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$\mathbf{GL(n,\mathbb{C})\hat{}}$ and $\mathbf{GL(n,\mathbb{R})\hat{}}$

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In honor of Stephen Gelbart for his 60^{th} birthday.

Abstract. In this paper, which is based on the 1985 preprint [T1], we present a relatively simple classification of the unitary duals of $GL(n,F)$, $F=\mathbb{R}$ or $\mathbb{C}$. The approach is uniform in the local field F, and we hope that it will be accessible not only to specialists in the field.

Introduction

Let G be a locally compact group. The *unitary dual* $\hat{G}$ of G, i.e. the set of equivalence classes of irreducible unitary representations of G, plays a dominant role in the realm of abstract harmonic analysis. It is of fundamental interest to describe $\hat{G}$ explicitly. The notion of the unitary dual stems from classical harmonics analysis and it is therefore natural to study it in the context of Lie groups. It turns out that the best known Lie groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ (especially the latter) admit a remarkably simple description of their unitary duals. To wit, the Gelfand-Naimark series already constructed in the 1940's ([GfN]), completed by Stein's complementary series constructed in [St] in 1960's, comprise the entire unitary dual of $GL(n,\mathbb{C})$. In particular, all the irreducible unitary representations of $GL(n,\mathbb{C})$ are induced from (not necessarily unitary) one-dimensional representations. Despite of this simple description, it is by no means easy to prove this classification, or for that matter, the classification of the unitary dual of any other reductive Lie group with almost simple derived group of split rank > 1.

The goal of this paper is to present a relatively simple classification of the unitary duals of the groups $GL(n,\mathbb{C})$ and $GL(n,\mathbb{R})$. Our hope is that it will be accessible not only to specialists in the field. The prerequisites from representation theory are rather modest, and can be viewed as standard. We note that for a non-archimedean local field F the classification of the unitary dual of $GL(n,F)$ was accomplished in [T3]. The description looks superficially more complicated than that in the archimedean case. However, one can still give a uniform classification

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statement which covers all local fields (archimedean or not). One of our main goals is to present a uniform strategy of proof for the classification Theorem.

This paper is a revised and simplified version of the 1985 preprint [T1] which follows the strategy of [T2], [T3] (cf. also [T4]). The approach in [T1] relied on the irreducibility of parabolic induction of unitary representations. In [Ki] Kirillov reduced this problem to a distributional statement. However, there was a serious gap in the argument for the latter statement. Therefore the classification of [T1] could not be considered complete at that time. A short time later Vogan obtained another classification of the unitary duals of general linear groups over archimedean fields (as well as over the quaternions) using completely different methods ([V2], Theorem 6.18). In fact, even the equivalence of Vogan's description of the unitary dual to that of [T1] is not straightforward (cf. [C]).[1]

Recently, Baruch obtained a complete proof of Kirillov's claim ([Ba]). This is a difficult result which is proved using and generalizing ideas from the proof of Harish-Chandra's regularity theorem for eigendistributions on reductive Lie groups.[2] Therefore, the classification in [T1] is now complete. Although the final result is not new, we feel that it merits revisiting in light of the central role of the group $GL(n)$ among Lie groups and the importance of the result.

Let us now describe the classification in detail. By F we shall denote the field $\mathbb{R}$ or $\mathbb{C}$. The standard absolute value on $\mathbb{R}$ will be denoted by $|\ \ |_{\mathbb{R}}$, while the square of the standard absolute value on $\mathbb{C}$ will be denoted by $|\ \ |_{\mathbb{C}}$ (in both cases $|\ \ |_F$ is the modulus character of F).

By standard parabolic subgroup of $GL(n,F)$ we shall mean a parabolic subgroup which contains the subgroup of upper triangular matrices. We shall consider Levi factors of standard parabolic subgroups which contain the subgroup of diagonal matrices. Any Levi subgroup M is isomorphic to a direct product of general linear groups, say $GL(n_i,F)$, $i=1,\dots,k$.

Given irreducible representations σ_i, $i=1,\dots,k$ of $GL(n_i,F)$ we view $\sigma \cong \sigma_1\otimes\sigma_2\otimes\cdots\otimes\sigma_k$ as a representation of $M = GL(n_1,F)\times\cdots\times GL(n_k,F)$ viewed as the Levi subgroup of a unique standard parabolic subgroup P of $GL(n,F)$ with $n=n_1+\cdots+n_k$. We shall denote by $\mathrm{Ind}_P^{GL(n,F)}(\sigma)$, or simply by $\mathrm{Ind}(\sigma)$ (keeping in mind that σ determines n), the representation of $GL(n,F)$ parabolically induced by σ from P. (The induction is always normalized, i.e., it preserves unitarity.)

For any unitary character δ of $F^\times$ let $u(\delta,n)=\delta\circ\det$ viewed as a character of $GL(n,F)$. For an irreducible representation δ of $GL(2,\mathbb{R})$ which is square integrable modulo the center, and a positive integer n, the parabolically induced representation

$$(1)\qquad \mathrm{Ind}\Big(\,|\det|_F^{(n-1)/2}\delta\otimes|\det|_F^{(n-1)/2-1}\delta\otimes\cdots\cdots\otimes|\det|_F^{-(n-1)/2}\delta\,\Big)$$

has a unique irreducible quotient, which will be denoted by $u(\delta,n)$. For $0<\alpha<1/2$ denote

$$(2)\qquad \pi(u(\delta,n),\alpha)=\mathrm{Ind}\ \big(\,|\det|_F^{\alpha}\,u(\delta,n)\,\otimes\,|\det|_F^{-\alpha}\,u(\delta,n)\,\big).$$

Denote by B the set of all possible representations $u(\delta,n)$ and $\pi(u(\delta,n),\alpha)$, where δ runs over the set of all unitary characters of $F^\times$ and in addition if $F=\mathbb{R}$

[1] We have not been able to find a complete reference in the literature for the equivalence of the two classifications.

[2] This method of attack is also suggested, but not pursued, in [T4], p.247.

over equivalence classes of irreducible square integrable modulo center representations of $GL(2, \mathbb{R})$, while n runs over all positive integers and $0 < \alpha < 1/2$.

Now we can state the classification of unitary duals of groups $GL(n, F)$

Theorem.

(1) *For $\sigma_1, \ldots, \sigma_k \in B$ the parabolically induced representation*

$$\mathrm{Ind}(\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k)$$

is an irreducible unitarizable representation of a general linear group over F. Further, if p is a permutation of $\{1, \ldots, k\}$, then

$$\mathrm{Ind}(\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k) \cong \mathrm{Ind}\left(\sigma_{p(1)} \otimes \sigma_{p(2)} \otimes \cdots \otimes \sigma_{p(k)}\right).$$

(2) *Each irreducible unitarizable representation π of a general linear group over F is equivalent to a parabolically induced representation from (1). Moreover, π determines the sequence $\sigma_1, \ldots, \sigma_k$ (in B) uniquely up to a permutation.*

More generally, we can define the set B in the following uniform way for any local field. To each irreducible square integrable representation δ of $GL(m, F)$ modulo the center and a positive integer n we define $u(\delta, n)$ to be the unique irreducible quotient of (1) (if $m = 1$, then δ is a character and $u(\delta, n)(g) = \delta(\det g)$). Define $\pi(u(\delta, n), \alpha)$ by formula (2). Denote by B the collection of all possible representations $u(\delta, n)$ and $\pi(u(\delta, n), \alpha)$, when δ runs over all equivalence classes of irreducible square integrable modulo center representations of $GL(m, F)$ for all positive integers m, while n runs over all positive integers and $0 < \alpha < 1/2$ (recall that in the case that δ is an irreducible square integrable modulo center representation of $GL(m, \mathbb{C})$, then m can be only 1 and δ is a unitary character of $\mathbb{C}^\times$; if δ is an irreducible square integrable modulo center representation of $GL(m, \mathbb{R})$, then m can be only 1 or 2; if it is 1, then δ is a unitary character of $\mathbb{R}^\times$, and if it is 2, then δ is an irreducible square integrable modulo center representation of $GL(2, \mathbb{R})$). With this set B and $| \ |_F$ the modulus character of F, the above classification theorem holds for any local field.

In this formulation the Theorem reduces the classification of the unitary dual of the general linear group to that of the square-integrable representations. The latter are treated as "black box". (They are of course much more simple in the archimedean cases.)

The proof of the classification Theorem has two main steps, which correspond roughly to the claims (1) and (2) of the theorem:

(i) construction of irreducible unitary representations,

(ii) their exhaustion.

The first step is essentially the proof of unitarizability of the representations $u(\delta, n)$ which are the building blocks for the unitary dual. The unitarizability is evident if δ is a character of $F^\times$. This already covers the complex case, but in the real case we also have to consider the case where δ is an irreducible square integrable representation of $GL(2, \mathbb{R})$. In this case Speh has shown that (a unitary twist of) $u(\delta, n)$ is a local component of a representation in the discrete spectrum of $GL(2n, \mathbb{A}_{\mathbb{Q}})$, and is therefore unitarizable. Jacquet extended Speh's result to the non-archimedean case to show that $u(\delta, n)$ are unitarizable in this case as well ([J2]).

There is also an interesting inductive procedure to handle the $u(\delta, n)$'s. First, note that $\delta = u(\delta, 1)$ which is certainly unitarizable. The complementary series

$\pi(\delta, \alpha)$, $0 \leq \alpha < 1/2$ (2) give rise to unitarizable representations. All irreducible subquotients at the end (i.e. for $\alpha = 1/2$) are unitarizable (by [M]). This implies that $u(\delta, 2)$ is unitarizable. We can continue this way to show inductively the unitarizability of $u(\delta, n)$, provided that we know that Ind $(u(\delta, n) \otimes u(\delta, n-2))$ are irreducible. In fact, this was carried out in the non-archimedean case. (See sec. 7 of [T4] for more details.)

The second step of proof is the exhaustion claimed in (ii). This means that each irreducible representation which is not in our list has to be shown to be non-unitary. The problem is that it is difficult to get hold of the entire set of irreducible representations, let alone to prove their non-unitarizability. Thus, such a proof is not only hard, but it also shifts our attention from the unitary dual to the much bigger set of all irreducible representations. The main feature of our exhaustion argument given here is that it is realized almost entirely in the setting of unitary representations. The idea is to show, firstly, that for any irreducible Hermitian (and in particular, for any unitarizable) representation π of a general linear group we can find σ_i's and τ_j's in B such that

$$\text{(3)} \qquad \mathrm{Ind}(\pi \otimes \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k) \quad \text{and} \quad \mathrm{Ind}(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_l)$$

have an irreducible subquotient in common. If π is unitarizable, these representations are irreducible, and therefore equivalent. The final step is to show that in fact $\pi \cong \mathrm{Ind}(\tau_{i_1} \otimes \tau_{i_2} \otimes \cdots \otimes \tau_{i_m})$ for some $1 \leq i_1 < i_2 < \cdots < i_m \leq l$, which imply the exhaustion.

Consider the direct sum R of the Grothendieck groups of the representations of $GL(n, F)$ of finite length, over all n. Parabolic induction gives a commutative ring structure on R. In the non-archimedean case this ring was introduced earlier by Bernstein and Zelevinsky, and it plays a central role in the proof. It turns out that R is a polynomial ring (in infinitely many indeterminates). The crux of the exhaustion part is to show that $u(\delta, n)$'s are irreducible in R. This is the most technically complicated part of the proof. It means that $u(\delta, n)$ is the "opposite" of fully induced, in the sense that its character cannot be expressed as parabolic induction of the character of a virtual representation of finite length of a proper Levi subgroup.

To summarize, our approach is a mixture of analytic methods (Baruch's result, complementary series, etc.), algebraic methods (analysis of the ring structure R) and input from the theory of automorphic forms (giving the unitary structure of $u(\delta, n)$ (although the possibility of a local proof is not overruled)), which in turn uses analytic and arithmetic techniques. It has the advantage that both in the statement and in the proof, it is not too sensitive in the local field, and it does not go too deep into the internal structure of the representations. This is why we call our approach "external".

In contrast, the classification given by Vogan (both the statement and the proof) are in terms of parabolic and cohomological induction. The latter is a very important algebraic tool in representations of real groups which is related to Langlands functoriality and can be explicitly described in terms of K-types. It is specific to the archimedean case (reflecting the fact that the absolute Galois group of $\mathbb{R}$ is much simpler than that of a non-archimedean local field). This makes the approach in [V2] specific to the archimedean case, although it does give a detailed information about the building blocks of the unitary dual. For example, the unitarity of the $u(\delta, n)$ is proved by algebraic methods.

Now we shall describe briefly the contents of the paper. After introducing the notation and recalling some basic general facts about representation theory of reductive groups we introduce the ring R alluded to above in the first section. The second is the heart of the proof. We reduce the classification of unitary duals of general linear groups to five statements. These statements are proved in section 3 for $F=\mathbb{C}$ and in sections 4 and 5 for $F=\mathbb{R}$.

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1. Algebra of representations

We shall denote by F either $\mathbb{R}$ or $\mathbb{C}$. The set of non- negative (resp. positive) integers is denoted by $\mathbb{Z}_+$, (resp. $\mathbb{N}$). Set

$$G_n = GL(n,F)\ , \quad n\in\mathbb{Z}_+.$$

Let K_n be a maximal compact subgroup of G_n. We shall take K_n to be $U(n)$ if $F=\mathbb{C}$ and $K_n=O(n)$ if $F=\mathbb{R}$. The groups G_n are considered as real Lie groups. Let $\mathfrak{g}_n$ be the Lie algebra of G_n. A $(\mathfrak{g}_n,K_n)$-module will be simply called Harish-Chandra module of G_n (the modules which show up in this paper are always isomorphic to the K_n-finite vectors of some continuous representation of G_n on a Hilbert space).

The category of all Harish-Chandra modules of G_n of finite length will be denoted by $\mathcal{HC}(G_n)$. The set of all equivalence classes of irreducible Harish-Chandra modules of G_n is denoted by $\tilde G_n$. We shall identify an irreducible Harish-Chandra module with its class. The set of all unitarizable classes in $\tilde G_n$ is denoted by $\hat G_n$. Let R_n be the Grothendieck group of $\mathcal{HC}(G_n)$. The set $\tilde G_n$ will be identified with a subset of R_n in a natural way. In that case, R_n is a free $\mathbb{Z}$-module over $\tilde G_n$. We have a natural map $\pi\mapsto\pi^{\mathrm{ss}}$, $\mathcal{HC}(G_n)\longrightarrow R_n$, which we shall call semi-simplification. For π in $\mathcal{HC}(G_n)$ and $\sigma\in\tilde G_n$, denote by $n(\sigma,\pi)$ the multiplicity of σ in π. Then

$$\pi^{\mathrm{ss}}=\textstyle\sum_{\sigma\in\tilde G_n} n(\sigma,\pi)\,\sigma.$$

We say that π contains $\sigma_0\in\tilde G_n$ (resp. π contains $\sigma_0\in\tilde G_n$ with multiplicity one) if $n(\sigma_0,\pi)\geq 1$ (resp. $n(\sigma,\pi)=1$). If π contains σ_0, we shall write $\sigma_0\leq\pi$.

Let $P=MN$ be the standard parabolic subgroup of $G_{n_1+n_2}$ given by

$$P_{(n_1,n_2)}=\{(g_{ij})\in G_{n_1+n_2}\ \ ;\ \ g_{ij}=0\ \text{ if }\ i>n_1\ \text{ and }\ j\leq n_1\}.$$

Denote by $M_{(n_1,n_2)}$ the block upper triangular matrices of type (n_1,n_2). The unipotent radical of $P_{(n_1,n_2)}$ is denoted by $N_{(n_1,n_2)}$. Let σ_i be an object in $\mathcal{HC}(G_{n_i})$ for $i=1,2$. The tensor product $\sigma_1\otimes\sigma_2$ is $(\mathfrak{g}_{n_1}\times\mathfrak{g}_{n_2},K_{n_1}\times K_{n_2})$-module. Since M is naturally isomorphic to $G_{n_1}\times G_{n_2}$; $\mathfrak{g}_{n_1}\times\mathfrak{g}_{n_2}$ is considered as Lie algebra of M and $K_{n_1}\times K_{n_2}$ is considered as a maximal compact subgroup in M. We shall denote by $\sigma_1\times\sigma_2$ the Harish-Chandra module parabolically induced by $\sigma_1\otimes\sigma_2$. Proposition 4.1.12 of [V1] implies that $\sigma_1\times\sigma_2$ is of the finite length. For Harish-Chandra modules $\sigma_1,\sigma_2,\sigma_3$ of finite length we have

$$\sigma_1\times(\sigma_2\times\sigma_3)\cong(\sigma_1\times\sigma_2)\times\sigma_3 \tag{1-1}$$

a relation which is implied by induction in stages (Proposition 4.1.18 of [V1]).

Let

$$R = \underset{n\geq 0}{\oplus} R_n.$$

Then R is a graded commutative group. We shall call this grading the standard one. We shall define the structure of a graded ring on R. For it, it is enough to define a $\mathbb{Z}$-bilinear mapping $\times : R_{n_1} \times R_{n_2} \longrightarrow R_{n_1+n_2}, n_1, n_2 \in \mathbb{Z}_+$. Take $s \in R_{n_1}$ and $t \in R_{n_2}$. Then we can write $s = \sum_{\sigma\in\tilde{G}_{n_1}} a_\sigma \sigma,\ \ a_\sigma \in \mathbb{Z},\ \ t = \sum_{\tau\in\tilde{G}_{n_2}} b_\tau \tau,\ \ b_\tau \in \mathbb{Z}$, where $a_\sigma \neq 0$ and $b_\tau \neq 0$ only for finitely many σ and τ. The above expressions are unique. Now we define

$$s \times t = \sum_{\sigma\in\tilde{G}_{n_1},\, \tau\in\tilde{G}_{n_2}} s_\sigma b_\tau\ (\sigma \times \tau)^{\mathrm{ss}}.$$

Recall $(\sigma_1 \times \sigma_2)^{\mathrm{ss}} \in R_{n_1+n_2}$. In this way R becomes an (associative) graded ring with unit (associativity follows from (1-1)) [3].

For irreducible Harish-Chandra modules σ_1 and σ_2 we have the following equality in R:

$$\sigma_1 \times \sigma_2 = \sigma_2 \times \sigma_1. \tag{1-2}$$

This relation is a consequence of Proposition 4.1.20 of [V1] about induction from associated parabolic subgroups. Note that if $\sigma_1 \times \sigma_2 \in \mathrm{Irr}$, then $\sigma_1 \times \sigma_2 \cong \sigma_2 \times \sigma_1$. The relation (1-2) implies the following

Proposition 1.1. The induction functor induces on R a structure of a commutative graded $\mathbb{Z}$-algebra.

For further information about the ring R, we describe the Langlands classification of irreducible $(\mathfrak{g}_n, K_n)$-modules. Denote

$$\mathrm{Irr} = \overset{\infty}{\underset{n=0}{\cup}} \tilde{G}_n, \qquad \mathrm{Irr}^u = \overset{\infty}{\underset{n=0}{\cup}} \hat{G}_n.$$

Clearly, Irr is basis of $\mathbb{Z}$-module R.

By abuse of language, we refer to square integrable representations as those which are square integrable modulo the center. The set of all square integrable classes in $\hat{G}_n$, $n \geq 1$, is denoted by $D^u(G_n)$ (this set is non-empty only if $n = 1$ for $F = \mathbb{C}$, and if $n = 1, 2$ for $F = \mathbb{R}$). Set $D^u = \widehat{\mathbb{C}^\times}$ if $F = \mathbb{C}$, and $D^u = \widehat{\mathbb{R}^\times} \cup D^u(G_2)$ if $F = \mathbb{R}$ (i.e. $D^u = \overset{\infty}{\underset{n=1}{\cup}} D^u(G_n)$). Let $|\ |_F$ be the normalized absolute value on F. In the case of $F = \mathbb{R}$, this is the standard absolute value, while in the complex case this is the square of the standard one. Define $\nu : G_n \to \mathbb{R}$, $\nu(g) = |\det g|_F$. Let $D(G_n) = \{\nu^\alpha\pi; \alpha \in \mathbb{R}, \pi \in D^u(G_n)\}$ and denote $D = \widetilde{\mathbb{C}^\times}$ if $F = \mathbb{C}$, and $D = \widetilde{\mathbb{R}^\times} \cup D(G_2)$ if $F = \mathbb{R}$ (i.e. $D = \overset{\infty}{\underset{n=0}{\cup}} D(G_n)$). If $\delta \in D$, then $e(\delta) \in \mathbb{R}$ and $\delta^u \in D^u$ are uniquely determined by the relation $\delta = \nu^{e(\delta)}\delta^u$.

Let X be a set. A function $f : X \to \mathbb{Z}_+$ with the finite support is called a finite multiset in X. The set of all finite multisets in X denoted by $M(X)$. The set $M(X)$ is an additive semigroup in a natural way. Let $f \in M(X)$. Suppose that $\{x_1, \dots, x_n\}$ is the support of f. Then we shall write f also in the following way

$$f = (\underbrace{x_1, \dots, x_1}_{f(x_1)\text{-times}},\ \underbrace{x_2, \dots, x_2}_{f(x_2)\text{-times}},\ \dots\ , \underbrace{x_n, \dots, x_n}_{f(x_n)\text{-times}}).$$

[3] Note that we could introduce R_n as the group of virtual characters of G_n. Then the multiplication in R corresponds to parabolic induction of characters.

If $f \in M(X)$, then we write $\mathrm{card}(f) = \sum_{x\in X} f(x)$. We call $\mathrm{card}(f)$ the cardinal number of the multiset f. The number $f(x)$ will be called the multiplicity of x in f.

Let $a \in M(D)$. Choose $\delta_i \in D(G_{n_i})$, $i = 1,\dots,k$ such that $a = (\delta_1,\dots,\delta_k)$. After a renumeration, we can assume that $e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_k)$. Now the Harish-Chandra module $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ has a unique irreducible quotient, which will be denoted by $L(a)$. This quotient is independent (up to an equivalence) of a renumeration which satisfies the above condition. Denote

$$\lambda(a) = \delta_1 \times \cdots \times \delta_k \in R.$$

Then $\lambda(a)$ contains $L(a)$. The mapping $a \mapsto L(a)$, $M(D) \to \mathrm{Irr}$ is a bijection. This is a version of Langlands' classification of non- unitary duals of $GL(n)$'s.

If π is in $\mathcal{HC}(G_n)$, then $\tilde{\pi}$ denotes the contragredient of π, and $\bar{\pi}$ the complex conjugate (module) of π. We denote $\tilde{\bar{\pi}}$ by π^+, and call it a Hermitian contragredient π. If π is isomorphic to π^+, then π is called a Hermitian module.

For $a = (\delta_1,\dots,\delta_n) \in M(D)$ and $\alpha \in \mathbb{R}$ denote

$$\tilde{a} = (\tilde{\delta}_1,\dots,\tilde{\delta}_n),\quad \bar{a} = (\bar{\delta}_1,\dots,\bar{\delta}_n),\quad a^+ = (\delta_1^+,\dots,\delta_n^+),\quad \nu^\alpha a = (\nu^\alpha\delta_1,\dots,\nu^\alpha\delta_n).$$

If $\delta \in D$, then $\delta = \nu^{e(\delta)}\delta^u$, $\bar{\delta} = \nu^{e(\delta)}(\bar{\delta^u})$, $\tilde{\delta} = \nu^{-e(\delta)}(\delta^u)\tilde{}$, $\delta^+ = \nu^{-e(\delta)}\delta^u$, $\nu^\alpha\delta = \nu^{e(\delta)+\alpha}\delta^u$.

Proposition 1.2. For $a \in M(D)$ and $\alpha \in \mathbb{R}$ we have

$$\overline{L(a)} = L(\bar{a}),\quad L(a)^+ = L(a^+),\quad L(a)\tilde{} = L(\tilde{a}),\quad \nu^\alpha L(a) = L(\nu^\alpha a).$$

Proof. The first relation is obvious and it implies that the second and the third relation are equivalent. The second relation is proved in the proof of Theorem 7 in [KnZu]. The fourth relation can be proved directly by constructing intertwining operator between induced modules $\nu^\alpha\lambda(a)$ and $\lambda(\nu^\alpha a)$, which induces an equivalence between $\nu^\alpha L(a)$ and $L(\nu^\alpha a)$. $\square$

Proposition 1.3. The ring R is a $\mathbb{Z}$-polynomial ring over indeterminates D. This means that $\{\lambda(a); a \in M(D)\}$ is a $\mathbb{Z}$-basis of R.

Proof. This is a well known fact because $\lambda(a)$, $a \in M(D)$ correspond to the standard characters which form a basis of the group of all virtual characters (for a fixed reductive Lie group). In fact, the proposition can be proved easily directly using [J1], and properties of Langlands' classification. Lemmas 3.3 and 4.5 of this paper also imply the proposition. $\square$

Corollary 1.4. (i) The ring R is factorial.

(ii) If $\delta \in D$, then δ is prime.
(iii) Let $\pi \in R$ be a homogeneous non-zero element of the graded ring R. Suppose that $\pi = \sigma_1 \times \sigma_2$ for some $\sigma_1, \sigma_2 \in R$. Then σ_1 and σ_2 are homogeneous elements.
(iv) The group of invertible elements in R is $\{L(\emptyset), -L(\emptyset)\}$. Note that $L(\emptyset)$ is identity in R.

Proof. Proposition 1.3 implies (i) and (ii). Proposition 1.3 implies that R is an integral domain. This implies (iii). From (iii) we obtain (iv) directly. $\square$

Remark 1.5. The mappings $\pi \to \bar{\pi},\ \pi \to \tilde{\pi},\ \pi \to \pi^+$ and $\pi \to \nu^\alpha \pi$ induce automorphisms of graded ring R (this follows from Proposition 1.2), which we shall denote respectively $-,\ \sim,\ +,\ \nu^\alpha\ :\ R \to R$. The first three automorphisms are involutions. Each of these four automorphisms can be described by a permutation of indeterminates D.

We shall say that $f \in R$ is Hermitian if $f = f^+$.

2. Formal approach to unitary dual of general linear group

In this section F is either $\mathbb{R}$ of $\mathbb{C}$. For $\delta \in D$ and $n \in \mathbb{N}$ denote

$$a(\delta, n) = (\nu^{\frac{n-1}{2}}\delta, \nu^{\frac{n-1}{2}-1}\delta, \dots, \nu^{-\frac{n-1}{2}}\delta), \quad u(\delta, n) = L(a(\delta, n)).$$

If $n = 0$, then we take $a(\delta, 0) = \emptyset$ and $u(\delta, 0) = L(\emptyset)$. Proposition 1.2 implies $\nu^\alpha u(\delta, n) = u(\nu^\alpha \delta, n)$ for $\alpha \in \mathbb{R}$. If $\delta \in D^u$, then $u(\delta, n)^+ = u(\delta, n)$. For $\sigma \in \mathrm{Irr}$ and $\alpha \in \mathbb{R}$ denote

$$\pi(\sigma, \alpha) = (\nu^\alpha \sigma) \times (\nu^{-\alpha}\sigma^+) \in R.$$

Clearly, $\pi(\sigma, \alpha)$ is a Hermitian element of R. Note that $\pi(\sigma, \alpha) = \pi(\sigma, -\alpha)$ if σ is Hermitian.

Now we shall introduce the following statements:

(U0) If $\sigma, \tau \in \mathrm{Irr}^u$ then $\sigma \times \tau \in \mathrm{Irr}^u$.

(U1) If $\delta \in D^u$ and $n \in \mathbb{N}$, then $u(\delta, n) \in \mathrm{Irr}^u$.

(U2) If $\delta \in D^u$, $n \in \mathbb{N}$ and $0 < \alpha < 1/2$, then $\pi(u(\delta, n), \alpha) \in \mathrm{Irr}^u$.

(U3) If $\delta \in D$ and $n \in \mathbb{N}$, then $u(\delta, n)$ is a prime element of the factorial ring R.

(U4) If $a, b \in M(D)$, then $L(a) \times L(b)$ contains $L(a + b)$ as a subquotient.

Assuming (U0) – (U4) to hold, we describe the unitary duals of general linear groups. In the following sections of this paper, we shall prove these statements.

The proof of the following proposition is in the section 8. of [T4]. For the sake of completeness, we present also here a (slightly modified) proof of it.

Proposition 2.1. Suppose that (U0) - (U4) holds. Then Irr^u is a multiplicative semigroup and it is a free abelian semigroup with a basis

$$B = \{u(\delta, n),\ \pi(u(\delta, n), \alpha);\ \delta \in D^u,\ \ n \in \mathbb{N}\ \text{ and }\ 0 < \alpha < 1/2\}.$$

In other words:

(i) If $\pi_1, \dots, \pi_i \in B$, then $\pi_1 \times \pi_2 \times \cdots \times \pi_i \in \mathrm{Irr}^u$

(ii) If $\pi \in \mathrm{Irr}^u$, then there exist $\pi_1, \dots, \pi_i \in \mathrm{Irr}^u$, unique up to a permutation, such that $\pi = \pi_1 \times \cdots \times \pi_i$.

Proof. By (U0), Irr^u is a multiplicative semigroup. The statements (U1) and (U2) imply $B \subseteq \mathrm{Irr}^u$. Therefore (i) holds. If $\pi_1 \times \cdots \times \pi_i = \sigma_1 \times \cdots \times \sigma_j$ for some $\pi_1, \cdots, \pi_i, \sigma_1, \cdots, \sigma_j \in B$, then (U3) implies that $i = j$ and that the sequences $\pi_1, \dots, \pi_i$ and $\sigma, \dots, \sigma_j$ differ up to a permutation. It remains to prove the existence of presentation in (ii).

Let $\pi \in \mathrm{Irr}^u$. Choose $a \in M(D)$ such that $\pi = L(a)$. Since π is unitarizable, π is Hermitian i.e. $\pi = \pi^+$. By Proposition 1.2 we have $a = a^+$. Recall that for $\delta = \nu^{e(\delta)}\delta^u \in D$ we have $\delta^+ = \nu^{-e(\delta)}\delta^u$. Therefore we can find $\gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_m \in D^u$, and positive numbers $\alpha_1, \dots, \alpha_m$, such that we have the following equality of multisets

$$a = (\gamma_1, \dots, \gamma_m) + \sum_{i=1}^m (\nu^{\alpha_i}\delta_i, \nu^{-\alpha_i}\delta_i)$$

(cases $m = 0$ or $n = 0$ are possible). After a change of enumeration, we can assume that $\alpha_1, \dots, \alpha_u \in (1/2)\mathbb{Z}$ and $\alpha_{u+1}, \dots, \alpha_m \notin (1/2)\mathbb{Z}$ for some $0 \le u \le m$. Now introduce $\sigma_1, \dots, \sigma_v \in D^u$ and positive numbers $\beta_1, \dots, \beta_v$ such that

$$a = (\gamma_1, \dots, \gamma_m) + \sum_{i=1}^u (\nu^{\alpha_i}\delta_i, \nu^{-\alpha_i}\delta_i) + \sum_{j=1}^v (\nu^{\beta_j}\sigma_j, \nu^{-\beta_j}\sigma_j).$$

Recall that $\gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_u, \sigma_1, \dots, \sigma_v \in D^u$ and $\alpha_1, \dots, \alpha_u, \beta_1, \dots, \beta_v$ are positive numbers such that $\alpha_1, \dots, \alpha_u \in (1/2)\mathbb{Z}$ and $\beta_1, \dots, \beta_v \notin (1/2)\mathbb{Z}$ (the case of $n = 0$ or $u = 0$ or $v = 0$ is possible). Take $r_1, \dots, r_v \in \mathbb{R}$ and $m_1, \dots, m_v \in (1/2)\mathbb{Z}$ such that

$$\beta_j = r_j + m_j \text{ and } 0 < r_j < 1/2 \text{ for } j = 1, \dots, v.$$

Clearly, $m_1, \dots, m_v \ge 0$.

One gets directly that

$$\begin{aligned}
(\nu^{\alpha_i}\delta_i, \nu^{-\alpha_i}\delta_i) + a(\delta_i, 2\alpha_i - 1) &= a(\delta_i, 2\alpha_i + 1), \qquad i = 1, \dots, u, \\
(\nu^{m_j}\delta_j) + \nu^{-1/2}a(\sigma_j, 2m_j) &= a(\sigma_j, 2m_j + 1), \qquad j = 1, \dots, v.
\end{aligned}$$

The second relation implies

$$\begin{aligned}
&(\nu^{\beta_j}\sigma_j, \nu^{-\beta_j}\sigma_j) + \nu^{r_j - 1/2}a(\sigma_j, 2m_j) + \nu^{1/2 - r_j}a(\sigma_j, 2m_j) \\
&\qquad = (\nu^{r_j + m_j}\sigma_j, \nu^{-r_j - m_j}\sigma_j) + \nu^{r_j - 1/2}a(\sigma_j, 2m_j) + \nu^{1/2 - r_j}a(\sigma_j, 2m_j) \\
&\qquad = \nu^{r_j}a(\sigma_j, 2m_j + 1) + \nu^{-r_j}a(\sigma_j, 2m_j + 1) \quad \text{for} \quad j = 1, \dots, v.
\end{aligned}$$

In the rest of the proof we shall use the following property. Let $a_1, a_2, \in M(D)$. Suppose that $L(a_1), L(a_2)$ are unitarizable. Then (U0) and (U4) implies $L(a_1) \times L(a_2) = L(a_1 + a_2)$. By induction we obtain that $L(a_1) \times L(a_2) \times \dots \times L(a_k) = L(a_1 + a_2 + \dots + a_k)$ if $a_1, \dots, a_k \in M(D)$ satisfy that $L(a_1), \dots, L(a_k) \in \mathrm{Irr}^u$.

Now we shall finish the proof. We compute, using (U2), (U3) and the above property

$$
\begin{aligned}
\pi \times & \prod_{i=1}^{u} u(\delta_, 2\alpha_i - 1) \times \prod_{j=1}^{v} \pi(u(\sigma_j, 2m_j), r_j - 1/2) \\
&= L\bigg((\gamma_1, \dots, \gamma_m) + \sum_{i=1}^{u} (\nu^{\alpha_i}\delta_i, \nu^{-\alpha_i}\delta_i) + \sum_{j=1}^{v} (\nu^{\beta_j}\sigma_j, \nu^{-\beta_j}\sigma_j) \bigg) \\
&\qquad\qquad\qquad \times \prod_{i=1}^{u} u(\delta_, 2\alpha_i - 1) \times \prod_{j=1}^{v} \pi(u(\sigma_j, 2m_j), r_j - 1/2) \\
&= L\bigg((\gamma_1, \dots, \gamma_n) + \sum_{i=1}^{u} [(\nu^{\alpha_i}\delta_i, \nu^{-\alpha_i}\delta_i) + a(\delta_i, 2\alpha_i - 1)] \\
&\qquad + \sum_{j=1}^{v} [(\nu^{-\beta_j}\sigma_j, \nu^{\beta_j}\sigma_j) + \nu^{r_j - 1/2} a(\sigma_j, 2m_j) + \nu^{1/2 - r_j} a(\sigma_j, 2m_j)] \bigg) \\
&= L\bigg((\gamma_1, \dots, \gamma_n) + \sum_{i=1}^{u} a(\delta_i, 2\alpha_i + 1) \\
&\qquad\qquad + \sum_{j=1}^{v} \big(\nu^{r_j} a(\sigma_j, 2m_j + 1) + \nu^{-r_j} a(\sigma_j, 2m_j + 1) \big) \bigg) \\
&= L(\gamma_1) \times \dots \times L(\gamma_n) \times \prod_{i=1}^{u} L(a(\delta_i, 2\alpha_i + 1)) \times \prod_{j=1}^{v} \pi(L(a(\sigma_j, 2m_j + 1)), r_j) \\
&= u(\gamma_1, 1) \times \dots \times u(\gamma_n, 1) \times \prod_{i=1}^{u} u(\delta_i, 2\alpha_i + 1) \times \prod_{j=1}^{v} \pi(u(\sigma_j, 2m_j + 1), r_j).
\end{aligned}
$$

Thus, π divides

$u(\gamma_1, 1) \times \dots \times u(\gamma_n, 1) \times \prod_{i=1}^{u} u(\delta_i, 2\alpha_i + 1) \times \prod_{j=1}^{v} \pi(u(\sigma_j, 2m_j + 1), r_j)$.

Now (U3) implies that π is a product of a subfamily of the modules

$$u(\gamma_i, 1), u(\delta_j, 2\alpha_j + 1), \nu^{r_k} u(\sigma_k, 2m_k + 1), \nu^{-r_k} u(\sigma_k, 2m_k + 1),$$

$i = 1, \dots, n,\ j = 1, \dots, u,\ k = 1, \dots, v$. The fact that π is Hermitian implies that π is a product of a subfamily of the modules $u(\gamma_i, 1), u(\delta_j, 2\alpha_j + 1), \pi(u(\sigma_k, 2m_k + 1), r_k)$. Thus, we have proved the existence of an expansion of π into a product of elements of B. This concludes the proof. □

Corollary 2.2. If (U0) – (U4) hold, then the mapping $\Theta : (\pi_1, \dots, \pi_n) \mapsto \pi_1 \times \dots \times \pi_n$, $M(B) \to \mathrm{Irr}^u$, is an isomorphism of semigroups.

In the rest of this paper we shall focus our attention on the proof of (U0) - (U4), or give a reference where one can find proofs. Theorem 0.3 of [Ba] implies (Kirillov's) Conjecture 0.1 of the same paper and Theorem 2.1 of [Sa] implies (U0).

Now we prove the remaining claims (U1) - (U4). We shall consider the complex and the real case separately.

3. Complex general linear group

In the preceding section we have shown that (U0) – (U4) imply a classification of the unitary dual of $GL(n, F)$. In this section we shall assume F to be $\mathbb{C}$ and we shall see that (U1) – (U4) hold in this case (we have noticed above that (U0) holds).

First we shall recall a number of basic facts from representation theory of $GL(n,\mathbb{C})$. We shall start with $GL(1,\mathbb{C})$ and $GL(2,\mathbb{C})$. It is well known that $D=\tilde{G}_1=(\mathbb{C}^\times)\tilde{}$, $D^u=\hat{G}_1=(\mathbb{C}^\times)\hat{}$ (as we already have noticed). Let $\delta\in D^u$. This means that δ is a unitary character of $\mathbb{C}^\times$. Then $u(\delta,n)$ is just a unitary character $g\to\delta(\det g)$, $G_n\to\mathbb{C}^\times$, i.e. an one-dimensional unitarizable module of G_n. Thus, (U1) holds. Further let $0<\alpha<1/2$. The module $\pi(u(\delta,n),\alpha)$ restricted to $SL(2n,\mathbb{C})$ is irreducible and unitarizable by [St][4]. This implies first $\pi(u(\delta,n),\alpha)\in\mathrm{Irr}$. The module $\pi(u(\delta,n),\alpha)$ is Hermitian, so its central character is unitary. Therefore $\pi(u(\delta,n),\alpha)$ is unitarizable (which means that (U2) holds).

Now we shall introduce two parameterizations of D. If $\delta\in D$, then there exist a unique $n\in\mathbb{Z}$ and $\beta\in\mathbb{C}$ such that $\delta(z)=|z|^{2\beta}\left(z/|z|\right)^n=|z|_{\mathbb{C}}^{\beta}\left(z/|z|\right)^n, z\in\mathbb{C}^\times=G_1$. In this case we shall write $\delta=\delta(\beta,n)$. Here $|\ |$ denotes the usual absolute value on $\mathbb{C}$, and we have $|\ |^2=|\ |_{\mathbb{C}}$. Note that $\delta(\beta,n)$ is a unitary character if and only if the real part of β is zero. The mapping $\mathbb{C}\times\mathbb{Z}\to(\mathbb{C}^\times)\tilde{}$ is an isomorphism, which gives a parameterization of D. Further, $\delta(\beta,u)^+=\delta(-\bar{\beta},n)$ and $e(\delta(\beta,n))=\mathrm{Re}\ \beta$.

For given $\beta\in\mathbb{C}$ and $n\in\mathbb{Z}$, there exist unique $x,y\in\mathbb{C}$ such that $x+y=2\beta$ and $x-y=n$. Then we shall write $\delta(\beta,n)=\gamma(x,y)$. Therefore

$$\gamma(x,y)(z)=|z|^{x+y}\cdot(z/|z|)^{x-y}.$$

In this way we obtain another parameterization of D by the set $\{(x,y)\in\mathbb{C}^2\ ;\ x-y\in\mathbb{Z}\}$. Note that $\gamma(x_1,y_1)\gamma(x_2,y_2)=\gamma(x_1+x_2,y_1+y_2)$, $\gamma(x,y)^+=\gamma(-\bar{y},-\bar{x})$ and $e(\gamma(x,y))=(1/2)\ \mathrm{Re}(x+y)$.

We shall say that $\delta_1,\delta_2\in D$ are linked if and only if $\delta_1\times\delta_2$ is reducible. The representation theory of $GL(2,\mathbb{C})$ implies that δ_1 and δ_2 are linked if and only if there exist $p,q\in\mathbb{Z}$ such that $pq>0$ and $(\delta_1\delta_2^{-1})(z)=z^p\bar{z}^q=\gamma(p,q)(z)$ for all $z\in\mathbb{C}^\times$ (see [JL]). In this case we have the equality $\delta_1\times\delta_2=L((\delta_1,\delta_2))+\nu_1\times\nu_2$, in R where $\nu_1,\nu_2\in D$ are defined by $\nu_1(z)=(\bar{z})^{-q}\delta_1(z)$, $\nu_2(z)=(\bar{z})^q\delta_2(z)$ ([JL]) and furthermore $\nu_1\times\nu_2$ is irreducible. If δ_1,δ_2,ν_1 and ν_2 are as above, we shall write $(\nu_1,\nu_2)\prec(\delta_1,\delta_2)$.

Now we shall interpret the above results in terms of the other parameterization of the characters of $\mathbb{C}^\times$. Let $\gamma(x_i,y_i)\in D$, $i=1,2$. Then $\gamma(x_1,y_1)$ and $\gamma(x_2,y_2)$ are linked if and only if

$$x_1-x_2\in\mathbb{Z},\quad\text{and}\quad(x_1-x_2)(y_1-y_2)>0.$$

If $\gamma(x_1,y_1)$ and $\gamma(x_2,y_2)$ are linked, then

$$(\gamma(x_1,y_2),\gamma(x_2,y_1))\prec(\gamma(x_1,y_1),\gamma(x_2,y_2)).$$

Let $(\delta_1,\dots,\delta_n)\in M(D)$. Suppose that δ_i and δ_j are linked for some $1\le i<j\le n$. Choose $\nu_i,\nu_j\in D$ such that $(\nu_i,\nu_j)\prec(\delta_i,\delta_j)$. Then we shall write

$$(\delta_1,\dots,\delta_{i-1},\nu_i,\delta_{i+1},\dots,\delta_{j-1},\nu_j,\delta_{j+1},\dots,\delta_n)\prec(\delta_1,\delta_2,\delta_3,\dots,\delta_n).$$

Let $a,b\in M(D)$. Then we write $a<b$ if there exist $c_1,\dots,c_k\in M(D)$, $k\ge2$, such that $a=c_1\prec c_2\prec c_2\prec\cdots\prec c_{k-1}\prec c_k=b$ (for $a,b\in M(D)$ we write $a\le b$ if $a=b$ or $a<b$). We shall see later that $\le$ is a partial ordering on $M(D)$ (and we shall examine some properties of $\le$).

[4]More precise, the Harish-Chandra module that we use to see the unitarizability of $\pi(u(\delta,n),\alpha)$ restricted to $SL(2n,\mathbb{C})$, is the module of K-finite vectors in the complementary series constructed in [St].

Let $a \in M(D)$. We say that $a = (\delta_1, \dots, \delta_n)$ is written in a standard order if $e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_n)$ (we shall usually write elements of $M(D)$ in a standard order). If this is the case we define

$$\mathbf{e}(a) = (e(\delta_1), e(\delta_2), \dots, e(\delta_n)) \in \mathbb{R}^n.$$

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_m) \in \mathbb{R}^m$. We shall write $x \leq y$ if and only if $n = m$ and $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ holds for each $k = 1, \dots, n$. It is obvious that $\leq$ is a partial ordering on $\cup_{n \geq 0} \mathbb{R}^n$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ denote $\mathrm{Tr}(x) = x_1 + \cdots + x_n$. For $a \in M(D)$, we define $\mathrm{Tr}(a)$ to be $\mathrm{Tr}(\mathbf{e}(a))$.

We have now a simple technical lemma regarding the notation that we have just introduced.

Lemma 3.1. *(i) Let $a, b \in M(D)$ and $a \leq b$. Then the gradings of $L(a)$ and $L(b)$ in R are the same (recall that $R = \oplus_{n=0}^{\infty} R_n$ is in a natural way a graded ring, and we can view Irr as a subset of R).*

(ii) *Fix $a \in M(D)$. The set of all $b \in M(D)$ such that $a \leq b$ or $b \leq a$ is finite.*

(iii) *Suppose that $a \leq b$ for $a, b \in M(D)$. Then $\mathbf{e}(a) \leq \mathbf{e}(b)$ and $\mathrm{Tr}\ a = \mathrm{Tr}\ b$. We have $a < b$ if and only if $\mathbf{e}(a) < \mathbf{e}(b)$.*

(iv) *The relation $a \leq b$ is a partial ordering on $M(D)$.*

(v) *Let $a_i, b_i \in M(D)$, $i = 1, 2$. Suppose that $a_i \leq b_i$, $i = 1, 2$. Then $a_1 + a_2 \leq b_1 + b_2$. We have $a_1 + a_2 = b_1 + b_2$ if and only if $a_i = b_i$ for $i = 1, 2$.*

Proof. The definition of $\leq$ on $M(D)$ implies (i). Let

$$a = (\gamma(x_1, y_1), \dots, \gamma(x_n, y_n)).$$

Suppose that $b = (\gamma(x_1^*, y_1^*), \dots, (x_n^*, y_n^*)) \in M(D)$ satisfies $a \leq b$ or $b \leq a$. Then $x_i^* \in \{x_1, \dots, x_n\}$ and $y_i^* \in \{y_1, \dots, y_n\}$. This implies (ii). Let $a, b \in M(D)$ satisfy $a \prec b$. Then a simple verification gives $\mathbf{e}(a) < \mathbf{e}(b)$ and $\mathrm{Tr}(a) = \mathrm{Tr}(b)$. This implies (iii). The claim (iii) implies (iv).

Let $a_i, b_i \in M(D)$, $a_i \leq b_i$, $i = 1, 2$. Then the definition of $\leq$ on $M(D)$ implies $a_1 + a_2 \leq b_1 + b_2$. If $a_i = b_i$, $i = 1, 2$, then clearly $a_1 + a_2 = b_1 + b_2$. Suppose now $a_1 + a_2 = b_1 + b_2$. Let $a_1 < b_1$. Then there exists $c \in M(D)$ such that $a_1 \leq c \prec b_1$, and thus $a_1 + a_2 \leq c + a_2 \prec b_1 + a_2 \leq b_1 + b_2$. Therefore $a_1 + a_2 < b_1 + b_2$, which contradicts to $a_1 + a_2 = b_1 + b_2$ since $\leq$ is a partial ordering on $M(D)$. This completes the proof of (v). □

Lemma 3.2. *Let $a, b \in M(D)$. Then $\lambda(a)$ contains $L(b)$ if and only if $b \leq a$.*

Proof. Suppose $b \leq a$. We shall prove that $\lambda(a)$ contains $L(b)$ by induction with respect to the partial ordering on $M(D)$ (this is possible by (ii) of Lemma 3.1). Let c be an element in $M(D)$. Then by definition of $L(c)$, $\lambda(c)$ contains $L(c)$. Suppose that a is an minimal element of $M(D)$. Then $b = a$ and by the above remark, $\lambda(a)$ contains $L(b) = L(a)$. Let $a^* \in M(D)$ be arbitrary. We suppose that the claim $\lambda(a)$ contains $L(b)$ holds for all $a \in M(D)$ such that $a < a^*$. Let $b \in M(D)$, $b \leq a^*$. If $b = a^*$, then $\lambda(a^*)$ contains $L(b) = L(a^*)$. Thus, we need to consider only the case of $b < a^*$. By the definition of $<$, there exists $c \in M(D)$ such that $b \leq c \prec a^*$. One sees directly, using commutativity and associativity of R, that each $\pi \in \mathrm{Irr}$ which is contained in $\lambda(c)$, is also contained in $\lambda(a^*)$. Now the induction hypothesis implies that $\lambda(a^*)$ contains $L(b)$.

Suppose now that $\lambda(a)$ contains $L(b)$. Applying Example 3.16 of [SpV] (or Corollary 3.15 of of the same paper), we know that either $L(a) = L(b)$ i.e. $a = b$, or $L(b)$ is in the kernel of some factor of the long intertwining operator. In our situation it means that either $L(a) = L(b)$ i.e. $a = b$ or there exist $c \prec a$ such that $\lambda(c)$ contains $L(b)$, because the kernels of the factors of the long intertwining operator attached to $\lambda(a)$ have form $\lambda(c)$ for $c \prec a$ (see Lemma 3.8 of [SpV]). Now we get $b \leq a$ using induction on a, with respect to the ordering of $M(D)$. □

LEMMA 3.3. *Fix $a \in M(D)$.*

(i) *There exist $m_b^a \in \mathbb{N}$ for $b \leq a$, such that $\lambda(a) = \sum_{b \leq a} m_b^a L(b)$ holds in R. Further, $m_a^a = 1$.*
(ii) *There exist $m_{(a,b)} \in \mathbb{Z}$ for $b \leq a$, such that $L(a) = \sum_{b\leq a} m_{(a,b)}\lambda(b)$. We have $m_{(a,a)} = 1$.*
(iii) *Suppose that $c \in M(D)$ satisfies $c < a$. Let c be adjacent to a, i.e. there does not exist $b \in M(D)$ such that $c < b < a$. Then $m_{(a,c)} \neq 0$.*
(iv) *Let $c \in M(D)$ satisfies $c < a$. Suppose that for $d \in M(D)$ such that $d \prec a$ we have $\mathbf{e}(c) \not< \mathbf{e}(d)$. Then a is adjacent to c (and $m_{(a,c)} \neq 0$).*

PROOF. Lemma 3.2 and the fact that $L(a)$ has multiplicity one in $\lambda(a)$, imply (i).

We shall show (ii) and (iii) simultaneously by induction on $a \in M(D)$. If $a \in M(D)$ is minimal, then $L(a) = \lambda(a)$ by (i). Therefore (ii) holds. Note that there does not exist c such that $c < a$. Therefore (iii) also holds. Let $a \in M(D)$ be arbitrary. From (i) and the induction hypothesis we have

$$L(a) = \lambda(a) - \sum_{b<a} m_b^a L(b) = \lambda(a) - \sum_{b<a} m_b^a \left(\lambda(b) + \sum_{d<b} m_{(b,d)}\lambda(d)\right). \tag{3-1}$$

Gathering the terms in the above expansion for $L(a)$, we obtain (ii). Suppose that $m_{(a,b)} = 0$. Since $m_b^a > 0$ for all $b < a$, (3-1) implies that there exist $b' \in M(D)$ such that $b < b' < a$. This proves (iii).

Suppose $c \in M(D)$ satisfies (iv). Assume that c is not adjacent to a. Then there exist $b, d \in M(D)$ such that $c < b \leq d \prec a$. Then $\mathbf{e}(c) < \mathbf{e}(d)$ by (iii) of Lemma 3.1. This contradiction proves (vi). □

The following proposition is just (U4).

PROPOSITION 3.4. If $a, b \in M(D)$, then $L(a) \times L(b)$ contains $L(a + b)$ as a subquotient. The multiplicity is one.

PROOF. We compute in R

$$\begin{aligned} L(a) \times L(b) &= \Big(\lambda(a) + \sum_{c<a} m_{(a,c)}\,\lambda(c)\Big) \times \Big(\lambda(b) + \sum_{d<b} m_{(b,d)}\,\lambda(d)\Big) \\ &= \lambda(a) \times \lambda(b) + \sum_{d<b} m_{(b,d)}\,\lambda(a) \times \lambda(d) + \sum_{c<a} m_{(a,c)}\,\lambda(c) \times \lambda(b) \\ &\qquad + \sum_{c<a,\ b<d} m_{(a,c)}\, m_{(b,d)}\,\lambda(c) \times \lambda(d) \\ &= L(a+b) + \sum_{u<a+b} m_u^{a+b} L(u) \\ &+ \sum_{d<b} m_{(b,d)} \Big(\sum_{u\leq a+d} m_u^{a+d} L(u)\Big) + \sum_{c<a} m_{(a,c)} \Big(\sum_{u\leq c+b} m_u^{c+b} L(u)\Big) \\ &\qquad + \sum_{c<a,\ d<b} m_{(a,c)}\, m_{(b,d)} \Big(\sum_{u\leq c+d} m_u^{c+d} L(u)\Big). \end{aligned}$$

This and (v) of Lemma 3.1 implies that $L(a) \times L(b)$ contains $L(a + b)$ with multiplicity one. □

Now we return to $u(\delta,n)$, $\delta\in D$. Let $\delta=\gamma(x,y)$. Then

$$\begin{aligned}a(\delta,n)&=(\nu^{\frac{n-1}{2}}\delta,\nu^{\frac{n-1}{2}-1}\delta,\dots,\nu^{-\frac{n-1}{2}}\delta)\\&=(\gamma(x+\tfrac{n-1}{2},y+\tfrac{n-1}{2}),\gamma(x+\tfrac{n-1}{2}-1,,y+\tfrac{n-1}{2}-1),\dots,\gamma(x-\tfrac{n-1}{2},y-\tfrac{n-1}{2})).\end{aligned}$$

In the rest of this section, we shall consider R as a polynomial ring in indeterminates D.

Lemma 3.5. *Let $\delta,\delta'\in D$. Then the degree of the polynomial $u(\delta,n)$ in the indeterminate δ' is either zero or one.*

Proof. We know $u(\delta,n)=\sum_{a\le a(\delta,n)}m_{(a(\delta,n),a)}\lambda(a)$. Let

$$a_0=(\gamma(x_1,y_1),\dots,\gamma(x_n,y_n))\le a(\delta,n).$$

The formula for $a\prec b$ (in terms of "γ-coordinates") implies $\{x_1,\dots,x_n\}=\{x+\frac{n-1}{2},x+\frac{n-1}{2}-1,\dots,x-\frac{n-1}{2}\}$. Therefore, $x_1,\dots,x_n$ are all different, which implies that $\gamma(x_1,y_1),\dots,\gamma(x_n,y_n)$ are all different. This implies the lemma. □

Denote

$$X_i=\gamma(x+\tfrac{n-1}{2}+1-i,\ y+\tfrac{n-1}{2}+1-i),\quad i=1,\dots,n.$$

For $1\le i<j\le n$ let $Y_i(i,j)=\gamma(x+\frac{n-1}{2}+1-i,y+\frac{n-1}{2}+1-j)$, $\quad Y_j(i,j)=\gamma(x+\frac{n-1}{2}+1-j,y+\frac{n-1}{2}+1-i)$.

Note that

$(Y_i(i,j),Y_j(i,j))\prec(X_i,X_j)$. Denote $a_0=a(\delta,n)=(X_1,X_2,\dots,X_n)$ and

$$a_{i,j}=(X_1,\dots,X_{i-1},Y_i(i,j),X_{i+1},\dots,X_{j-1},Y_j(i,j),X_{j+1},\dots,X_n).$$

Then $a_{i,j}\prec a_0$ for all $1\le i<j\le n$.

Lemma 3.6. *For $1\le p\le n-1$, $a_{p,p+1}$ is adjacent to a_0.*

Proof. First note

$$\mathbf{e}(a_0)=(\tfrac{n-1}{2},\tfrac{n-1}{2}-1,\cdots,-\tfrac{n-1}{2})+\mathrm{Re}(\tfrac{x+y}{2})\,(1,1,\dots,1),$$

$$\mathbf{e}(Y_i(i,j),Y_j(i,j))=(\tfrac{n-1}{2}-\tfrac{i+j}{2}+1,\tfrac{n-1}{2}-\tfrac{i+j}{2}+1)+\mathrm{Re}\,(\tfrac{x+y}{2})\,(1,1). \tag{3-2}$$

Suppose that there exists some $d\in M(D)$ such that $a_{p,p+1}<d\prec a_0$. Then $\mathbf{e}(a_{p,p+1})<\mathbf{e}(d)$ by (iii) of Lemma 3.1. There exists $1\le i<j\le n$ such that $d=a_{i,j}$. Thus $\mathbf{e}(a_{p,p+1})<\mathbf{e}(a_{i,j})$. The definition of the ordering on $\mathbb{R}^n$ and (3-2) implies $p\le i$. Using the fact that Tr $a_{p,p+1}=$ Tr $a_{i,j}$ we obtain that $j\le p+1$. Therefore $(i,j)=\{p,p+1\}$ i.e. $i=p$, $j=p+1$, which implies $a_{p,p+1}=a_{i,j}$ and $\mathbf{e}(a_{p,p+1})=\mathbf{e}(d)$. This contradicts our assumption. The proof of the lemma is now complete. □

Before continuing with the proof of (U3) we need to make a small digression.

Suppose that $\mathcal{R}$ is a polynomial ring over the set of indeterminates $\mathcal{D}$. Let $\psi:\mathcal{D}\to\mathbb{Z}$ be any function. We can then define a grading gr_ψ on $\mathcal{R}$ specifying $\mathrm{gr}_\psi(d)=\psi(d)$ for $d\in\mathcal{D}$. In this way we get a $\mathbb{Z}$-grading on $\mathcal{R}$, which we shall call ψ-grading.

If we take $\psi\equiv1$, then we get the usual grading by total degree of polynomials. If we take for ψ the characteristic function of a fixed $d\in\mathcal{D}$, then we get the grading by the degree in d.

Lemma 3.7. *Suppose that a non-constant $T \in \mathcal{R}$ has the property that each $d \in \mathcal{D}$ has the degree in T equal to 0 or 1. Let V be the set of $d \in \mathcal{D}$ which have degree one in T and let $T = \sum_{m\in S} c_m m$ be a shortest expansion of T as a sum of monomials $d_1 d_2 \dots d_l$, where $l \in \mathbb{N}$ and $d_i \in \mathcal{D}$. Suppose that T factors as a product of two non-constant polynomials P_1 and P_2 in $\mathcal{R}$. Then there exists a partition $V = V_1 \cup V_2$ into two non-empty sets which satisfies the following condition: if T is homogeneous for some ψ-grading, then the function $m \mapsto \sum_{d\in V_1, d|m} \psi(d)$ on S is constant (here $d|m$ denotes: d divides m).*

Proof. Denote by V_i the set of all $d \in \mathcal{D}$ which have degree at least one in P_i (then the degree of $d \in V_i$ is exactly one, since $\mathcal{R}$ is an integral domain). The condition on degrees of $d \in \mathcal{D}$ in T implies that V_1 and V_2 are disjoint ($\mathcal{R}$ is an integral domain). Clearly, $V = V_1 \cup V_2$. Since P_i are non- constant, V_i are non-empty. Suppose that T is homogeneous for some ψ-grading. Since $\mathcal{R}$ is integral domain, P_i must be then also homogeneous for ψ-grading. This easily implies that V_1 satisfies the condition of the lemma. □

Now we shall go back to our study of $u(\delta, n) \in R$, where $\delta = \gamma(x, y)$. We shall consider two gradings below. First consider the ψ-grading for $\psi \equiv 1$. Now (ii) of Lemma 3.3 implies that elements of Irr are homogeneous for this grading. This is the same grading as the standard one.

Define $\phi : \gamma(x', y') \mapsto x' - y',\ D \to \mathbb{Z}$. First one checks directly that $a \prec b$ implies $\mathrm{gr}_\phi(\lambda(a)) = \mathrm{gr}_\phi(\lambda(b))$. This implies $\mathrm{gr}_\phi(\lambda(a)) = \mathrm{gr}_\phi(\lambda(b))$ if $a \leq b$. This and (ii) of Lemma 3.3 imply that elements of Irr are homogeneous also for ϕ-grading.

Proposition 3.8. Elements $u(\delta, n)$ are prime in R.

Proof. Suppose that some $u(\delta, n)$ is not prime. Then $u(\delta, n) = P_1 \times P_2$ for some $P_i \in R$ which are not invertible in R. Note that P_i must be homogeneous for the standard grading of $R = \oplus_{n=0}^{\infty} R_n$. Write a shortest expansion $u(\delta, n) = \sum_{m\in S} c_m\, m$ as in Lemma 3.7. First note that by (ii) of Lemma 3.3, at least one c_m is 1. Therefore, P_i are non-constant polynomials (homogeneous for the standard grading). We remind the reader that R is a $\mathbb{Z}$-polynomial ring. By (ii) of Corollary 1.4 we know that $n \geq 2$.

The above discussion and Lemma 3.5 imply that $T = u(\delta, n)$ satisfies the conclusion of Lemma 3.7. Then there exist V_1 and V_2 as in Lemma 3.7.

Further note that $c_{\lambda(a_0)} \neq 0$ by (i) of Lemma 3.3, and $c_{a_{p,p+1}} \neq 0$ for $p = 1, \dots, n-1$ by Lemma 3.6 and (iii) of Lemma 3.3. In other words, a_0 and $a_{p,p+1}$ are in S.

Denote

$$v_1(\psi) = \mathrm{card}\,\{i; 1 \leq i \leq n \text{ and } X_i \in V_1\} = \sum_{d\in V_1, d|\lambda(a_0)} 1.$$

If this would be 0, then V_1 would be empty (since this is independent of $\lambda(a) \in S$ by Lemma 3.7). This and the fact that V_2 is non-empty imply $1 \leq v_1(\psi) \leq n-1$. Therefore there exists $p \in \{1, \dots, n-1\}$ such that $\{X_p, X_{p+1}\} \not\subseteq V_i$ for $i = 1, 2$. Then $X_p \in V_1$ and $X_{p+1} \in V_2$, or $X_p \in V_2$ and $X_{p+1} \in V_1$. Without lost of generality, we can assume that the first possibility holds.

We know by the property of V_1 from Lemma 3.7 that

$$v_1(\psi) = \textstyle\sum_{d\in V_1,\, d|\lambda(a_{p,p+1})} 1 = \sum_{d\in V_1\setminus\{X_p, X_{p+1}\},\, d|\lambda(a_{p,p+1})} 1$$

$$= \sum_{d\in V_1\setminus\{X_p,X_{p+1}\},\, d|\lambda(a_0)} 1 + \sum_{j\in\{p,p+1\},\, Y_j(p,p+1)\in V_1} 1.$$

This implies

$$\big((\{X_1,\dots,X_n\}\cap V_1)\setminus\{X_p\}\big)\cup\{Y_j(p,p+1)\} = \{d\in V_1; d|\lambda(a_{p.p+1})\}$$

for some $j\in\{p,p+1\}$. Now applying Lemma 3.7 to ϕ-grading, we get

$$\sum_{d\in((\{X_1,\dots,X_n\}\cap V_1)\setminus\{X_p\})\cup\{Y_j(p,p+1)\}} \phi(d) = \sum_{d\in\{X_1,\dots,X_n\}\cap V_1} \phi(d),$$

which implies $\phi(Y_j(p,p+1)) = \phi(X_p)$, for some $j\in\{p,p+1\}$. Note that

$$\phi(X_p) = x-y,\ \ \phi(Y_p(p,p+1)) = x-y+1 \ \text{ and }\ \phi(Y_{p+1}(p,p+1)) = x-y-1.$$

From this we see that we cannot have $\phi(Y_j(p,p+1)) = \phi(X_p)$. This contradiction completes the proof. □

We have seen that all (U0) – (U4) hold in the complex case.

4. Real general linear group I

For the rest of this paper, we shall assume that $F=\mathbb{R}$. First we shall parameterize D. The signum characters of $\mathbb{R}^\times$ will be denoted by sgn (clearly $\operatorname{sgn}\in D^u$). Now we shall recall of some simple facts from the representation theory of $GL(2,\mathbb{R})$ (see [JL]).

Let $\delta_1,\delta_2\in\tilde G_1$. If $\delta_1\times\delta_2$ is irreducible, then $\delta_1\times\delta_2 = L((\delta_1,\delta_2))$. Further, $\delta_1\times\delta_2$ is reducible if and only if there exist $p\in\mathbb{Z}\setminus\{0\}$ such that $\delta_1(t)\delta_2(t)^{-1} = t^p\operatorname{sgn}(t)$ for $t\in\mathbb{R}^\times = G_1$.

If $\delta_1\times\delta_2$ is reducible, then there exist $\gamma(\delta_1,\delta_2)\in D$ such that $\delta_1\times\delta_2 = L((\delta_1,\delta_2)) + \gamma(\delta_1,\delta_2)$. The mapping $(\delta_1,\delta_2)\to\gamma(\delta_1,\delta_2)$ from the set of all pair in $\tilde G_1\times\tilde G_1$ such that $\delta_1\times\delta_2$ reduces, into $D\setminus\tilde G_1$, is surjective. We have

$$\gamma(\delta_1,\delta_2) = \gamma(\delta_1',\delta_2') \text{ if and only if } \{\delta_1,\delta_2\} = \{\delta_1',\delta_2'\} \text{ or } \{\delta_1,\delta_2\} = \{\delta_1'\operatorname{sgn},\delta_2'\operatorname{sgn}\}.$$

The relation $\delta_1\times\delta_2 = L((\delta_1,\delta_2))+\gamma(\delta_1,\delta_2)$ implies $\gamma(\delta_1,\delta_2)^+ = \gamma(\delta_1^+,\delta_2^+)$. We have also $e(\gamma(\delta_1,\delta_2)) = \frac12(e(\delta_1)+e(\delta_2))$. Thus $\gamma(\delta_1,\delta_2)\in D^u$ if and only if $e(\delta_1)+e(\delta_2) = 0$.

Now we shall introduce another parameterization of elements of D. First we shall make a short preparation. Let $\gamma(\delta_1,\delta_2)\in D$, where δ_1,δ_2 are characters of $\mathbb{R}^\times$. By the definition of $\gamma(\delta_1,\delta_2)$ we know that $(\delta_1\delta_2^{-1})(t) = t^p\operatorname{sgn}(t)$ for some $p\in\mathbb{Z}\setminus\{0\}$ (here $t\in\mathbb{R}^\times$). Write $\delta_i(t) = |t|^{\alpha_i}(\operatorname{sgn}(t))^{m_i}$, where $\alpha_i\in\mathbb{C}$, $m_i\in\{0,1\}$ for $i=1,2$. Now $(\delta_1\delta_2^{-1})(t) = t^p\operatorname{sgn}(t)$ implies $|t|^{\alpha_1-\alpha_2}\operatorname{sgn}(t)^{m_1-m_2} = t^p\operatorname{sgn}(t) = |t|^p(\operatorname{sgn}(t))^{p+1}$. Thus $\alpha_1-\alpha_2 = p$ and $(\operatorname{sgn})^{m_1-m_2} = (\operatorname{sgn})^{p+1}$. The last relation implies $m_1-m_2\equiv p+1 \pmod 2)$. Therefore, if we denote by $\delta_1^*(t) = |t|^{\alpha_1}$, then $\gamma(\delta_1,\delta_2) = \gamma(\delta_1^*,\delta_2^*)$ where $\delta_2^* = \delta_2(\operatorname{sgn})^{m_1}$. Note that $\delta_2^*(t) = |t|^{\alpha_2}\operatorname{sgn}(t)^{m_2}\cdot\operatorname{sgn}(t)^{m_1} = |t|^{\alpha_2}\operatorname{sgn}(t)^{m_1-m_2} = |t|^{\alpha_2}\operatorname{sgn}(t)^{\alpha_1-\alpha_2+1}$.

Fix $x,y\in\mathbb{C}$ satisfying $x-y\in\mathbb{Z}\setminus\{0\}$. Set $\delta_1(t) = |t|^x$, $\delta_2(t) = |t|^y\operatorname{sgn}(t)^{x-y+1}$. Then $\delta_1\delta_2^{-1}(t) = t^{x-y}\operatorname{sgn}(t)$, and therefore we can define $\gamma(x,y)$ by

$$\gamma(x,y) = \gamma(\delta_1,\delta_2).$$

From the above discussion it follows that $\gamma(x,y) = \gamma(y,x)$ (since $\gamma(\delta_1,\delta_2) = \gamma(\delta_2\operatorname{sgn},\delta_1\operatorname{sgn})$). Further

$$\gamma(x,y) = \gamma(x',y') \iff \{x,y\} = \{x',y'\}.$$

We have

$$\gamma(x,y)^+ = \gamma(-\bar x,-\bar y),\ \ e(\gamma(x,y)) = \operatorname{Re}(\frac{x+y}{2}),\ \ \nu^\alpha\gamma(x,y) = \gamma(x+\alpha,y+\alpha),\ \alpha\in\mathbb{R}.$$

For $x \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ denote

$$\gamma_\varepsilon(x)(t) = |t|^x(\mathrm{sgn}(t))^\varepsilon, \quad t \in G_1.$$

Now $(x,y) \to \gamma_\varepsilon(x)$, $\mathbb{C} \times (\mathbb{Z}/2\mathbb{Z}) \to \tilde{G}_1$ is a bijection, which parameterizes $\tilde{G}_1$. Clearly $e(\gamma_\varepsilon(x)) = \mathrm{Re}(x)$.

Now the following lemma holds.

Lemma 4.1. *The representation*

$$\gamma_\varepsilon(x) \times \gamma_{\varepsilon'}(x') \notin \mathrm{Irr} \iff x - x' \in \mathbb{Z}\backslash\{0\} \text{ and } x - x' + 1 \equiv \varepsilon - \varepsilon' \pmod 2.$$

If $\gamma_\varepsilon(x)\times\gamma_{\varepsilon'}(x')$ reduces, then we have $\gamma_\varepsilon(x)\times\gamma_{\varepsilon'}(x') = L((\gamma_\varepsilon(x),\gamma_{\varepsilon'}(x'))) + \gamma(x,x')$.

Now we shall describe the infinitesimal characters of the modules $L(a)$, $a \in M(D)$. Let $\delta \in D$. Then either $\delta = \gamma_\varepsilon(x)$ for some $x \in \mathbb{C}$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, or $\delta = \gamma(x',y')$ for some $x',y' \in \mathbb{C}$, $x'-y' \in \mathbb{Z}\backslash\{0\}$. Now we define $\chi(\delta) \in M(\mathbb{C})$ by $\chi(\gamma_\varepsilon(x)) = (x)$, $\chi(\gamma(x',y')) = (x',y')$. For $a = (\delta_1,\ldots,\delta_n) \in M(D)$ we define $\chi(a)$ by the formula

$$\chi(a) = \chi(\delta_1) + \cdots + \chi(\delta_n).$$

We consider the standard grading on R. If $\pi \in R_n$, then we shall write $n = \mathrm{gr}(\pi)$. For $a \in M(D)$, define $\mathrm{gr}(a) = \mathrm{gr}(L(a))$. With this definition, we have $\mathrm{gr}(a) = \mathrm{card}\chi(a)$.

Let $\mathfrak{a}_n \subseteq \mathfrak{g}_n$ be the Lie algebra of the subgroup A_n of all diagonal elements in G_n. Let $\mathfrak{a}_n^{\mathbb{C}}$ and $\mathfrak{g}_n^{\mathbb{C}}$ be complexifications of these two algebras. The universal enveloping algebras of $\mathfrak{a}_n^{\mathbb{C}}$ and $\mathfrak{g}_n^{\mathbb{C}}$ are denoted by $\mathcal{U}(\mathfrak{a}_n^{\mathbb{C}})$ and $\mathcal{U}(\mathfrak{g}_n^{\mathbb{C}})$ respectively. The center of the algebra $\mathcal{U}(\mathfrak{g}_n^{\mathbb{C}})$ is denoted by $\mathfrak{Z}(\mathfrak{g}_n^{\mathbb{C}})$. We consider the Harish-Chandra homomorphism $\xi : \mathfrak{Z}(\mathfrak{g}_n^{\mathbb{C}}) \to \mathcal{U}(\mathfrak{a}_n^{\mathbb{C}})$. Let $(\mathfrak{a}_n^{\mathbb{C}})^*$ be the space of all complex linear functionals on $\mathfrak{a}_n^{\mathbb{C}}$. For $\lambda \in (\mathfrak{a}_n^{\mathbb{C}})^*$, let $\xi_\lambda : \mathfrak{Z}(\mathfrak{g}_n^{\mathbb{C}}) \to \mathbb{C}$ be the composition of ξ with the evaluation at λ.

Let A_n^0 be the connected component of the group A_n containing identity, and M the torsion subgroup of A_n. The normalizer of A_n in K_n is denoted by M'. Set $W = M'/M$. Now W acts on $\mathfrak{g}_n^{\mathbb{C}}$ and $\mathfrak{a}_n^{\mathbb{C}}$. As it is well known, every homomorphism of $\mathfrak{Z}(\mathfrak{g}_n^{\mathbb{C}})$ into $\mathbb{C}$ is obtained as ξ_λ for some $\lambda \in (\mathfrak{a}_n^{\mathbb{C}})^*$. Also $\xi_\lambda = \xi_\mu$ if and only if $W\lambda = W\mu$.

We identify $\mathfrak{g}_n^{\mathbb{C}}$ in a natural way with the Lie algebra of all complex $n \times n$ matrices. Then $\mathfrak{a}_n^{\mathbb{C}}$ is the subalgebra of all diagonal matrices in $\mathfrak{g}_n^{\mathbb{C}}$. Now W acts on $\mathfrak{a}_n^{\mathbb{C}}$ by permutations of diagonal elements and W is isomorphic to the permutation group of order n.

Let $\lambda \in (\mathfrak{a}_n^{\mathbb{C}})^*$. Then there exist $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ such that $\lambda : \mathrm{diag}(x_1,\ldots,x_n) \mapsto \lambda_1 x_1 + \cdots + \lambda_n x_n$ Note that $\lambda \mapsto (\lambda_1,\ldots,\lambda_n)$ is an isomorphism of $(\mathfrak{a}_n^{\mathbb{C}})^*$ onto $\mathbb{C}^n$. We shall identify these two vector spaces using this isomorphism. In this identification, W acts by permuting of coordinates. Therefore $(\mathfrak{a}_n^{\mathbb{C}})^*/W$ can be naturally identified with the set of all multisets $(\lambda_1,\ldots,\lambda_n)$ in $\mathbb{C}$ (of cardinal number n).

For $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n$, let $\lambda^* : A_n^0 \to \mathbb{C}^\times$ be the character

$$\lambda^* : \mathrm{diag}(a_1,\ldots,a_n) \mapsto a_1^{\lambda_1}\ldots a_n^{\lambda_n}.$$

The mapping $\lambda \mapsto \lambda^*$ is a group isomorphism of $(\mathfrak{a}_n^{\mathbb{C}})^*$ onto the group $(A_n^0)\tilde{}$ of all continuous homomorphisms of A_n^0 into $\mathbb{C}^\times$.

Fix $a = (\gamma_{\varepsilon_1}(x_1),\ldots,\gamma_{\varepsilon_n}(x_n)) \in M(\tilde{G}_1)$. Let $\mu \in (\mathfrak{a}_n^{\mathbb{C}})^*$ corresponds to $(x_1,\ldots,x_n) \in \mathbb{C}^n$ under the above identification. Then by Lemma 4.1.8 of [V1], $\lambda(a)$ has infinitesimal character, and it is equal to ξ_μ.

For $a \in M(D)$ such that $\mathrm{gr}(a) = n$, denote $\chi(a) = (x_1, \ldots, x_n)$. Then by Lemma 4.1 there exist $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Z}/2\mathbb{Z}$ such that $\pi \in \mathrm{Irr}$ which is contained in $\lambda(a)$, is contained in $\lambda((\gamma_{\varepsilon_1}(x_1), \ldots, \gamma_{\varepsilon_n}(x_n)))$. Therefore, the infinitesimal character of $\lambda(a)$ is ξ_μ, where μ is as above. This implies:

Lemma 4.2. *Let $a, b \in M(D)$ with $\mathrm{gr}(a) = \mathrm{gr}(b)$. Then $L(a)$ and $L(b)$ have the same infinitesimal character if and only if $\chi(a) = \chi(b)$.* □

Now we shall describe a necessary conditions that $\lambda(a)$ contains $L(b)$ for $a, b \in M(D)$. Let $a = (\delta_1, \ldots, \delta_m) \in M(D)$. We say that $a = (\delta_1, \ldots, \delta_m)$ is written in a standard order if $e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_m)$. Suppose that $a = (\delta_1, \ldots, \delta_m)$ is written in a standard order. Let $\mathrm{gr}(a) = n$. Define

$$\mathbf{e}(a) = (\underbrace{e(\delta_1), \ldots, e(\delta_1)}_{\mathrm{gr}(\delta_1)\text{ times}}, \underbrace{e(\delta_2), \ldots, e(\delta_2)}_{\mathrm{gr}(\delta_2)\text{ times}}, \ \ldots\ , \underbrace{e(\delta_m), \ldots, e(\delta_m)}_{\mathrm{gr}(\delta_m)\text{ times}}) \in \mathbb{R}^n.$$

Clearly, $\mathbf{e}(a)$ is uniquely determined by a.

We define a partial ordering on $\mathbb{R}^n$ as before (recall $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \iff \sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for all $1 \leq k \leq n$). For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define $\mathrm{Tr}(x) = x_1 + \cdots + x_n$. If $a \in M(D)$, then we define $\mathrm{Tr}(a) = \mathrm{Tr}(\mathbf{e}(a))$.

Lemma 4.3. *Let $a, b \in M(D)$. Suppose that $L(b)$ is contained in $\lambda(a)$. Then $\mathrm{gr}(a) = \mathrm{gr}(b)$, $\chi(a) = \chi(b)$, $\mathrm{Tr}(a) = \mathrm{Tr}(b)$ and $\mathbf{e}(b) \leq \mathbf{e}(a)$. Further, $a \neq b$ if and only if $\mathbf{e}(b) < \mathbf{e}(a)$.*

Proof. The claim $\mathrm{gr}(a) = \mathrm{gr}(b)$ is obvious. Since $L(b)$ and $\lambda(a)$ have the same infinitesimal character, $\chi(a) = \chi(b)$ is a consequence of Lemma 4.2. Let $c = (\delta_1, \ldots, \delta_m) \in M(D)$. After a renumeration we can suppose that $\delta_i = \gamma(x_i, y_i)$ for $1 \leq i < k$, and $\delta_i = \gamma_{\varepsilon_i}(z_i)$ for $k \leq i \leq m$, for some $1 \leq k \leq m+1$, $x_i, y_i, z_i \in \mathbb{C}$, $\varepsilon_i \in \mathbb{Z}/2\mathbb{Z}$. Then $\chi(c) = (x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}, z_k, \ldots, z_m)$. Now we have $\mathrm{Tr}(c) = \mathrm{Re}(\sum_{i=1}^{k-1}(x_i + y_i) + \sum_{i=k}^m z_i)$. Therefore, $\mathrm{Tr}(c)$ depends only on $\chi(c)$. Now $\chi(a) = \chi(b)$ (which we have observed) implies $\mathrm{Tr}(a) = \mathrm{Tr}(b)$.

Let $\mathrm{gr}(a) = n$. Denote

$$\beta_i = (\underbrace{1, 1, \ldots, 1}_{i-\text{times}}, 0, \ldots, 0) - \tfrac{i}{n}(1, 1, \ldots, 1, 1, \ldots, 1) \in \mathbb{R}^n, \quad i = 1, \ldots, n-1.$$

We consider on $\mathbb{R}^n$ the standard scalar product $< (x_1, \ldots, x_n), (y_1, \ldots, y_n) >= \sum_{i=1}^n x_i y_i$. Now by Proposition 4.13 of [BoW] we have $< \beta_i, \mathbf{e}(b) > \leq < \beta_i, \mathbf{e}(a) >$ for $i = 1, \ldots, n-1$. Also, if all $n-1$ above inequalities are actually equalities, then $a = b$ by the same proposition. Note that $< (x_1, \ldots, x_n), \beta_i >= x_1 + \cdots + x_i - \frac{i}{n}\mathrm{Tr}((x_1, \ldots, x_n))$. Since $\mathrm{Tr}(a) = \mathrm{Tr}(b)$, we obtain that $\mathbf{e}(b) \leq \mathbf{e}(a)$, and further that $a \neq b$ if and only if $\mathbf{e}(b) < \mathbf{e}(a)$. □

Lemma 4.4. *(i) Let $a \in M(D)$. The set of all $b \in M(D)$ such that $\chi(a) = \chi(b)$, is finite.*

(ii) *Let $a_i, b_i \in M(D)$, $i = 1, 2$. Suppose that $\mathbf{e}(b_i) \leq \mathbf{e}(a_i)$ for $i = 1, 2$. If $\mathbf{e}(a_1 + a_2) = \mathbf{e}(b_1 + b_2)$, then $\mathbf{e}(a_i) = \mathbf{e}(b_i)$ for $1 = 1, 2$.*

Proof. The claim (i) is a direct consequence of the definition of $\chi(b)$, $b \in M(D)$. Suppose that $a_i, b_i \in M(D)$ satisfy $\mathbf{e}(b_i) \leq \mathbf{e}(a_i)$ for $i = 1, 2$. Let $\mathbf{e}(a_1) \neq \mathbf{e}(b_1)$ or $\mathbf{e}(a_2) \neq \mathbf{e}(b_2)$. Without lost of generality, we can assume that $\mathbf{e}(a_1) \neq \mathbf{e}(b_1)$. Now $\mathbf{e}(b_1) < \mathbf{e}(a_1)$. Write $\mathbf{e}(a_1) = (e_1, \ldots, e_n)$, $\mathbf{e}(a_2) = (e_{n+1}, \ldots, e_{m+n})$, $\mathbf{e}(b_1) = (f_1, \ldots, f_n)$, $\mathbf{e}(b_2) = (f_{n+1}, \ldots, f_{m+n})$. Choose a permutation σ of the set $\{1, \ldots, n+m\}$, such that $f_{\sigma^{-1}(1)} \geq f_{\sigma^{-1}(2)} \geq \cdots \geq f_{\sigma^{-1}(n+m)}$. Now

$$\mathbf{e}(b_1+b_2)=(f_{\sigma^{-1}(1)},\dots,f_{\sigma^{-1}(n+m)}).$$

We can choose σ as above, which satisfies $1\le\sigma^{-1}(i)<\sigma^{-1}(j)\le n\Rightarrow i<j$, and $m+1\le\sigma^{-1}(i)<\sigma^{-1}(j)\le n+m\Rightarrow i<j$. Now $\mathbf{e}(b_1)<\mathbf{e}(a_1)$ implies that $\mathbf{e}(b_1+b_2)=(f_{\sigma^{-1}(1)},\dots,f_{\sigma^{-1}(n+m)})<(e_{\sigma^{-1}(1)},\dots,e_{\sigma^{-1}(n+m)})$. Since $(e_{\sigma^{-1}(1)},\dots,e_{\sigma^{-1}(n+m)})\le\mathbf{e}(a_1+a_2)$ (which one can easily check), we have $\mathbf{e}(b_1+b_2)<\mathbf{e}(a_1+a_2)$. This proves (ii) of the lemma. □

Let $a=(\delta_1,\dots,\delta_k)\in M(D)$. For $b\in M(D)$ we shall say $b\prec a$ if there exist $1\le i<j\le k$ and $c\in M(D)$ such that $b=(\delta_1,\delta_{i-1},\delta_{i+1},\dots,\delta_{j-1},\delta_{j+1},\dots,\delta_k)+c$, where $L(c)$ is a subquotient of $\delta_i\times\delta_j$ satisfying $c\ne(\delta_i,\delta_j)$. Using Lemma 4.3, it is a simple combinatorial exercise to show that $\mathbf{e}(b)<\mathbf{e}(a)$. This implies that if we generate by $\prec$ an ordering $<$ on $M(D)$, that it is really an ordering.

Suppose that $\lambda(a)$ contains $L(b)$ and $a\ne b$ $(a,b\in M(D))$. The end of §3 of [SpV] (more precisely, Corollary 3.15, Lemma 3.8 and Theorem 3.7 of [SpV]) implies that $L(b)$ is contained in $\lambda(a_1)$, for some $a_1\prec a$. Continuing this analysis as in the second paragraph of the proof of Lemma 3.2, we would get $b<a$.

Lemma 4.5. *Fix $a\in M(D)$.*

(i) *There exist $m_b^a\in\mathbb{Z}_+$ for $b\in M(D)$, $b\le a$, so that $\lambda(a)=\sum_{b\le a}m_b^aL(b)$. Further, $m_a^a=1$.*

(ii) *There exist numbers $m_{(a,b)}\in\mathbb{Z}$ for $b\in M(D)$, $b\le a$, such that $L(a)=\sum_{b\le a}m_{(a,b)}\lambda(b)$. We have $m_{(a,a)}=1$.*

Proof. The fact that $L(a)$ has multiplicity one in $\lambda(a)$ ([BoW]), Lemma 4.3 and the discussion preceding the proposition imply (i). We prove (ii) in the same way as (ii) of Lemma 3.3 (by induction in a finite set). □

Now we can see that (U4) holds.

Proposition 4.6. If $a,b\in M(D)$, then $L(a)\times L(b)$ contains $L(a+b)$, and the multiplicity is one.

Proof. The proof of the above proposition is very similar to the proof of Proposition 3.4 (use (ii) of Lemma 4.4 instead of (v) of Lemma 3.1). □

We shall now get additional information regarding expansions considered in Lemma 4.5.

Lemma 4.7. *Let $a,b,c\in M(D)$. If $b\prec a$, then $m_b^a\ge1$. Let $c\prec a$ and suppose that c satisfies the following condition*

(4-1) *for each $d\in M(D)$, such that $d\prec a$ (and $d\ne c$), we have $\mathbf{e}(c)\not<\mathbf{e}(d)$.*

Then $m_{(a,c)}\ne0$.

Proof. The fact that $m_b^a\ge1$ if $b\prec a$ follows directly from the last proposition (or the factorization of the long intertwining operator). For the other claim of the lemma we shall use relation (3-1) (which obviously holds also here). Fix some $c\in M(D)$ for which $c\prec a$ and $m_{(a,c)}=0$. Then $m_c^a\ge1$, and (3-1) implies that there must exist $d\in M(D)$ satisfying $c<d<a$ and $m_d^a\ge1$. Suppose additionally that c satisfies (4-1). Take d' such that $d\le d'\prec a$. Then $d'\ne c$ since $c<d\le d'$. The last inequality implies $\mathbf{e}(c)<\mathbf{e}(d')$. This contradicts (4-1). Thus, $m_{(a,c)}\ne0$. □

Now we shall observe that (U1) holds for $GL(n,\mathbb{R})$.

Proposition 4.8. Representations $u(\delta, n)$ are unitarizable if $\delta \in D^u$ and $n \in \mathbb{N}$.

Proof. If $\mathrm{gr}(\delta) = 1$, then $u(\delta, n)$ is a unitary character of G_n obtained by composing δ with the determinant homomorphism. Thus, $u(\delta, n)$ is unitarizable. It remains to consider the case $\mathrm{gr}(\delta) = 2$. Then $\delta = \gamma(x, y)$ for some $x, y \in \mathbb{C}$ such that $x - y \in \mathbb{Z}\backslash\{0\}$ and $\mathrm{Re}(x + y) = 0$. If $x + y = 0$, then Theorem 3.5.3 of [Sp2] implies unitarizability of $u(\gamma(x, y), n)$. From $\gamma(x, y) = \nu^{\frac{x+y}{2}} \gamma(\frac{x-y}{2}, \frac{y-x}{2})$ we obtain $u(\gamma(x, y), n) = \nu^{\frac{x+y}{2}} u(\gamma(\frac{x-y}{2}, \frac{y-x}{2}), n)$. Since $\nu^{\frac{x+y}{2}}$ is a unitary character of $G_{2n}, u(\gamma(x, y), n)$ is unitarizable. □

Remark 4.9. In [Sp2] Speh has shown that representations $u(\delta, n)$ are constructed from square integrable representations in a way compatible with the way representations in the residual spectrum of adelic $GL(n)$ are constructed from the cuspidal automorphic representations. From this she easily concluded the unitarizability of $u(\delta, n)$. Such general construction of the residual spectrum was done by Jacquet in [J2]. An "elementary" way to prove the unitarizability of $u(\delta, n)$'s, based on Miličić's result in [M], would be to prove irreducibility of $u(\delta, n) \times u(\delta, n - 2)$'s (see section 7. of [T4]).

Now we shall consider (U2).

Proposition 4.10. For $\delta \in D^u, 0 < \alpha < 1/2$ and $n \in \mathbb{N}$, $\pi(u(\delta, n), \alpha)$ are unitarizable .

Proof. Before we go to prove the proposition, we shall prove two facts.

Fact 1: Let $a = (\delta_1, \ldots, \delta_u), b = (\delta_{u+1}, \ldots, \delta_{u+v}) \in M(D)$. Suppose that $\delta_i \times \delta_j \in \mathrm{Irr}$ for all $1 \leq i \leq u$, $u + 1 \leq j \leq u + v$. Then $L(a) \times L(b) \in \mathrm{Irr}$.

This result was proved in [Ze] by Zelevinsky in the case of $GL(n)$ over non-archimedean field. His proof applies, after necessary modifications, also to the archimedean case. We shall sketch it here. We can suppose that multisets $a = (\delta_1, \ldots, \delta_u)$ and $b = (\delta_{u+1}, \ldots, \delta_{u+v})$ are written in a standard order. Let $\mathrm{gr}(a+b) = n$. Suppose that σ is a permutation of $\{1, \ldots, u + v\}$ which satisfy the following assumptions:

$$1 \leq \sigma(i) < \sigma(j) \leq u \Rightarrow i < j, \quad \text{and} \quad u + 1 \leq \sigma(i) < \sigma(j) \leq u + v \Rightarrow i < j.$$

Let $\pi_\sigma = \delta_{\sigma(1)} \times \delta_{\sigma(2)} \times \cdots \times \delta_{\sigma(u+v)}$. Induction by stages and $\delta_i \times \delta_j \in \mathrm{Irr}$ for all $1 \leq i \leq u$, $u+1 \leq j \leq u+v$, imply that all π_σ are isomorphic. Note that $L(a) \times L(b)$ is a quotient of π_{id} (id denotes the identity permutation). The consideration above implies that $L(a) \times L(b)$ has a unique irreducible quotient, and this quotient is isomorphic to $L(a + b)$. Repeating the above considerations with $\tilde{a}$ and $\tilde{b}$, applying the contragredient functor (and Proposition 1.2), one obtains that $L(a) \times L(b)$ has a unique irreducible submodule, which is isomorphic to $L(a + b)$. The multiplicity one of $L(a + b)$ in $\lambda(a + b)$ implies that $L(a) \times L(b)$ is irreducible. This finishes the proof of the first fact.

Fact 2: Let $\gamma(x_i, y_i) \in D$ for $i = 1, 2$. Suppose $x_1 - x_2 \notin \mathbb{Z}$. Then $\gamma(x_1, y_1) \times \gamma(x_2, y_2)$ is irreducible.

Suppose that $\delta_1 \times \delta_2 \notin \mathrm{Irr}$. Set $a_0 = (\gamma(x_1, y_1), \gamma(x_2, y_2))$. Since $\gamma(x_1, y_1) \times \gamma(x_2, y_2)$ is not irreducible, there exists $a \in M(D)$ such that $\chi(a) = \chi(a_0)$ and $\mathbf{e}(a) < \mathbf{e}(a_0)$. The condition $\chi(a) = \chi(a_0)$ implies that a is one of the following multisets $(\gamma(x_1, y_1), \gamma(x_2, y_2))$, $(\gamma_{\varepsilon_1}(x_1), \gamma_{\varepsilon_2}(y_1), \gamma(x_2, y_2))$, $(\gamma(x_1, y_1), \gamma_{\varepsilon_3}(x_2), \gamma_{\varepsilon_4}(y_2))$,

$(\gamma_{\varepsilon_1}(x_1), \gamma_{\varepsilon_2}(y_1), \gamma_{\varepsilon_3}(x_2), \gamma_{\varepsilon_4}(y_2))$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{Z}/2\mathbb{Z}$. Direct verification implies $\mathbf{e}(a) \geq \mathbf{e}(a_0)$ in all four cases. This is a contradiction.

An immediate consequence of the preceding two remarks on irreducibility, and Proposition 4.1, is the fact that $\pi(u(\delta,n),\alpha)$ is irreducible for $0 < \alpha < 1/2$. We know that $\pi(u(\delta,n),0) = u(\delta,n) \times u(\delta,n) \in \mathrm{Irr}^u$ by (U0). Thus $\pi(u(\delta,n),\alpha) \in \mathrm{Irr}$, for $0 \leq \alpha < 1/2$ and $\pi(u(\delta,n),0)$ is unitarizable. Well-known analytic properties of intertwining operators imply now that $\pi(u(\delta,n),\alpha)$ is unitarizable for $0 < \alpha < 1/2$. (Apply Speh's criterion in §3 of [Sp1] for existence of complementary series.) □

5. Real general linear group II

In this section, we shall prove that $u(\delta,n), \delta \in D$, are prime (we continue to assume that $F = \mathbb{R}$).

Lemma 5.1. *For $\delta \in \tilde{G}_1$ and $n \in \mathbb{N}$, $u(\delta,n)$ is a prime element of R.*

Proof. Note that it is enough to consider the case of $n \geq 2$ by Corollary 1.4. We have $\delta = \gamma_\varepsilon(x)$ for some $x \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Now $u(\delta,n) = L(a(\delta,n))$, where

$$a(\delta,n) = (\gamma_\varepsilon(x + \tfrac{n-1}{2}), \gamma_\varepsilon(x + \tfrac{n-1}{2} - 1), \dots, \gamma_\varepsilon(x - \tfrac{n-1}{2})).$$

We consider $u(\delta,n)$ as a polynomial in indeterminates D. Since $\chi(a(\delta,n))$ consists of n different elements, we see that the degree of $u(\delta,n)$ in any indeterminate (from D) is either 0 or 1.

Denote $a_0 = a(\delta,n) = (X_1,\dots,X_m)$, $X_i = \gamma_\varepsilon(x + \frac{n-1}{2} + 1 - i)$, $1 \leq i \leq n$,

$$X_{i,j} = \gamma(x + \tfrac{n-1}{2} + 1 - i, x + \tfrac{n-1}{2} + 1 - j), \quad 1 \leq i < j \leq n,\ j - i \equiv 1 \pmod 2,$$

$$a_{i,j} = (X_1,\dots,X_{i-1},X_{i,j},X_{i+1},\dots,X_{j-1},X_{j+1},\dots,X_n),$$
$$1 \leq i < j \leq n, j - i \equiv 1 \pmod 2.$$

Fix $1 \leq i < n$. Now we shall show that $a_{i,i+1}$ satisfies the condition (4-1) of Lemma 4.7 with respect to a_0. Note that $a_{i,i+1} \prec a_0$ (this implies that $\lambda(a_0)$ contains $L(a_{i,i+1})$). Suppose that $d \in M(D)$ satisfies $d \prec a_0$ and $d \neq a_{i,i+1}$. Then by Lemma 4.1 there exist $1 \leq j < k \leq n$, $j - k \equiv 1 \pmod 2$, such that $d = a_{j,k}$ (which implies $\mathbf{e}(d) = \mathbf{e}(a_{j,k})$). Thus for the proof of the condition (4-1) for $a_{i,i+1}$, it is enough to see that $\mathbf{e}(a_{i,i+1}) \not< \mathbf{e}(a_{j,k})$ for all $1 \leq j < k \leq n$. Assume on the contrary that such j,k exist, i.e. $\mathbf{e}(a_{i,i+1}) < \mathbf{e}(a_{j,k})$.

Note that $\mathbf{e}(a_0) = (\frac{n-1}{2}, \frac{n-3}{2}, \frac{n-5}{2}, \dots, -\frac{n-1}{2}) + \mathrm{Re}(x)(1,1,\dots,1)$,

$$\mathbf{e}(a_{i,i+1}) = (\tfrac{n-1}{2}, \tfrac{n-3}{2}, \dots, \tfrac{n+3-2i}{2}, \tfrac{n-2i}{2}, \tfrac{n-2i}{2}, \tfrac{n-3-2i}{2}, \dots, -\tfrac{n-1}{2})$$
$$+ \mathrm{Re}(x)(1,1,\dots,1).$$

Now one obtains directly that $\mathbf{e}(a_{i,i+1}) < \mathbf{e}(a_{j,k})$ implies $i \leq j$ and $k \leq i+1$. Thus $i = j$ and $i+1 = k$. This is a contradiction. Therefore, $a_{i,i+1}$ and a_0 satisfy (4-1).

Suppose that $u(\delta,n)$ is not prime. Let $u(\delta,n) = P \times Q$ be a non-trivial decomposition (i.e. neither P nor Q is invertible in R). Since coefficient of $\lambda(a_0)$ is 1, P and Q must be non-constant polynomials (recall that R is polynomial $\mathbb{Z}$-ring). Therefore, we can apply Lemma 3.7. We shall apply it using the notation of that lemma. Note that neither $\{X_1,X_2,\dots,X_n\} \subseteq V_1$ nor $\{X_1,X_2,\dots,X_n\} \cap V_1 = \emptyset$ (otherwise, $V_2 = \emptyset$ or $V_1 = \emptyset$). Choose $1 \leq i \leq n-1$ such that $\{X_i, X_{i+1}\} \not\subseteq V_1$ and $\{X_i, X_{i+1}\} \not\subseteq V_2$. Without lost of generality we can assume $X_i \in V_1$.

Note that we have proved in the first part of the proof that $a_{i,i+1} \in S$. Therefore, $\sum_{d\in V_1, d|\lambda(a_0)} \mathrm{gr}(d) = \sum_{d\in V_1, d|\lambda(a_{i,i+1})} \mathrm{gr}(d)$. Clearly $\sum_{d\in V_1, d|\lambda(a_0)} \mathrm{gr}(d) = \sum_{d\in V_1, d|\lambda(a_0)} 1$. This implies

$$\begin{aligned} &1 + \textstyle\sum_{d\in V_1\setminus\{X_i, X_{i+1}, X_{i,i+1}\}, d|\lambda(a_0)} 1 \\ &\qquad\qquad = 2\,\mathrm{ch}_{V_1}(X_{i,i+1}) + \textstyle\sum_{d\in V_1\setminus\{X_i, X_{i+1}, X_{i,i+1}\}, d|\lambda(a_{i,i+1})} 1, \end{aligned}$$

where ch_{V_1} denotes the characteristic function of V_1 in V. Therefore $2\,\mathrm{ch}_{V_1}(X_{i,i+1}) = 1$, which obviously cannot hold. This contradiction completes the proof. $\square$

The following lemma is related to the ordering in $M(D)$. We shall need it in the proof that $u(\delta, n)$ are prime for $\delta \in D \cap \tilde{G}_2$.

Lemma 5.2. *Let $x, y \in \mathbb{C}$. Suppose that $x - y = k \in \mathbb{N}$. Let $a_0 = (\gamma(x,y), \gamma(x+1, y+1))$. Then:*

(i) *If $k \geq 3$, then $\lambda(a_0) = L(a_0) + m\,\gamma(x, y+1) \times \gamma(x+1, y)$ for some $m \in \mathbb{N}$.*

(ii) *If $k = 1$, then $\lambda(a_0) = L(a_0) + m\,\gamma_0(x) \times \gamma_1(y+1) \times \gamma(x+1, y)$ for some $m \in \mathbb{N}$ (note that $x = y + 1$ in this case).*

(iii) *If $k = 2$, then there exist $m_0, m_1, m_2 \in \mathbb{Z}_+$ satisfying $m_1 + m_2 + m_3 \geq 1$, such that*

$$\begin{aligned} \lambda(a_0) &= L(a_0) + m_2\,\gamma(x, y+1) \times \gamma(x+1, y) \\ &\quad + m_0\,L((\gamma_0(x), \gamma_0(y+1), \gamma(y, x+1))) + m_1\,L((\gamma_1(x), \gamma_1(y+1), \gamma(y, x+1))). \end{aligned}$$

Before we prove the above lemma, we shall prove the following elementary, but technical, lemma.

Lemma 5.3. *Let $x, y \in \mathbb{C}$ and $r \in \mathbb{N}$. Suppose that $x - y = k \in \mathbb{N}$. Denote $a_0 = (\gamma(x,y), \gamma(x+r, y+r))$. Let $a \in M(D)$. Suppose that $\chi(a) = \chi(a_0)$ and $\mathbf{e}(a) < \mathbf{e}(a_0)$. Then:*

(i) $\mathbf{e}(a_0) = (r, r, 0, 0) + \mathrm{Re}(\frac{x+y}{2})(1,1,1,1)$,
$$\mathbf{e}(a_0) > (\tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}) + \mathrm{Re}(\tfrac{x+y}{2})\ \ (1,1,1,1).$$

(ii) *If $k > 2r$ or $r = k$, then $\mathbf{e}(a) = (\frac{r}{2}, \frac{r}{2}, \frac{r}{2}, \frac{r}{2}) + \mathrm{Re}(\frac{x+y}{2})(1,1,1,1)$.*

(iii) *If $2r \geq k > r$, then $\mathbf{e}(a)$ equals to one of the following terms*
$$(\tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}) + \mathrm{Re}(\tfrac{x+y}{2})(1,1,1,1), \quad (\tfrac{k}{2}, \tfrac{r}{2}, \tfrac{r}{2}, r - \tfrac{k}{2}) + \mathrm{Re}(\tfrac{x+y}{2})(1,1,1,1).$$

(iv) *If $r > k$, then $\mathbf{e}(a)$ equals to one of the following terms*
$$(\tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}) + \mathrm{Re}(\tfrac{x+y}{2})(1,1,1,1), \quad (r - \tfrac{k}{2}, \tfrac{r}{2}, \tfrac{r}{2}, \tfrac{k}{2}) + \mathrm{Re}(\tfrac{x+y}{2})(1,1,1,1),$$
$$(\tfrac{r+k}{2}, \tfrac{r+k}{2}, \tfrac{r-k}{2}, \tfrac{r-k}{2}) + \mathrm{Re}(\tfrac{x+y}{2})(1,1,1,1).$$

Proof. The conditions $\chi(a) = \chi(a_0)$ and $a \neq a_0$ imply that a equals to one of the following elements

$$
\begin{aligned}
&a_1 = (\gamma(x+r,y), \gamma(y+r,x)), \quad \text{if } x+r \neq y \text{ and } y+r \neq x,\\
&a_2 = (\gamma(x+r,y), \gamma_{\varepsilon_1}(y+r), \gamma_{\varepsilon_2}(x)), \quad \text{if } y \neq x+r,\\
&a_3 = (\gamma_{\varepsilon_1}(x+r), \gamma_{\varepsilon_2}(y), \gamma(y+r,x)), \quad \text{if } x \neq y+r,\\
&a_4 = (\gamma_{\varepsilon_1}(x+r), \gamma_{\varepsilon_2}(y), \gamma_{\varepsilon_3}(y+r), \gamma_{\varepsilon_4}(x)),\\
&a_5 = (\gamma(x+r,x), \gamma(y+r,y)),\\
&a_6 = (\gamma_{\varepsilon_1}(x+r), \gamma_{\varepsilon_2}(x), \gamma(y+r,y))\\
&a_7 = (\gamma(x+r,x), \gamma_{\varepsilon_1}(y+r), \gamma_{\varepsilon_2}(y)),\\
&a_8 = (\gamma_{\varepsilon_1}(x), \gamma_{\varepsilon_2}(y), \gamma(x+r,y+r)),\\
&a_9 = (\gamma(x,y), \gamma_{\varepsilon_1}(x+y), \gamma_{\varepsilon_2}(y+r)),
\end{aligned}
$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{Z}/2\mathbb{Z}$. One gets directly $\mathrm{Re}(x) = \mathrm{Re}(\frac{x+y}{2} + \frac{x-y}{2}) = \frac{k}{2} + \mathrm{Re}\,\frac{x+y}{2}$, $\mathrm{Re}(y) = -\frac{k}{2} + \mathrm{Re}\,\frac{x+y}{2}$, $\mathbf{e}(a_0) = (r,r,0,0) + \mathrm{Re}(\frac{x+y}{2})(1,1,1,1)$, $\mathrm{Tr}(a) = 2(r + \mathrm{Re}(x+y))$. Note that

$$
\begin{aligned}
\text{(5-1)} \qquad \mathbf{e}(a_1) &= (\tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}, \tfrac{r}{2}) + \mathrm{Re}(\tfrac{x+y}{2})(1,1,1,1),\\
\mathbf{e}(a_5) &= (\tfrac{r+k}{2}, \tfrac{r+k}{2}, \tfrac{r-k}{2}, \tfrac{r-k}{2}) + \mathrm{Re}(\tfrac{x+y}{2})(1,1,1,1).
\end{aligned}
$$

Observe that (i) directly follows. Since $e(\gamma_\varepsilon(x+r)) = r + \frac{k}{2} + \mathrm{Re}\,\frac{x+y}{2}$, the condition $\mathbf{e}(a) < \mathbf{e}(a_0)$ implies that a can not be equal to a_3, a_4, a_6 or a_9. Since $e(\gamma_\varepsilon(y)) = -\frac{k}{2} + \mathrm{Re}\,\frac{x+y}{2}$, the condition $\mathbf{e}(a) < \mathbf{e}(a_0)$ implies $a \neq a_7, a_8$. Thus a is equal to a_1, a_2 or a_5.

Suppose $r > k$. Then $\mathbf{e}(a_2) = (r-\frac{k}{2}, \frac{r}{2}, \frac{r}{2}, \frac{k}{2}) + \mathrm{Re}(\frac{x+y}{2})(1,1,1,1)$. This, together with (5-1), implies (iv).

Suppose that $k > r$. Then (5-1) implies $\mathbf{e}(a_0) \not> \mathbf{e}(a_5)$. Thus if $k > r$, then $a = a_1$ of $a = a_2$. This implies (iii).

Let $k > 2r$. Then $k > r$ and thus $a = a_1$ or $a = a_2$. Now $\mathbf{e}(a_2) = (\frac{k}{2}, \frac{r}{2}, \frac{r}{2}, r - \frac{k}{2}) + \mathrm{Re}(\frac{x+y}{2})(1,1,1,1)$. Since $r - \frac{k}{2} < 0$, it can not be $\mathbf{e}(a_2) < \mathbf{e}(a_0)$. Thus $a = a_1$. Therefore (ii) holds if $k > 2r$.

Let $k = r$. We have shown that $a = a_1$, $a = a_2$ or $a = a_5$. Now $\mathbf{e}(a_5) = (r,r,0,0) + \mathrm{Re}(\frac{x+y}{2})(1,1,1,1)$, and thus $\mathbf{e}(a_5) = \mathbf{e}(a_0)$. Therefore $a \neq a_5$. Thus $a = a_1$ of $a = a_2$. Since $k = r$, we have $y + r = x$. Therefore, a_1 is not defined in this situation. This implies $a = a_2$. Now the rest of (ii) is obvious. The proof of Lemma 5.3 is now complete. □

Proof. Now we shall give proof of Lemma 5.2. In the proof of this lemma, we shall use the notation introduced in the proof of Lemma 5.3. By §2 of [Sp1], $\gamma(x,y) \times \gamma(x+1, y+1)$ is reducible (see also §5 of [Sp1]). We know that $L(a_0)$ is a composition factor of $\lambda(a_0)$ with multiplicity one. Therefore there exists $a \in M(D)$ such that $L(a)$ is contained in $\lambda(a_0)$ and $a \neq a_0$. This implies that $\chi(a) = \chi(a_0)$ and $\mathbf{e}(a) < \mathbf{e}(a_0)$.

Let $k \geq 3$. Then $k > 2r = 2$. Now the proof of Lemma 5.3 implies $a = a_1 = (\gamma(x+1,y), \gamma(y+1,x))$. Further, Lemma 5.3 implies easily that $\gamma(x,y+1), \gamma(x+1,y)$ is irreducible. This proves (i) (of Lemma 5.2).

Suppose that $k = 1$. Then $k = r$ (see Lemma 5.3), and the proof of Lemma 5.3 implies $a = a_2 = (\gamma(x+1,y), \gamma_{\varepsilon_1}(y+1), \gamma_{\varepsilon_2}(x)) = (\gamma(x+1,y), \gamma_{\varepsilon_1}(x), \gamma_{\varepsilon_2}(x))$

since $y+1=x$. We need to determine which ε_1 and ε_2 can show up. Note that $\gamma(x,y)$ is a composition factor of $\gamma_0(x)\times\gamma_0(y)$ and $\gamma(x+1,y+1)$ is a composition factor of $\gamma_0(x+1)\times\gamma_0(y+1)$. Therefore, $L(a)$ is a composition factor of $\gamma_0(x)\times\gamma_0(y)\times\gamma_0(x+1)\times\gamma_0(y+1)$. Let I_4 be identity of G_4. Then $-I_4$ acts in the above module trivially (as identity), and thus $-I_4$ acts in $L(a)$ trivially. Now $\gamma(y,x+1)$ is a composition factor of $\gamma_0(x+1)\times\gamma_1(y)$. Thus $L(a_2)$ is contained in $\gamma_0(x+1)\times\gamma_1(y)\times\gamma_{\varepsilon_1}(x)\times\gamma_{\varepsilon_2}(y+1)$. Here $-I_4$ act as a multiplication by $(-1)^{1+\varepsilon_1+\varepsilon_2}$. Thus $-I_4$ acts in $L(a_2)$ multiplying by $(-1)^{1+\varepsilon_1+\varepsilon_2}$. Since $L(a)=L(a_2)$, we have $\varepsilon_1+\varepsilon_2\equiv 1 \pmod 2$. Therefore $a=(\gamma_1(x),\gamma_0(x),\gamma(y,x+1))$. Again Lemma 5.3 easily implies that $\lambda(a)$ is irreducible. This proves (ii) (note that $e(\gamma_1(x))=e(\gamma_0(x))=e(\gamma(y,x+1))$).

We obtain (iii) in the same way as (i) and (ii). This ends the proof. □

We shall prove (U3).

PROPOSITION 5.4. For $\delta\in D$ and $n\in\mathbb{N}$, $u(\delta,n)$ is a prime element of R.

To prove above proposition, we note that Lemma 5.1, implies that it is enough to prove the proposition if $\mathrm{gr}(\delta)=2$. Choose $x,y\in\mathbb{C}$ such that $x-y=k\in\mathbb{N}$ and $\delta=\gamma(x,y)$. Without lost of generality, we can assume $\mathrm{Re}(x+y)=0$ (if $\alpha\in\mathbb{R}$, then $L(a)\to\nu^\alpha L(a)$ lifts to a multiplicative automorphism of R; see Remark 1.5). We introduce this assumption to simplify notation only. By Corollary 1.4, we can suppose that $n\geq 2$. The rest of the paper is the proof of the above proposition (for this $\delta=\gamma(x,y)$).

Let

$$X_i=\gamma(x+\tfrac{n-1}{2}+1-i,y+\tfrac{n-1}{2}+1-i),\quad i=1,\dots,n,$$

and $a_0=(X_1,\dots,X_n)$. Clearly, $a_0=a(\gamma(x,y),n)=a(\delta,n)$ and $u(\delta,n)=L(a_0)$.

We shall suppose that $u(\delta,n)=L(a_0)$ is not prime. Let

$$L(a_0)=P\times Q$$

be a non-trivial decomposition, i.e. P and Q are not invertible in R. As before, (ii) of Lemma 4.5 implies that P and Q are non-constant polynomials. Corollary 1.4 implies that P and Q are homogeneous (with respect to the standard grading on R). Write

$$P=\sum_{a\in M(D)}m_{(P,a)}\lambda(a),\quad Q=\sum_{a\in M(D)}m_{(Q,a)}\lambda(a).$$

Let $S_P=\{a\in M(D);m_{(P,a)}\neq 0\}$ and $S_Q=\{a\in M(D);m_{(Q,a)}\neq 0\}$. By (ii) of Lemma 4.5 we have

$$L(a_0)=X_1\times X_2\times\cdots\times X_n+\textstyle\sum_{a<a_0}m_{(a_0,a)}\lambda(a)$$

(note $\lambda(a_0)=X_1\times X_2\times\cdots\times X_n$). The definition of multiplication in R implies that there exist $a'\in S_P$ and $b'\in S_Q$ such that

$$a'+b'=a_0.$$

Since $\mathrm{gr}(P)>0$ and $\mathrm{gr}(Q)>0$, we have $a'\neq\emptyset$ and $b'\neq\emptyset$. Take a partition $\{\phi_1(1),\dots,\phi_1(p)\}\cup\{\phi_2(1),\dots,\phi_2(q)\}$ of $\{1,2,\dots,n\}$, such that

$$a'=(X_{\phi_1(1)},\dots,X_{\phi_1(p)}),\quad b'=(X_{\phi_2(1)},\dots,X_{\phi_2(q)})$$

and $p+q=n$. Then $p\geq 1,q\geq 1$. Further, $\mathrm{gr}(P)=2p$ and $\mathrm{gr}(Q)=2q$.

Chose $1\leq t\leq n-1$, $1\leq i\leq p$ and $1\leq j\leq q$ such that $\{t,t+1\}=\{\phi_1(i),\phi_2(j)\}$. After a renumeration, we can assume that $i=p$ and $j=q$, i.e. $\{t,t+1\}=\{\phi_1(p),\phi_2(q)\}$.

Recall that $X_t \times X_{t+1}$ reduces. Therefore, we can fix $a_1 \in M(D)$ which satisfies: $a_1 \prec (X_t, X_{t+1})$ and if $b \prec (X_t, X_{t+1})$, then $a_1 \not< b$ (it is enough to take $a_1 \prec (X_t, X_{t+1})$ with maximal $\mathbf{e}(a_1)$). Denote

$$a_{t,t+1} = a_1 + (X_1, X_2, \ldots, X_{t-1}, X_{t+2}, \ldots, X_n).$$

LEMMA 5.5.
$$m_{(a_0, a_{t,t+1})} \neq 0.$$

PROOF. First note

$$\mathbf{e}(a_0) = (\tfrac{n-1}{2}, \tfrac{n-1}{2}, \tfrac{n-3}{2}, \tfrac{n-3}{2}, \cdots\cdots, -\tfrac{n+1}{2}, -\tfrac{n+1}{2}, -\tfrac{n-1}{2}, -\tfrac{n-1}{2}).$$

Now we shall show that $a_{t,t+1}$ and a_0 satisfy (4-1). Suppose that $a_{t,t+1} < b$ and $b \prec a_0$. First there exists $1 \leq i < j \leq n$ and $b_1 \prec (X_i, X_j)$ such that

$$b = b_1 + (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots X_{j-1}, X_{j+1}, \ldots, X_n).$$

Further $\mathbf{e}(a_{t,t+1}) < \mathbf{e}(b)$. This and Lemma 5.3 imply $t \leq i$. Considering $\mathrm{Tr}(b)$ and $\mathrm{Tr}(a_0)$, similarly as in the proof of Lemma 5.1, we get $j \leq t+1$. Therefore, $i = t$ and $j = t+1$, which implies $b \prec a_{t,t+1}$. This contradicts our choice of a_1. Thus (4-1) holds. Now Lemma 4.7 implies the claim of above lemma. $\square$

Now we shall continue the proof of the proposition. First we shall describe a_1 more explicitly using Lemma 5.2 (the elements a_1 that we shall describe below will satisfy the condition that if $b \prec (X_t, X_{t+1})$, then $a \not< b$). Denote

$$Y_t = \gamma(x + \tfrac{n-1}{2} + 1 - t, y + \tfrac{n-1}{2} - t),\ Y_{t+1} = \gamma(x + \tfrac{n-1}{2} - t, y + \tfrac{n-1}{2} + 1 - t),$$

where Y_{t+1} is defined only if $x - y = k \geq 2$ (see Lemma 4.1).

Let $k \geq 3$. Then Lemma 5.2 implies that the only possibility for a_1 is $a_1 = (Y_t, Y_{t+1})$.

Suppose $k = 2$. Then (iii) of Lemma 5.2 implies that (exactly) one of the following two possibilities hold. The first case will be called non-standard, while the other one will be called standard.

The first case happens if there exists $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ such that if we denote

$$Y_{t+1}^{(x)} = \gamma_\varepsilon(x + \tfrac{n-1}{2} - t), \quad Y_{t+1}^{(y)} = \gamma_\varepsilon(y + \tfrac{n-1}{2} + 1 - t),$$

then $(Y_t, Y_{t+1}^{(x)}, Y_{t+1}^{(y)}) \leq (X_t, X_{t+1})$. In this case we take $a_1 = (Y_1, Y_{t+1}^{(x)}, Y_{t+1}^{(y)})$.

The standard case happens if there is no ε as above. Then we take $a_1 = (Y_t, Y_{t+1})$ (as we did in the case of $k \geq 3$).

Let $k = 1$. Denote

$$Y_{t+1}^{(0)} = \gamma_0(x + \tfrac{n-1}{2} - t), \quad Y_{t+1}^{(1)} = \gamma_1(x + \tfrac{n-1}{2} - t).$$

We take $a_1 = (Y_t, Y_{t+1}^{(0)}, Y_{t+1}^{(1)})$. In this case we define Y_{t+1} to be $(Y_{t+1}^{(0)}, Y_{t+1}^{(1)}) \in M(D)$.

If $k = 2$ and we are in the non-standard case, then we shall not use Y_{t+1} as it is defined above. In this case, it will be convenient to as to take Y_{t+1} to be $(Y_{t+1}^{(x)}, Y_{t+1}^{(y)}) \in M(D)$.

Since $m_{(a_0, a_{t,t+1})} \neq 0$ by Lemma 5.5, there exist $c' \in S_P$ and $d' \in S_Q$ such that

$$c' + d' = a_{t,t+1}$$

Clearly $\mathrm{gr}(c') = 2p$ and $\mathrm{gr}(d') = 2q$. Without lost of generality we can assume that $Y_t | \lambda(c')$ in R.

Let $k = 1$. Since the gradings of c' and d' are even, $Y_{t+1}^{(0)} | \lambda(c')$ if and only if $Y_{t+1}^{(1)} | \lambda(c')$. If $k = 2$ and we are in the non-standard case, then the same observation holds for $Y_{t+1}^{(x)}$ and $Y_{t+1}^{(y)}$. Therefore in these two cases, either $\lambda(Y_{t+1})|\lambda(c')$, or $\lambda(Y_{t+1})$ and $\lambda(c')$ are relatively prime. Note that in the remaining two cases ($k \geq 3$ and the standard case for $k=2$), we have $\lambda(Y_{t+1}) = Y_{t+1}$.

We consider now two possibilities. The first is $\lambda(Y_{t+1})|\lambda(c')$, and the other one is $\lambda(Y_{t+1}) \not|\lambda(c')$. We shall analyze now the first possibility. Suppose $\lambda(Y_{t+1})|\lambda(c')$. Obviously, we can decompose

$$\{1,2,\dots,t-1,t+2,\dots,n\} = \{\psi_1(1),\dots,\psi_1(p-2)\} \cup \{\psi_2(1),\dots,\psi_2(q)\}$$

such that

$$c' = (X_{\psi_1(1)},\dots,X_{\psi_1(p-2)},Y_t) + Y'_{t+1}, \quad d' = (X_{\psi_2(1)},\dots,X_{\psi_2(q)}),$$

where $Y'_{t+1} = (Y_{t+1}) \in M(D)$ if $Y_{t+1} \in D$, and $Y'_{t+1} = Y_{t+1}$ if $Y_{t+1} \in M(D)$. Note that $\{\psi_2(1),\dots,\psi_2(q)\} \not\subseteq \{\phi_2(1),\dots,\phi_2(q)\}$, since t or $t+1$ is contained in the right hand side, but neither t nor $t+1$ is contained in the left hand side. This implies directly $\{\psi_2(1),\dots,\psi_2(q)\} \cap \{\phi_1(1),\dots,\phi_1(p)\} \neq \emptyset$. If we denote $T = \{\phi_1(1),\dots,\phi_1(p)\} \cup \{\psi_2(1),\dots,\psi_2(q)\} \subseteq \{1,\dots,n\}$, then the last relation implies $T \neq \{1,\dots,n\}$. We shall denote by $\deg_T \mathcal{F}$ the total degree of $\mathcal{F} \in R$ in the indeterminates $\{X_i, i \in T\}$. We know that $\deg_T P \geq p$, $\deg_T Q \geq q$. Thus $\deg_T L(a_0) \geq p+q = n$. Considering the standard grading, one obtains $\deg_T L(a_0) = n$. Take $b_0 \in M(D)$ satisfying $\deg_T \lambda(b_0) = n$, and $m_{(a_0,b_0)} \neq 0$. Now Lemma 4.3 implies

$$\mathrm{gr}(b_0) = 2n, \quad \chi(b_0) = \chi(a_0) \quad (\text{and } \mathbf{e}(b_0) < \mathbf{e}(a_0)).$$

Then $\lambda(b_0) = X_1^{\alpha_1} \times \cdots \times X_n^{\alpha_n}$, where $\alpha_i \in \mathbb{Z}_+$, $i=1,\dots,n$, satisfy $\alpha_1 + \cdots + \alpha_n = n$. Since $T \neq \{1,\dots,n\}$, there exists some i such that $\alpha_i \neq 1$. Let $i_0 = \min\{i; \alpha_i \neq 1\}$. Obviously

$$\chi(a_0) = (x+\tfrac{n-1}{2}, x+\tfrac{n-3}{2}, x+\tfrac{n-5}{2}, \dots, x-\tfrac{n-1}{2}, y+\tfrac{n-1}{2}, y+\tfrac{n-3}{2}, \dots, y-\tfrac{n-1}{2}). \tag{5-2}$$

Now one sees directly that $x + \frac{n-1}{2} + 1 - i_0$ cannot have the same multiplicity in $\chi(a_0)$ and $\chi(b_0)$, which is a contradiction. This contradiction implies $Y_{t,t+1} \not|\lambda(c')$. It remains to analyze this case (this is the only remaining case).

Suppose $\lambda(Y_{t+1}) \not|\lambda(c')$. Choose a partition

$$\{1,\dots,t-1,t+2,\dots,n\} = \{\psi_1(1),\dots,\psi_1(p-1)\} \cup \{\psi_2(1),\dots,\psi_2(q-1)\}$$

such that

$$c' = (X_{\psi_1(1)},\dots,X_{\psi_1(p-1)},Y_t), \ d' = (X_{\psi_2(1)},\dots,X_{\psi_2(q-1)}) + Y'_{t+1},$$

where Y'_{t+1} has the same meaning as above: $Y'_{t+1} = (Y_{t+1})$ if $Y_{t+1} \in D$ and $Y'_{t+1} = Y_{t+1}$ if $Y_{t+1} \in M(D)$. Let $T = \{\psi_1(1),\dots,\psi_1(p-1),\phi_2(1),\dots,\phi_2(q)\}$ (this is a subset of $\{1,\dots,n\}$). Then clearly $T \neq \{1,\dots,n\}$. The total degree of $f \in R$ in the indeterminates $\{Y_t\} \cup \{X_i, i \in T\}$ will be denoted by $\deg_T^*$. Now in the same way as in the previous case, we obtain $\deg_T^* L(a_0) = n$. Thus we can find $b_0 \in M(D)$ such that $\deg_T^* \lambda(b_0) = n$ and $b_0 < a_0$ (and $m_{(a_0,b_0)} \neq 0$). Now Lemma 4.3 implies $\chi(b_0) = \chi(a_0)$ and $\mathrm{Tr}(a_0) = \mathrm{Tr}(b_0)$. Since $\mathrm{gr}(\lambda(b_0)) = 2n$, $\mathrm{gr}(X_i) = 2$, $i=1,\dots,n$ and $\mathrm{gr}(Y_t) = 2$, we get that $\lambda(b_0) = X_1^{\alpha_1} \times \cdots \times X_n^{\alpha_n} \times Y_t^{\alpha}$ for some $\alpha_i \in \mathbb{Z}_+$, $i=1,\dots,n$ and $\alpha \in \mathbb{Z}_+$ which must satisfy $\alpha_1 + \cdots + \alpha_n + \alpha = n$. Since $T \neq \{1,\dots,n\}$, there exists i such that $\alpha_i = 0$.

If $\alpha = 0$, then we get in the same way as in the preceding case that $\chi(b_0) \neq \chi(a_0)$. Therefore, $\alpha \geq 1$.

Consider $\chi(a_0)$ (see (5-2)). Since multiplicities in $\chi(a_0)$ are at most two, one gets $\alpha \leq 2$.

Suppose $\alpha = 1$. Note that $2\ \mathrm{Tr}\ (X_i) \in (n-1)+2\mathbb{Z},\ i = 1,\dots,n$, and $2\ \mathrm{Tr}(Y_t) \in n+2\mathbb{Z}$. This implies $2\ \mathrm{Tr}(a_0) \in n(n-1)+2\mathbb{Z} = 2\mathbb{Z}$ and $2\ \mathrm{Tr}(b_0) \in (n-1)(n-1)+n+2\mathbb{Z} = 1+2\mathbb{Z}$. This contradicts $\mathrm{Tr}(a_0) = \mathrm{Tr}(b_0)$. Therefore, α must be 2.

Suppose $\alpha = 2$. Consider first the case $n = 2$. Then multiplicity of each of two elements in $\chi(b_0)$ are 2, while there exist elements in $\chi(a_0)$ with multiplicity one. Therefore, $\chi(a_0) \neq \chi(b_0)$. This contradicts $\chi(a_0) = \chi(b_0)$. Thus, $n \geq 3$. Let $i_0 = \min\{i; \alpha_i \neq 1\}$. Considering multiplicity of $x + \frac{n-1}{2} + 1 - t$ in $\chi(a_0)$ and $\chi(b_0)$ (must be the same), we get directly $i_0 \geq t$. This and the assumption $\alpha = 2$, imply that the multiplicity of $x + \frac{n-1}{2} + 1 - t$ in $\chi(b_0)$ is strictly greater than the multiplicity of this element in $\chi(a_0)$ (recall $Y_t = \gamma\left(x + \frac{n-1}{2} + 1 - t, y + \frac{n-1}{2} - t\right),\ X_i = \gamma\left(x + \frac{n-1}{2} + 1 - i, y + \frac{n-1}{2} + 1 - i\right)$). Thus, $\chi(a_0) \neq \chi(b_0)$. This contradiction ends the proof of the proposition.

At the end, note that we have seen now that all (U0) – (U4) hold also in the real case.

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