = k + \ell \geq 2$. Condition (c) implies that for all $j = 1, 2, \ldots, \ell - 1$, between β_j and β_{j+1} is at least one $1 - \alpha_i$. From this follows $\ell - 1 \leq k$.

Suppose that the cardinality in (b) is strictly positive. Recall that it is even. Therefore $1 - \alpha_{k-1}, 1 - \alpha_k < \beta_1$. Remove from π the factors corresponding to α_k and α_{k-1}, and denote the obtained representation by π'. Recall that $l = 0$. Consider the family

$$\nu^x \times \nu^{\alpha_{k-1}} \rtimes \pi', \quad \alpha_{k-1} \leq x \leq \alpha_k.$$

This family contains π. Further, this is a continuous family of irreducible Hermitian representations. This follows from Proposition 5.1, 4.2 and 4.3. For example, by (4-2) reducibility could happen if $x + \beta_j = 1$, which imply $\alpha_{k-1} \leq 1 - \beta_j \leq \alpha_k$, and in particular $\beta_j \leq 1 - \alpha_{k-1}$. This implies $\beta_1 < 1 - \alpha_{k-1}$ which is impossible (since we have obtained above that $1 - \alpha_{k-1} < \beta_1$). Other conditions in Proposition 5.1 are obvious (they follow from (a), $0 < \alpha_i \leq 1/2$, $1/2 < \beta_j < 1$ and 4.2, 4.3).

Now π' is unitarizable by inductive assumption. Therefore, $\nu^{\alpha_{k-1}} \times \nu^{\alpha_{k-1}} \rtimes \pi' \cong (\nu^{-\alpha_{k-1}} \times \nu^{\alpha_{k-1}}) \rtimes \pi'$ is unitarizable. So the whole family consists of unitarizable representations. Therefore, π is unitarizable.

It remains to consider the case when the set in (b) is empty, i.e. $\beta_1 < 1 - \alpha_k$ (this implies also $a_k < 1/2$ since $1/2 < \beta_1$). Similarly as above, remove from π the factors corresponding to α_k and β_1, and denote the obtained representation by π'. Consider the family of representations

$$\nu^{\alpha_k} \times \nu^x \rtimes \pi', \quad \alpha_k \leq x \leq \beta_1$$

containing π. From $\beta_1 < 1 - \alpha_k$ and Proposition 5.1, 4.2 and 4.3 one gets easily that this is irreducible family. For example if $x + \alpha_i = 1$, then $1 - \alpha_i \leq \beta_1$, and so $1 - \alpha_k \leq \beta_1$ which is impossible. If $x + \beta_j = 1$ for $j \geq 2$, then $\alpha_k \leq 1 - \beta_j$. Note that by (c) there exists some α_i satisfying $\beta_1 < 1 - \alpha_i < \beta_j$. All this implies $\alpha_k \leq 1 - \beta_j < \alpha_i$, which is again impossible since α_k is maximal among α_i's. Other conditions for applying Proposition 5.1 are obviously satisfied.

So we have continuous family of irreducible Hermitian representations. Now in the same way as in the previous case unitarizability follows.

Now we shall prove the exhaustion in (ii) of the theorem. It remains to see that each irreducible generic unitarizable representations π satisfies conditions in (i) of Theorem 5.1. By Vogan's result [V], each irreducible generic representations is irreducible standard module (in our case this is irreducible principal series), so we can write π as

$$\pi \cong \nu^{c_1}\chi_1 \times \nu^{c_2}\chi_2 \times \cdots \times \nu^{c_s}\chi_m \rtimes \tau.$$

where $\chi_i \in (\mathbb{C}^\times)\hat{}$, $c_i > 0$ and $\tau \in T$. Since π is Hermitian, formula (5-1) implies that

$$\pi \cong \nu^{a_1} \times \cdots \times \nu^{a_m} \times (\nu^{b_1}\varphi_1 \times \nu^{b_1}\bar{\varphi}_1) \times \ldots \times (\nu^{b_l}\varphi_l \times \nu^{b_l}\bar{\varphi}_l) \rtimes \tau$$

for some $\varphi_i \in (\mathbb{C}^\times)\hat{}\backslash\{1_{\mathbb{C}^\times}\}$, $a_i, b_j > 0$ and $\tau \in T$. Now (2) of above lemma implies that $\nu^{a_i} \rtimes 1$ is unitarizable. Now 4.3 implies $a_i < 1$. Further, (1) of the same lemma implies $b_j < 1/2$. Since $0 < a_i < 1$, we can write exponents $a_1, \ldots, a_m$ as a sequence

$$0 < \alpha_1 \leq \cdots \leq \alpha_k \leq 1/2 < \beta_1 \leq \cdots \leq \beta_\ell < 1$$

Proposition 5.1 and 4.2 (together with relations of the second and third sections) imply that (a) holds, since π is irreducible.

Further, using unitary parabolic reduction we get that $\nu^{a_1} \times \cdots \times \nu^{a_m} \rtimes 1$ is irreducible and unitarizable. Denote this representation again by π.

Consider the case $\ell \geq 2$. Suppose that the cardinality in (c) is ≥ 2 for some j. Take $\alpha_u \leq \alpha_v$ from that set ($u \neq v$). Instead of ν^{α_v} in π put ν^x with $\alpha_u \leq x \leq \alpha_v$ and denote this representation by π_x. Since α_u and α_v belong to the same set in (c), representations π_x are irreducible. Therefore $\pi_x, \alpha_u \leq x \leq \alpha_v$ is a continuous family of irreducible Hermitian representations. Now (D) implies that π_{α_u} is unitarizable. Now $\pi_{\alpha_u} \cong (\nu^{\alpha_u} \times \nu^{-\alpha_u}) \rtimes \pi'$ (we get π' from π by removing ν^{α_u} and ν^{α_v}). Using parabolic reduction we get that π' is unitarizable.

In this way we have got a unitary representation with cardinality in (c) decreased for two. Continuing this procedure, we can suppose that this cardinality is 0 or 1. Suppose that it is 0. Then in the same way as above, where we have deformed α_v

to α_u , we can deform irreducibly β_{j+1} to β_j in π (it is deformation by irreducible representations since between β_j and β_{j+1} there is no $(1-\alpha_i)$'s), and get a representation π'', which is unitarizable by (D). Applying unitary parabolic reduction to π'', we would get that $\nu^{\beta_j}\times\nu^{-\beta_j}$ is unitarizable. This is impossible since $\beta_j > 1/2$ (see 4.2). Therefore, cardinalities in (c) are odd. Note that with odd cardinalities in (c), we have also obtained strict inequalities between β_j's.

Suppose that the cardinality in (b) is ≥ 2. Let α_u, α_v $(u \neq v)$ be in that set. Now we can deform α_v to α_u in π (it is deformation by irreducible representations since α_u and α_v both belong to the set in (b)). Again using unitary parabolic reduction, we get a representation which has the cardinality in (b) decreased for two. Continuing this procedure, we can come to the case when this cardinality is 0 or 1. To finish the proof, we need to show that this number is not 1.

Let the cardinality in (b) be 1. Then $1-\alpha_k < \beta_1$. Further by our assumption $1-\alpha_i > \beta_1$ for all $i \leq k-1$. Let $\alpha_k \leq y \leq \beta_1$. Denote by π_y the representation that we get if we put ν^y instead of ν^{α_k} in π. Applying Proposition 5,1 we shall check that π_y is irreducible. Suppose $y+\alpha_i = 1$ for some $i \leq k-1$. Then $1-\alpha_i \leq \beta_1$, which is impossible. Suppose $y+\beta_j = 1$ for some j. Then $\alpha_k \leq 1-\beta_j$, which implies $\beta_1 \leq \beta_j \leq 1-\alpha_k$. This contradicts to our assumption $1-\alpha_k < \beta_1$. From this we conclude irreducibility of π_y. Thus, we have continuous family of irreducible Hermitian representations, which contains π. So π_{β_1} is unitarizable. Now $\pi_{\beta_1} \cong \nu^{\beta_1}\times\nu^{-\beta_1}\rtimes\pi'$ where π' is Hermitian. Using unitary parabolic reduction we conclude that $\nu^{\beta_1}\times\nu^{-\beta_1}$ is unitarizable. This is impossible since $\beta_1 > 1/2$ (see 4.2).

This ends the proof of the exhaustion, and completes the proof of the theorem. □

References

[GN] Gelfand I.M. and Naimark M.A., *Unitary representations of the Lorentz group*, Izvestiya Akad. Nauk SSSR, Ser. Mat. (Russian) **11** (1947), 411-504.

[LMT] Lapid E., Muić G. and Tadić M., *On the generic unitary dual of quasi-split classical groups*, IMRN **26** (2004), 1335–1354.

[T] Tadić M., *An external approach to unitary representations*, Bulletin Amer. Math. Soc. **28** (1993), no. 2, 215-252.

[V] Vogan D.A., *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48** (1978), 75-98.

Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

E-mail address: `tadic@math.hr`