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Unramified Unitary Duals for Split Classical p–adic Groups; The topology and Isolated Representations

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For Freydoon Shahidi, with our admiration and appreciation

Introduction

This paper is our attempt to understand the work of Barbasch and Moy [**BbMo2**] on unramified unitary dual for split classical p-adic groups from a different point of view.

The classification of irreducible unitary representations of reductive groups over local fields is a fundamental problem of harmonic analysis with various possible applications, like those in number theory and the theory of automorphic forms. The class of unramified unitary representations is especially important in the aforementioned applications. These representations occur in the following set-up that we fix in this paper. Let F be a non–Archimedean local field of an arbitrary characteristic and $\mathcal{O}$ is its ring of integers. When we work with the classical groups we are obliged to require that the characteristic of F is different than 2. Let G be the group of the F–points of a F–split reductive group $\mathbf{G}$. An irreducible (complex) representation π of G is unramified if it has a vector fixed under $\mathbf{G}(\mathcal{O})$. The set of equivalence classes $\mathrm{Irr}^{unr}(G)$ of unramified irreducible representations of G is usually described by the Satake classification (see [**Cr**]). This classification is essentially the Langlands classification for those representations. We write $\mathrm{Irr}^{u,unr}(G) \subset \mathrm{Irr}^{unr}(G)$ for the subset consisting of unramified unitarizable representations. We equip that set with the topology of the uniform convergence of matrix coefficients on compact subsets ([**F**], [**Di**]; see also [**T1**], [**T6**]). A good understanding of unramified unitarizable representations is fundamental for the theory of automorphic forms since almost all components of cuspidal and residual automorphic representations are unramified and unitarizable. By a good understanding we mean the following:

(1) To have an explicit classification of unramified unitary duals with explicit parameters and with Satake parameters easily computed from them.

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The converse to (1) is not trivial and it is important. More precisely, from the point of view of the theory of automorphic forms, it would be important to have a way to decide from Satake parameters if a representation is unitarizable. Even for general linear groups we need a simple algorithm. This leads to the following:

(2) To have an effective way (an algorithm) for testing unitarity of an arbitrary unramified representation in $\mathrm{Irr}^{unr}(G)$ given by its Satake parameter.
(3) To understand the topology in terms of the classification, especially isolated points in $\mathrm{Irr}^{u,unr}(G)$, which are exactly the isolated points in the whole unitary dual which are unramified representations. This would be particularly interesting to understand from the point of view of automorphic spectra.

If $\mathbf{G} = \mathrm{GL}(n)$, then those tasks and much more were accomplished in the works of the second–named author ([**T4**], [**T5**]; see also [**T2**], [**T3**]) more than twenty years ago. In Section 4 of the present paper we give a simple solution due to the second named author to those problems based entirely on [**Ze**], without use of a result of Bernstein on irreducibility of unitary parabolic induction proved in [**Be2**] and used in the earlier proof of the classification (see Theorem 4-1.)

In this paper we give the solutions to problems (1)–(3) in the case of the split classical groups $G = S_n$ where S_n is one of the groups $\mathrm{Sp}(2n, F), \mathrm{SO}(2n + 1, F), \mathrm{O}(2n, F)$ (see Section 1 for the precise description of the groups). Regarding (3), we have an explicit description of isolated points, and (2) gives an algorithm for getting limit points for a given sequence in the dual. Further, the algorithm from (2) gives parameters in (1) in the case of unitarizability (the other direction is obvious).

This paper is the end of a long effort ([**M4**], [**M6**]). The approach to the problem (1) is motivated and inspired by the earlier work [**LMT**], in creation of which ideas of E. Lapid played an important role (see also [**T12**] which is a special case of [**LMT**]). On a formal level, the formulation of the solution (see Theorem 0-8) to the problem (1) is "dual" to that of the one in [**LMT**], but in our unramified case it has a much more satisfying formulation. This is not surprising since we are dealing with very explicit representations. On the other hand, the proofs are more involved. For example, the proof of the unitarity of the basic "building blocks" (see Theorem 0-4 below) requires complicated arguments with the poles of degenerate Eisenstein series (see [**M6**]). The problems (2) and (3) were not yet considered for split classical groups. A characteristic of our approach is that at no point in the proofs does the explicit internal structure of representations play a role. This is the reason that this can be considered as an external approach to the unramified unitary duals (of classical groups), which is a kind of a continuation of such approaches in [**T3**], [**T2**], [**LMT**], etc.

We expect that our approach has a natural Archimedean version similar to the way that [**LMT**] covers both non-Archimedean and Archimedean cases, or the way the earlier paper [**T3**] has a corresponding Archimedean version [**T2**], with the same description of unitary duals for general linear groups and proofs along the same lines.

Now, we describe our results. They are stated in Section 5 in more detail than here. After being acquainted with the basic notation in Section 1, the reader may proceed to read Section 5 directly.

In the introduction, we use classical notation for induced representations. In the rest of the paper we shall use notation adapted to the case of general linear and classical groups which very often substantially simplifies arguments in proofs. A part of the exposition below makes perfect sense also for Archimedean fields (we shall comment on this later).

We fix the absolute value $|\ |$ of F which satisfies $d(ax) = |a|dx$. Let χ be an unramified character of $F^\times$ and $l \in \mathbb{Z}_{>0}$. Then we consider the following induced representation of $GL(l, F)$:

$$\mathrm{Ind}_{P_\emptyset}^{GL(l,F)}(|\ |^{\frac{l-1}{2}}\chi \otimes |\ |^{\frac{l-1}{2}-1}\chi \otimes \cdots \otimes |\ |^{-\frac{l-1}{2}}\chi)$$

which has a character $\chi \circ \det$ as the unique irreducible quotient. We denote this character by:

$$\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)} \rangle.$$

We introduce Langlands dual groups as follows:

$$\begin{array}{ll} G = S_n = \mathrm{SO}(2n+1, F) & \hat{G}(\mathbb{C}) = \mathrm{Sp}(2n, \mathbb{C}) \subset GL(N, \mathbb{C}); N = 2n \\ G = S_n = \mathrm{O}(2n, F) & \hat{G}(\mathbb{C}) = \mathrm{O}(2n, \mathbb{C}) \subset GL(N, \mathbb{C}); N = 2n \\ G = S_n = \mathrm{Sp}(2n, F) & \hat{G}(\mathbb{C}) = \mathrm{SO}(2n+1, \mathbb{C}) \subset GL(N, \mathbb{C}); N = 2n+1. \end{array}$$

The local functorial lift $\sigma^{GL(N,F)}$ of $\sigma \in \mathrm{Irr}^{unr}(S_n)$ to $GL(N, F)$ is always defined and it is an unramified representation. (See (10-1) in Section 10 for the precise description.) It is an easy exercise to check that the map $\sigma \mapsto \sigma^{GL(N,F)}$ is injective. This lift plays the key role in the solutions to problems (2) and (3).

In order to describe $\mathrm{Irr}^{u,unr}(S_n)$ we need to introduce more notation. Let $\mathbf{sgn}_u$ be the unique unramified character of order two of $F^\times$ and let $\mathbf{1}_{F^\times}$ be the trivial character of $F^\times$. Let $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. Then we define α_χ as follows:

$$\begin{array}{l} \text{if } S_n = \mathrm{O}(2n, F), \text{ then } \alpha_\chi = 0 \\ \text{if } S_n = \mathrm{SO}(2n+1, F), \text{ then } \alpha_\chi = \dfrac{1}{2} \\ \text{if } S_n = \mathrm{Sp}(2n, F), \text{ then } \alpha_{\mathbf{sgn}_u} = 0 \text{ and } \alpha_{\mathbf{1}_{F^\times}} = 1. \end{array}$$

We refer to Remark 5-3 for an explanation of this definition in terms of rank–one reducibility.

A pair (m, χ), where $m \in \mathbb{Z}_{>0}$ and χ is an unramified unitary character of $F^\times$ is called a Jordan block. The following definition can be found in [**M4**] (see also Definition 5-4 in Section 5):

Definition 0-1. *Let $n > 0$. We denote by $Jord_{sn}(n)$ the collection of all the sets $Jord$, which consist of Jordan blocks, such that the following hold:*

$$\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\} \textit{ and } m - (2\alpha_\chi + 1) \in 2\mathbb{Z} \textit{ for all } (m,\chi) \in \mathrm{Jord}$$
$$\sum_{(m,\chi)\in \mathrm{Jord}} m = \begin{cases} 2n & \textit{if } S_n = \mathrm{SO}(2n+1,F) \textit{ or } S_n = \mathrm{O}(2n,F); \\ 2n+1 & \textit{if } S_n = \mathrm{Sp}(2n,F), \end{cases}$$
and, additionally, if $\alpha_\chi = 0$*, then* $\mathrm{card}\,\{k; (k,\chi) \in \mathrm{Jord}\} \in 2\mathbb{Z}$.

Let $\mathrm{Jord} \in \mathrm{Jord}_{sn}(n)$. Then, for $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, we let
$$\mathrm{Jord}_\chi = \{k; (k,\chi) \in \mathrm{Jord}\}.$$
We let
$$\mathrm{Jord}'_\chi = \begin{cases} \mathrm{Jord}(\chi); & \mathrm{card}(\mathrm{Jord}_\chi) \text{ is even}; \\ \mathrm{Jord}(\chi) \cup \{-2\alpha_\chi + 1\}; & \mathrm{card}(\mathrm{Jord}_\chi) \text{ is odd}. \end{cases}$$
We write elements of Jord'_χ in the following way (the case $l_{1_{F^\times}} = 0$ or $l_{sgn_u} = 0$ is not excluded):
$$\begin{cases} \text{for } \chi = 1_{F^\times} & \text{as } \ a_1 < a_2 < \cdots < a_{2l_{1_{F^\times}}} \\ \text{for } \chi = sgn_u & \text{as } b_1 < b_2 < \cdots < b_{2l_{sgn_u}}. \end{cases}$$

Next, we associate to $\mathrm{Jord} \in \mathrm{Jord}_{sn}(n)$, the unramified representation $\sigma(\mathrm{Jord})$ of S_n defined as the unique irreducible unramified subquotient of the representation parabolically induced from the representation
$$\left(\otimes_{i=1}^{l_{\mathbf{1}_{F^\times}}} \langle [-\frac{a_{2i}-1}{2}, \frac{a_{2i-1}-1}{2}]^{(\mathbf{1}_{F^\times})} \rangle\right) \otimes \left(\otimes_{j=1}^{l_{\mathbf{sgn}_u}} \langle [-\frac{b_{2j}-1}{2}, \frac{b_{2j-1}-1}{2}]^{(\mathbf{sgn}_u)} \rangle\right).$$

Recall that irreducible tempered (resp., square integrable) representations of a reductive group can be characterized by satisfying certain inequalities (resp., strict inequalities). In [**M4**], the first author defines negative (resp., strongly negative) irreducible representations as those which satisfy the reverse inequalities (resp., strict inequalities). An unramified representation is strongly negative if its Aubert dual is in the discrete series. See [**M4**] for more details. We have the following result (see [**M4**]; Theorem 5-8 in Section 5 of this paper):

Theorem 0-2. *Let* $n \in \mathbb{Z}_{>0}$. *The map* $\mathrm{Jord} \mapsto \sigma(\mathrm{Jord})$ *defines a one–to–one correspondence between the set* $\mathrm{Jord}_{sn}(n)$ *and the set of all strongly negative unramified representations of* S_n.

The inverse mapping to $\mathrm{Jord} \mapsto \sigma(\mathrm{Jord})$ will be denoted by $\sigma \mapsto \mathrm{Jord}(\sigma)$. Let us note that the set $\mathrm{Jord}_{sn}(n)$ also parameterizes the generic irreducible square integrable representations with Iwahori fixed vector.

An unramified representation is negative if its Aubert dual is tempered. Negative representations are classified in terms of strongly negative as follows ([**M4**]; Theorem 5-10 in Section 5 of this paper):

Theorem 0-3. *Let* $\sigma_{neg} \in \mathrm{Irr}^{unr}(S_n)$ *be a negative representation. Then there exists a sequence of pairs* $(l_1,\chi_1), \ldots, (l_k,\chi_k)$ *(*$l_i \in \mathbb{Z}_{\geq 1}$*,* χ_i *is an unramified unitary character of* $F^\times$*), unique up to a permutation and taking inverses of characters, and* *a unique strongly negative representation σ_{sn} such that σ_{neg} is a subrepresentation of the parabolically induced representation*

$$\mathrm{Ind}^{S_n}\left(\langle[-\frac{l_1-1}{2},\frac{l_1-1}{2}]^{(\chi_1)}\rangle\otimes\cdots\otimes\langle[-\frac{l_k-1}{2},\frac{l_k-1}{2}]^{(\chi_k)}\rangle\otimes\sigma_{sn}\right).$$

Conversely, for a sequence of pairs $(l_1,\chi_1),\ldots,(l_k,\chi_k)$ ($l_i\in\mathbb{Z}_{>0}$, χ_i an unramified unitary character of $F^\times$) and a strongly negative representation σ_{sn}, the unique irreducible unramified subquotient of

$$\mathrm{Ind}^{S_n}\left(\langle[-\frac{l_1-1}{2},\frac{l_1-1}{2}]^{(\chi_1)}\rangle\otimes\cdots\otimes\langle[-\frac{l_k-1}{2},\frac{l_k-1}{2}]^{(\chi_k)}\rangle\otimes\sigma_{sn}\right)$$

is negative and it is a subrepresentation.

We let $\mathrm{Jord}(\sigma_{neg})$ to be the multiset

$$\mathrm{Jord}(\sigma_{sn})+\sum_{i=1}^{k}\{(l_i,\chi_i),(l_i,\chi_i^{-1})\}$$

(multisets are sets where multiplicities are allowed).

The proofs of Theorems 0-2 and 0-3 given in [**M4**] are obtained with Jacquet modules techniques enabling the results to hold for F of any characteristic different than two.

The key result for this paper is the following result of the first author (see [**M6**]; see Theorem 5-11):

Theorem 0-4. *Every negative representation is unitarizable. Every strongly negative representation is a local component of a global representation appearing in the residual spectrum of a split classical group defined over a global field.*

The unitarizability of negative representations was obtained earlier by D. Barbasch and A. Moy. It follows from their unitarity criterion in [**BbMo**] and [**BbMo1**], which says that unitarizability can already be detected on Iwahori fixed vectors.

The following theorem is a consequence of above results:

Theorem 0-5. *Let $\sigma\in\mathrm{Irr}^{unr}(S_n)$ be a negative representation. Then its lift to $GL(N,F)$ is given by:*

$$\sigma^{GL(N,F)}\simeq\times_{(l,\chi)\in\mathrm{Jord}(\sigma)}\ \langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\chi)}\rangle.$$

Moreover, its Arthur parameter $W_F\times SL(2,\mathbb{C})\times SL(2,\mathbb{C})\to\hat{G}(\mathbb{C})\subset GL(N,\mathbb{C})$ is given by:

$$\oplus_{(l,\chi)\in\mathrm{Jord}(\sigma)}\ \chi\otimes V_1\otimes V_l,$$

where V_l is the unique algebraic representation of $SL(2,\mathbb{C})$ of dimension l.

In order to describe the whole $\mathrm{Irr}^{u,unr}(S_n)$, we need to introduce more notation. We write $\mathcal{M}^{unr}(S_n)$ for the set of pairs $(\mathbf{e},\sigma_{neg})$, where:

- $\mathbf{e}$ is a (perhaps empty) multiset consisting of a finite number of triples (l, χ, α) where $l \in \mathbb{Z}_{>0}$, χ is an unramified unitary character of $F^\times$, and $\alpha \in \mathbb{R}_{>0}$.
- $\sigma_{neg} \in \mathrm{Irr}\ S_{n_{neg}}$ (this defines n_{neg}) is negative satisfying:

$$n = \sum_{(l,\chi)} l \cdot \mathrm{card}\ \mathbf{e}(l, \chi) + n_{neg}.$$

For $l \in \mathbb{Z}_{>0}$ and an unramified unitary character χ of $F^\times$, we denote by $\mathbf{e}(l, \chi)$ the submultiset of $\mathbf{e}$ consisting of all positive real numbers α (counted with multiplicity) such that $(l, \chi, \alpha) \in \mathbf{e}$.

We attach $\sigma \in \mathrm{Irr}^{unr}(S_n)$ to $(\mathbf{e}, \sigma_{neg})$ in a canonical way. By definition, σ is the unique irreducible unramified subquotient of the following induced representation:

$$\mathrm{Ind}^{S_n}\left(\left(\otimes_{(l,\chi,\alpha)\in\mathbf{e}}\ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(|\ |^\alpha\chi)}\rangle\right) \otimes \sigma_{neg}\right). \tag{0-6}$$

We remark that the definition of σ does not depend on the choice of ordering of elements in $\mathbf{e}$.

In order to obtain unitary representations, we impose further conditions on $\mathbf{e}$ in the following definition (see Definition 5-13):

DEFINITION 0-7. *Let $\mathcal{M}^{u,unr}(S_n)$ be the subset of $\mathcal{M}^{unr}(S_n)$ consisting of the pairs $(\mathbf{e}, \sigma_{neg})$ satisfying the following conditions:*

(1) *If $\chi \notin \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, then $\mathbf{e}(l, \chi) = \mathbf{e}(l, \chi^{-1})$ and $0 < \alpha < \frac{1}{2}$ for all $\alpha \in \mathbf{e}(l, \chi)$.*
(2) *If $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$, then $0 < \alpha < \frac{1}{2}$ for all $\alpha \in \mathbf{e}(l, \chi)$.*
(3) *If $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l - (2\alpha_\chi + 1) \in 2\mathbb{Z}$, then $0 < \alpha < 1$ for all $\alpha \in \mathbf{e}(l, \chi)$. Moreover, if we write the exponents that belong to $\mathbf{e}(l, \chi)$ as follows:*

$$0 < \alpha_1 \leq \cdots \leq \alpha_u \leq \frac{1}{2} < \beta_1 \leq \cdots \leq \beta_v < 1.$$

(We allow $u = 0$ or $v = 0$.) Then we must have the following:
   (a) *If $(l, \chi) \notin \mathrm{Jord}(\sigma_{neg})$, then $u + v$ is even.*
   (b) *If $u > 1$, then $\alpha_{u-1} \neq \frac{1}{2}$.*
   (c) *If $v \geq 2$, then $\beta_1 < \cdots < \beta_v$.*
   (d) *$\alpha_i \notin \{1 - \beta_1, \ldots, 1 - \beta_v\}$ for all i.*
   (e) *If $v \geq 1$, then the number of indices i such that $\alpha_i \in]1 - \beta_1, \frac{1}{2}]$ is even.*
   (f) *If $v \geq 2$, then the number of indices i such that $\alpha_i \in]1 - \beta_{j+1}, 1 - \beta_j[$ is odd.*

The main result of the paper is the following explicit description of $\mathrm{Irr}^{u,unr}(S_n)$ (see Theorem 5-14):

THEOREM 0-8. *Let* $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$. *Then*

$$Ind^{S_n}\left(\left(\otimes_{(l,\chi,\alpha)\in\mathbf{e}} \ \ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(|\ |^{\alpha}\chi)}\rangle\right) \otimes \sigma_{neg}\right)$$

is irreducible. Moreover, the map

$$(\mathbf{e}, \sigma_{neg}) \longmapsto \mathrm{Ind}^{S_n}\left(\left(\otimes_{(l,\chi,\alpha)\in\mathbf{e}} \ \ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(|\ |^{\alpha}\chi)}\rangle\right) \otimes \sigma_{neg}\right)$$

is a one–to–one correspondence between $\mathcal{M}^{u,unr}(S_n)$ *and* $\mathrm{Irr}^{u,unr}(S_n)$.

This result is proved in Sections 7, 8, and 9. The preparation for the proof is done in the first two of these three sections. In Section 2, where we recall (from [**T8**]) some general principles for proving unitarity and non–unitarity, we also prove a new criterion for non–unitarity (see (RP) in Section 2). In Section 6 we describe all necessary reducibility facts explicitly (most of them are already established in [**M4**]).

Theorem 0-8 clearly solves problem (1) for the split classical groups. In Section 10 we describe a simple algorithm that has:
INPUT: an arbitrary irreducible unramified representation $\sigma \in \mathrm{Irr}^{unr}(S_n)$ given by its Satake parameter.
OUTPUT: tests the unitarity of σ and at the same time constructs the corresponding pair $(\mathbf{e}, \sigma_{neg})$ if σ is unitary.

The algorithm is based on the observation that, for a unitarizable σ, the Zelevinsky data (see [**Ze**]; or Theorem 1-7 here) of the lift $\sigma^{GL(N,F)}$ is easy to describe from the datum $(\mathbf{e}, \sigma_{neg})$ of σ. (See Lemma 10-7.) This solves problem (2) stated above. We observe that this problem is almost trivial for $GL(n, F)$. (See Theorem 4-1.)

The algorithm is very simple, and one can go almost directly to the algorithm in Section 10, to check if some irreducible unramified representation given in terms of Satake parameters is unitarizable (Definition 5-13 is relevant for the algorithm). In Section 12 we give examples of the use of this algorithm. The algorithm has ten steps some of them quite easy, but usually only a few of them enter the test (see Section 12). It would be fairly easy to write a computer program, possible to handle classical groups of ranks exceeding tens of thousands, for determining unitarizability in terms of Satake parameters.

Finally, we come to problem (3). In Section 3 we show that $\mathrm{Irr}^{u,unr}(S_n)$ is naturally homeomorphic to a compact subset of the complex manifold consisting of all Satake parameters for S_n (see Theorem 3-5 and Theorem 3-7). The results of this section almost directly follow from [**T6**]. In Section 11 we determine the isolated points in $\mathrm{Irr}^{u,unr}(S_n)$. To describe the result, we introduce more notation. Let $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ be a strongly negative representation. Let $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. Then we write $\mathrm{Jord}(\sigma)_\chi$ for the set of l such that $(l, \chi) \in \mathrm{Jord}(\sigma)$. If $a \in \mathrm{Jord}(\sigma)_\chi$ is not the minimum, then we write a_- for the greatest $b \in \mathrm{Jord}(\sigma)_\chi$ such that $b < a$. We have the following:

$$a - a_- \text{ is even (whenever } a_- \text{ is defined).}$$

Now, we are ready to state the main result of Section 11. It is the following theorem (see Theorem 11-3):

THEOREM 0-9. *A representation* $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ *is isolated if and only if* σ *is strongly negative, and for every* $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ *such that* $\mathrm{Jord}(\sigma)_\chi \neq \emptyset$ *the following holds:*

(1) $a - a_- \geq 4$*, for all* $a \in \mathrm{Jord}(\sigma)_\chi$ *whenever* a_- *is defined.*
(2) *If* $\mathrm{Jord}_\chi \neq \{1\}$*, then* $\min \mathrm{Jord}_\chi \setminus \{1\} \geq 4$.

(We do not claim that $1 \in \mathrm{Jord}_\chi$ in (2). If $1 \notin \mathrm{Jord}_\chi$, then (2) claims that $\min \mathrm{Jord}_\chi \geq 4$.)

Since $\mathrm{Irr}^{u,unr}(S_n)$ is an open subset of $\mathrm{Irr}^{u}(S_n)$, the above theorem also classifies the isolated representations in the whole unitary dual $\mathrm{Irr}^{u}(S_n)$, which are unramified.

As an example, let $S_1 = \mathrm{Sp}(2,F) = \mathrm{SL}(2,F)$. Then the trivial representation $\mathbf{1}_{SL(2,F)}$ is strongly negative and $\mathrm{Jord}(\mathbf{1}_{\mathrm{SL}(2,F)}) = \{(3, \mathbf{1}_{F^\times})\}$. As is well–known, it is not isolated and this theorem confirms that. Let $n \geq 2$ and let $S_n = \mathrm{Sp}(2n,F)$. Then the trivial representation $\mathbf{1}_{\mathrm{Sp}(2n,F)}$ is strongly negative and $\mathrm{Jord}(\mathbf{1}_{\mathrm{Sp}(2n,F)}) = \{(2n+1, \mathbf{1}_{F^\times})\}$. Clearly, $\mathbf{1}_{\mathrm{Sp}(2n,F)}$ is isolated (as it is well–known from [**K**]). We may consider the degenerate case $n = 0$. Then $\mathrm{Sp}(0,F)$ is the trivial group and $\mathbf{1}_{Sp(0,F)}$ is its trivial representation. It is reasonable to call such representation strongly negative and let $\mathrm{Jord}(\mathbf{1}_{\mathrm{Sp}(0,F)}) = \{(1, \mathbf{1}_{F^\times})\}$. Apart from that case, one always has $\mathrm{Jord}_\chi \neq \{1\}$.

Similarly, if we let $S_n = \mathrm{SO}(2n+1,F)$ $(n > 0)$, then $\mathbf{1}_{\mathrm{SO}(2n+1,F)}$ is strongly negative. We have $\mathrm{Jord}(\mathbf{1}_{\mathrm{SO}(2n+1,F)}) = \{(2n, \mathbf{1}_{F^\times})\}$. As is well–known, it is not isolated for $n = 1$ and this theorem confirms that. It is isolated for $n \geq 2$ (as it is well–known from [**K**]).

We close this introduction with several comments. First recall that in [**BbMo2**], D. Barbasch and A. Moy address the first of the three problems that we consider in our paper. Their related paper [**BbMo**] contains some very deep fundamental results on unitarizability, like the fact that the Iwahori-Matsumoto involution preserves unitarity in the Iwahori fixed vector case. Their Hecke algebra methods are opposite to our methods. Their approach is based on a careful study of the internal structure of representations on Iwahori fixed vectors, based on the Kazhdan-Lusztig theory [**KLu**]. Their main result –Theorem A on page 23 of [**BbMo2**]– states that a parameter of any irreducible unitarizable unramified representation of a classical group is a "complementary series from an induced from a tempered representation tensored with a GL-complementary series". In other words, that it comes from a complementary series starting with a representation induced by a negative representation (from Theorem 0-4) tensored with a GL-complementary series. They do not determine parameters explicitly (they observe that "the parameters are hard to describe explicitly"; see page 23 of their paper). They get the unitarizability of negative representations by local (Hecke algebra) methods, but do not relate them to the automorphic spectrum like Theorem 0-4 does. Summing up without going into the details, the description in [**BbMo2**] partially covers Theorem 0-8.

There are complementary series in a number of cases, but their paper does not give the full picture: the explicit parameterization of the unramified unitary duals. In [**Bb**] there is also a description of unramified unitary dual (see the very beginning of that paper for more precise description of the content of that paper). On http:/www.liegroups.org there is an implementation of an algorithm for unitarity based on an earlier version of Barbasch's paper [**Bb**] (one can find more information regarding this on that site).

Our quite different approach gives explicit parameters of different type, and these explicit parameters have a relatively simple combinatorial description. We observe that all that we need for describing unitary duals are one-dimensional unramified unitary characters of general linear groups, parabolic induction, and taking irreducible unramified subquotients.

Let us note that except motivations coming directly from the theory of automorphic forms, like applications to analytic properties of L-functions, etc., a motivation for us to get explicit classification of unramified unitary duals was to be able to answer question (2) (which would be definitely important also for the study of automorphic forms), and to classify isolated points in (3) (which can be significant in the study of automorphic spectra).

Recall that in two important cases where the unitarizability is understood (the case of general linear groups and the case of generic representations of quasi-split classical groups), the classification theorem is uniform for Archimedean case as well as for the non-Archimedean case (see [**T3**], [**T2**] and [**LMT**]). Moreover, the proofs are essentially the same (not only analogous). Therefore, it is natural to expect this to be the case for unramified unitarizable representations of classical split groups. Having this in mind, we shall comment briefly the Archimedean case. Assume $F = \mathbb{R}$ or $F = \mathbb{C}$. Let $|\ |$ be the ordinary absolute value (resp., square of it) if $F = \mathbb{R}$ (resp., $F = \mathbb{C}$), i.e., the modulus character of F (like in the non-Archimedean case). If we fix some suitable maximal compact subgroup K of S_n, then we may consider K–spherical representations, and call them unramified. Then the above constructions and statements make sense. More precisely, define $\mathrm{Jord}_{sn}(n)$ using only $\mathbf{1}_{F^\times}$ (i.e., all unramified characters χ of $F^\times$ which satisfy $\chi = \chi^{-1}$). Call representations from $\{\sigma(\mathrm{Jord});\ \mathrm{Jord} \in \mathrm{Jord}_{sn}\}$ strongly negative (define $\sigma(\mathrm{Jord})$ in the same way as in the non-Archimedean case). Define negative representations as those which come as irreducible unramified subrepresentations of representations displayed in Theorem 0-3. Then Theorem 0-4 follows from [**M6**] (where is the uniform proof for non-Archimedean and Archimedean case). Now, it is natural to ask if Theorem 0-8 is also true in that set–up. We have not been able to check that using the results of [**Bb**]. But there are a number of facts which suggest this. The first is Theorem 0-4. Second is that number of arguments in the proof of Theorem 0-8 make sense in the Archimedean case (as in [**LMT**]). The complex case shows a particular similarity (consult [**T12**]). We expect that the approach of this paper will be extended to the Archimedean case. We also expect that Theorem 0-9 describing isolated representations holds in the Archimedean case, with a similar proof. We plan to address the Archimedean case in the future.

At the end, let us note that one possible strategy to get the answer to (1) (but not to (2) and (3)) would be to try to get Theorem 0-8 from the classification in [**LMT**], using the Barbasch-Moy fundamental result that the Iwahori-Matsumoto involution preserves unitarity in the Iwahori fixed vector case (the proof of which is based on the Kazhdan-Lusztig theory [**KLu**]). This much less direct approach would still leave a number of questions to be solved. Furthermore, we expect that the approach that we present in our paper has a much greater chance for generalization than the one that we just discussed above as was the case for general linear groups, where the classification of unramified irreducible unitary representations was first obtained using the Zelevinsky classification, which very soon led to the classification of general irreducible unitary representations in terms of the Zelevinsky, as well as the Langlands, classification.

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1. Preliminary Results

Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively.

Let F be a non–Archimedean field of characteristic different from 2. We write $\mathcal{O}$ for the maximal compact subring of F. Let $\mathfrak{p}$ the unique maximal ideal in $\mathcal{O}$. Let ϖ be a fixed generator of $\mathfrak{p}$ and let q be the number of elements in the corresponding residue field of $\mathcal{O}$. We write ν for the normalized absolute value of F. Let χ be a character of $F^\times$. We can uniquely write $\chi = \nu^{e(\chi)}\chi^u$ where χ^u is a unitary character and $e(\chi) \in \mathbb{R}$.

Let G be an l–group (see [**BeZ**]). We will consider smooth representations of G on complex vector spaces. We simply call them representations. If σ is a representation of G, then we write V_σ for its space. Its contragredient representation is denoted by $\widetilde{\sigma}$ and the corresponding non–degenerate canonical pairing by $\langle\ ,\ \rangle : V_{\widetilde{\sigma}} \times V_\sigma \to \mathbb{C}$. If σ_1 and σ_2 are representations of G, then we write $\mathrm{Hom}_G(\sigma_1, \sigma_2)$ for the space of all G–intertwining maps $\sigma_1 \to \sigma_2$. We say that σ_1 and σ_2 are equivalent, $\sigma_1 \simeq \sigma_2$, if there is a bijective $\varphi \in \mathrm{Hom}_G(\sigma_1, \sigma_2)$. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible admissible representations of G. Let $\mathrm{R}(G)$ be the Grothendieck group of the category $\mathcal{M}_{adm.fin.leng.}(G)$ of all admissible representations of finite length of G. If σ is an object of $\mathcal{M}_{adm.fin.leng.}(G)$, then we write $s.s.(\sigma)$ for its semi–simplification in $\mathrm{R}(G)$. Frequently, in computations we simply write σ instead of $s.s.(\sigma)$. If G is the trivial group, then we write its unique irreducible representation as $\mathbf{1}$.

Next, we fix the notation for the general linear group $\mathrm{GL}(n, F)$. Let I_n the identity matrix in $\mathrm{GL}(n, F)$. Let ${}^t g$ be the transposed matrix of $g \in \mathrm{GL}(n, F)$. The transposed matrix of $g \in GL(n, F)$ with respect to the second diagonal will be denoted by ${}^{\tau} g$. If χ is a character of $F^\times$ and π is a representation of $GL(n, F)$, then the representation $(\chi \circ \det) \otimes \pi$ of $GL(n, F)$ will be written as $\chi\pi$.

We fix the minimal parabolic subgroup $P_{min}^{GL_n}$ of $\mathrm{GL}(n, F)$ consisting of all upper triangular matrices in $\mathrm{GL}(n, F)$. A standard parabolic subgroup P of $\mathrm{GL}(n, F)$ is a parabolic subgroup containing $P_{min}^{GL_n}$. There is a one–to–one correspondence between the set of all ordered partitions α of n, $\alpha = (n_1, \ldots, n_k)$ ($n_i \in \mathbb{Z}_{>0}$), and the set of standard parabolic subgroups of $\mathrm{GL}(n, F)$, attaching to a partition α the parabolic subgroup P_α consisting of all block–upper triangular matrices:

$$p = (p_{ij})_{1 \le i,j \le k},\ p_{ij} \text{ is an } n_i \times n_j\text{–matrix},\ \ p_{ij} = 0\ (i > j).$$

The parabolic subgroup P_α admits a Levi decomposition $P_\alpha = M_\alpha N_\alpha$, where

$$\begin{aligned} M_\alpha &= \{\mathrm{diag}(g_1, \ldots, g_k);\ g_i \in \mathrm{GL}(n_i, F)\ (1 \le i \le k)\} \\ N_\alpha &= \{p \in P_\alpha;\ p_{ii} = I_{n_i}\ (1 \le i \le k)\}. \end{aligned}$$

Let π_i be a representation of $\mathrm{GL}(n_i, F)$ $(1 \le i \le k)$. Then we consider $\pi_1 \otimes \cdots \otimes \pi_k$ as a representation of M_α as usual:

$$\pi_1 \otimes \cdots \otimes \pi_k(\mathrm{diag}(g_1, \ldots, g_k)) = \pi_1(g_1) \otimes \cdots \otimes \pi_k(g_k),$$

and extend it trivially across N_α to the representation of P_α denoted by the same letter. Then we form (normalized) induction written as follows (see [**BeZ1**], [**Ze**]):

$$\pi_1 \times \cdots \times \pi_k = i_{n,\alpha}(\pi_1 \otimes \cdots \otimes \pi_k) := \mathrm{Ind}_{P_\alpha}^{\mathrm{GL}(n,F)}(\pi_1 \otimes \cdots \otimes \pi_k)$$

In this way obtain the functor $\mathcal{M}_{adm.fin.leng.}(M_\alpha) \xrightarrow{i_{n,\alpha}} \mathcal{M}_{adm.fin.leng.}(\mathrm{GL}(n, F))$ and a group homomorphism $\mathrm{R}(M_\alpha) \xrightarrow{i_{n,\alpha}} \mathrm{R}(\mathrm{GL}(n, F))$. Next, if π is a representation of $\mathrm{GL}(n, F)$, then we form the normalized Jacquet module $r_{\alpha,n}(\pi)$ of π (see [**BeZ1**]). It is a representation of M_α. In this way we obtain a functor $\mathcal{M}_{adm.fin.leng.}(\mathrm{GL}(n, F)) \xrightarrow{r_{\alpha,n}} \mathcal{M}_{adm.fin.leng.}(M_\alpha)$ and a group homomorphism $\mathrm{R}(\mathrm{GL}(n, F)) \xrightarrow{r_{\alpha,n}} \mathrm{R}(M_\alpha)$. The functors $i_{n,\alpha}$ and $r_{\alpha,n}$ are related by the Frobenius reciprocity:

$$\mathrm{Hom}_{\mathrm{GL}(n,F)}(\pi,\ i_{n,\alpha}(\pi_1 \otimes \cdots \otimes \pi_k)) \simeq \mathrm{Hom}_{M_\alpha}(r_{\alpha,n}(\pi),\ \pi_1 \otimes \cdots \otimes \pi_k).$$

We list some additional basic properties of induction:

$$\pi_1 \times (\pi_2 \times \pi_3) \simeq (\pi_1 \times \pi_2) \times \pi_3,$$
$$\pi_1 \times \pi_2 \text{ and } \pi_2 \times \pi_1 \text{ have the same composition series},$$
$$\text{if } \pi_1 \times \pi_2 \text{ is irreducible, then } \pi_1 \times \pi_2 \simeq \pi_2 \times \pi_1,$$
$$\chi(\pi_1 \times \pi_2) \simeq (\chi\pi_1) \times (\chi\pi_2), \qquad \text{for a character } \chi \text{ of } F^\times,$$
$$\widetilde{\pi_1 \times \pi_2} \simeq \tilde{\pi}_1 \times \tilde{\pi}_2.$$

We take $\mathrm{GL}(0, F)$ to be the trivial group (we consider formally the unique element of this group as a 0×0 matrix and the determinant map $\mathrm{GL}(0, F) \to F^\times$). We extend $\times$ formally as follows: $\pi \times \mathbf{1} = \mathbf{1} \times \pi := \pi$ for every representation π of $\mathrm{GL}(n, F)$. The listed properties hold in this extended setting. We also let $r_{(0),\ 0}(\mathbf{1}) = \mathbf{1}$.

Now, we fix the basic notation for the split classical groups. Let

$$J_n = \begin{bmatrix} 00 & \ldots & 01 \\ 00 & \ldots & 10 \\ \vdots & & \\ 10 & \ldots & 0 \end{bmatrix} \in \mathrm{GL}(n, F).$$

The symplectic group (of rank $n \ge 1$) is defined as follows:

$$\mathrm{Sp}(2n, F) = \left\{ g \in \mathrm{GL}(2n, F);\ g \cdot \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \cdot {}^t g = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\}.$$

Next, the split orthogonal groups special odd-orthogonal groups (both of rank $n \geq 1$) are defined by

$$\begin{aligned} \mathrm{SO}(n, F) &= \{g \in \mathrm{SL}(n, F);\ g \cdot J_n \cdot {}^t g = J_n\} \\ \mathrm{O}(n, F) &= \{g \in \mathrm{GL}(n, F);\ g \cdot J_n \cdot {}^t g = J_n\} \end{aligned}$$

We take $\mathrm{Sp}(0, F), \mathrm{SO}(0, F), \mathrm{O}(0, F)$ to be the trivial groups (we consider their unique element formally as a 0×0 matrix). In the sequel, we fix one of the following three series of the groups:

$$\begin{aligned} S_n &= \mathrm{Sp}(2n, F),\ n \geq 0 \\ S_n &= \mathrm{O}(2n, F),\ n \geq 0 \\ S_n &= \mathrm{SO}(2n+1, F),\ n \geq 0. \end{aligned}$$

Let $n > 0$. Then the minimal parabolic subgroup $P_{\min}^{S_n}$ of S_n consisting of all upper triangular matrices is fixed. A standard parabolic subgroup P of S_n is a parabolic subgroup containing $P_{min}^{S_n}$. There is a one–to–one correspondence between the set of all finite sequences of positive integers of total mass $\leq n$ and the set of standard parabolic subgroups of S_n defined as follows. For $\alpha = (m_1, \ldots, m_k)$ of total mass $m := \sum_{i=1}^{l} m_i \leq n$, we let

$$P_\alpha^{S_n} := \begin{cases} P_{(m_1, \ldots, m_k,\ 2(n-m),\ m_k, m_{k-1}, \ldots, m_1)} \cap S_n;\ S_n = \mathrm{Sp}(2n, F),\ \mathrm{O}(2n, F) \\ P_{(m_1, \ldots, m_k,\ 2(n-m)+1,\ m_k, m_{k-1}, \ldots, m_1)} \cap S_n;\ S_n = \mathrm{SO}(2n+1, F). \end{cases}$$

(The middle term $2(n-m)$ is omitted if $m = n$.) The parabolic subgroup $P_\alpha^{S_n}$ admits a Levi decomposition $P_\alpha = M_\alpha^{S_n} N_\alpha^{S_n}$, where

$$\begin{aligned} M_\alpha^{S_n} &= \{\mathrm{diag}(g_1, \ldots, g_k, g,\ {}^\tau g_k^{-1}, \ldots, {}^\tau g_1^{-1});\ g_i \in \mathrm{GL}(m_i, F)\ (1 \leq i \leq k),\ g \in S_{n-m}\} \\ N_\alpha^{S_n} &= \{p \in_\alpha^{S_n};\ p_{ii} = I_{n_i}\ \forall i\} \end{aligned}$$

Let π_i be a representation of $\mathrm{GL}(n_i, F)$ $(1 \leq i \leq k)$. Let σ be a representation of S_{n-m}. Then we consider $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$ as a representation of $M_\alpha^{S_n}$ as usual:

$$\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma(\mathrm{diag}(g_1, \ldots, g_k, g,\ {}^\tau g_k^{-1}, \ldots, {}^\tau g_1^{-1})) = \pi_1(g_1) \otimes \cdots \otimes \pi_k(g_k) \otimes \sigma(g),$$

and extend it trivially across $N_\alpha^{S_n}$ to the representation of $P_\alpha^{S_n}$ denoted by the same letter. Then we form (normalized) induction written as follows (see [**T9**]):

$$\pi_1 \times \cdots \times \pi_k \rtimes \sigma = \mathrm{I}_{n,\alpha}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma) := \mathrm{Ind}_{P_\alpha^{S_n}}^{S_n}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma)$$

In this way obtain a functor $\mathcal{M}_{adm.fin.leng.}(M_\alpha^{S_n}) \xrightarrow{I_{n,\alpha}} \mathcal{M}_{adm.fin.leng.}(S_n)$ and a group homomorphism $\mathrm{R}(M_\alpha^{S_n}) \xrightarrow{I_{n,\alpha}} \mathrm{R}(S_n)$. Next, if π is a representation of S_n, then we form the normalized Jacquet module $\mathrm{Jacq}_{\alpha,n}(\pi)$ of π. It is a representation of $M_\alpha^{S_n}$. In this way obtain a functor $\mathcal{M}_{adm.fin.leng.}(S_n) \xrightarrow{\mathrm{Jacq}_{\alpha,n}} \mathcal{M}_{adm.fin.leng.}(M_\alpha^{S_n})$ and a group homomorphism $\mathrm{R}(S_n) \xrightarrow{\mathrm{Jacq}_{\alpha,n}} \mathrm{R}(M_\alpha^{S_n})$. Here Frobenius reciprocity implies

$$\mathrm{Hom}_{S_n}(\pi,\ \mathrm{I}_{n,\alpha}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma)) \simeq \mathrm{Hom}_{M_\alpha^{S_n}}(\mathrm{Jacq}_{\alpha,n}(\pi),\ \pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma).$$

Further

$$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \simeq (\pi_1 \times \pi_2) \rtimes \sigma,$$
$$\widetilde{\pi \rtimes \sigma} \simeq \widetilde{\pi} \rtimes \widetilde{\sigma}$$
$$\pi \rtimes \sigma \text{ and } \tilde{\pi} \rtimes \sigma \text{ have the same composition series,}$$
$$\text{if } \pi \rtimes \sigma \text{ is irreducible, then } \pi \rtimes \sigma \simeq \tilde{\pi} \rtimes \sigma.$$

We remark that the third listed property follows, for example, from the general result ([**BeDeK**], Lemma 5.4), but in our case there is a proof that is simpler and based on the following result of Waldspurger (see [**MœViW**]):

$$\begin{aligned} &\widetilde{\sigma} \simeq \sigma, \quad && S_n = \mathrm{SO}(2n+1, F),\ O(2n, F) \\ &\widetilde{\sigma} \simeq \sigma^x, && S_n = \mathrm{Sp}(2n, F), \end{aligned}$$

where $x \in GL(2n, F)$ satisfies $x \cdot \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \cdot {}^t x = (-1) \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}$, and $\sigma^x(g) = \sigma(x^{-1}gx)$, $g \in S_n$. Now, the stated property is obvious for $S_n = \mathrm{SO}(2n+1, F)$ and $O(2n, F)$. Let $S_n = \mathrm{Sp}(2n, F)$. Then

$$(\pi \rtimes \sigma)^y \simeq \pi \rtimes \sigma^x,$$

where $y = \mathrm{diag}(I_m, x, I_m)$ (π is a representation of $GL(m, F)$). If $\pi \rtimes \widetilde{\sigma} = \sum m_\rho \rho$ is a decomposition into irreducible representations in $\mathrm{R}(S_n)$, then $\widetilde{\pi \rtimes \widetilde{\sigma}} = \sum m_\rho \widetilde{\rho}$, and, by a result of Waldspurger, we have the following:

$$\pi \rtimes \sigma = \pi \rtimes \widetilde{\sigma}^x = (\pi \rtimes \widetilde{\sigma})^y = \sum m_\rho \rho^y = \sum m_\rho \widetilde{\rho} = \widetilde{\pi \rtimes \widetilde{\sigma}} = \widetilde{\pi} \rtimes \sigma. \tag{1-1}$$

In this paper we work mostly with unramified representations. Let $n \geq 1$. If G is one of the groups $\mathrm{GL}(n, F)$, $\mathrm{Sp}(2n, F)$, $\mathrm{O}(2n, F)$, or $\mathrm{SO}(2n+1, F)$, then we let K be its maximal compact subgroup of the form $\mathrm{GL}(n, \mathcal{O})$, $\mathrm{Sp}(2n, \mathcal{O})$, $\mathrm{O}(2n, \mathcal{O})$, or $\mathrm{SO}(2n+1, \mathcal{O})$, respectively. We say that $\sigma \in \mathrm{Irr}(G)$ is unramified if it has a non–zero vector invariant under K. Unramified representations of G are classified using the Satake classification.

To explain the Satake classification, we let $P_{\min} = M_{\min} N_{\min}$ be the minimal parabolic subgroup of G as described above:

$$\begin{aligned} M_{\min} &= \{\mathrm{diag}(x_1, \ldots, x_n)\}; && G = \mathrm{GL}(n, F) \\ M_{\min} &= \{\mathrm{diag}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1})\}; && G = \mathrm{Sp}(2n, F), \mathrm{O}(2n, F) \\ M_{\min} &= \{\mathrm{diag}(x_1, \ldots, x_n, 1, x_n^{-1}, \ldots, x_1^{-1})\}; && G = \mathrm{SO}(2n+1, F). \end{aligned}$$

Let $W = N_G(M_{\min})/M_{\min}$ be the Weyl group of G. It acts on $M_{\min}$ by conjugation: $w.m = wmw^{-1}$, $w \in W$, $m \in M_{\min}$. This action extends to an action on the characters χ of $M_{\min}$ in the usual way: $w(\chi)(m) = \chi(w^{-1}mw)$, $w \in W$, $m \in M_{\min}$.

Explicitly, using the above description of $M_{\min}$, we fix the isomorphism $M_{\min} \simeq (F^\times)^n$ (considering only the first n–coordinates). If $G = \mathrm{GL}(n, F)$, then W acts on $M_{\min}$ as the group of permutations of n–letters. If G is one of the groups $\mathrm{Sp}(2n, F)$, $\mathrm{O}(2n, F)$, or $\mathrm{SO}(2n+1, F)$, then W acts on $M_{\min}$ as a group generated by the group of the permutations of n–letters and the following transformation:

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1^{-1}, x_2, \ldots, x_n).$$

We have the following classification result (see [**Cr**]; [**R**] for $\mathrm{O}(2n, F)$):

Theorem 1-2. (i) *Let $\chi_1, \dots, \chi_n$ be a sequence of unramified characters of $F^\times$. Then the induced representation* $\mathrm{Ind}_{P_{\min}}^G(\chi_1 \otimes \cdots \otimes \chi_n)$ *contains a unique unramified irreducible subquotient, denoted $\sigma^G(\chi_1, \dots, \chi_n)$.*

(ii) *Assume that $\chi_1, \dots, \chi_n$ and $\chi'_1, \dots, \chi'_n$ are two sequences of unramified characters of $F^\times$. Then $\sigma^G(\chi_1, \dots, \chi_n) \simeq \sigma^G(\chi'_1, \dots, \chi'_n)$ if and only if there is $w \in W$ such that $\chi'_1 \otimes \cdots \otimes \chi'_n = w(\chi_1 \otimes \cdots \otimes \chi_n)$. In another words, if and only if there is a permutation α of and a sequence $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ such that $\chi'_i = \chi_{\alpha(i)}^{\epsilon_i}$, $i = 1, \dots, n$. ($\epsilon_1 = 1, \dots, \epsilon_n = 1$ for $G = \mathrm{GL}(n, F)$.)*

(iii) *Assume that $\sigma \in \mathrm{Irr}(G)$ is an unramified representation. Then there exists a sequence $(\chi_1, \dots, \chi_n)$ of unramified characters of $F^\times$ such that $\sigma \simeq \sigma^G(\chi_1, \dots, \chi_n)$. Every such sequence we call a supercuspidal support of σ.*

We let $\mathrm{Irr}^{unr}(G)$ be the set of equivalence classes of irreducible unramified representations of G. We consider the trivial representation of the trivial group to be unramified. We let

$$\begin{cases} \mathrm{Irr}^{unr}(\mathrm{GL}) = \cup_{n \geq 0} \mathrm{Irr}^{unr}(GL(n, F)) \\ \mathrm{Irr}^{unr}(S) = \cup_{n \geq 0} \mathrm{Irr}^{unr}(S_n). \end{cases} \tag{1-3}$$

There is another more precise classification of the elements of $\mathrm{Irr}^{unr}(\mathrm{GL})$ that we describe (see [**Ze**]).

In order to write down the Zelevinsky classification we introduce some notation. Let χ be an unramified character of $F^\times$, and let $n_1, n_2 \in \mathbb{R}$, $n_2 - n_1 \in \mathbb{Z}_{\geq 0}$. We denote by

$$[\nu^{n_1}\chi, \nu^{n_2}\chi] \qquad \text{or} \qquad [n_1, n_2]^{(\chi)}$$

the set $\{\nu^{n_1}\chi, \nu^{n_1+1}\chi, \dots, \nu^{n_2}\chi\}$, and call it a segment of unramified characters. To such a segment $\Delta = [\nu^{n_1}\chi, \nu^{n_2}\chi]$, Zelevinsky has attached a representation which is the unique irreducible subrepresentation of $\nu^{n_1}\chi \times \nu^{n_1+1}\chi \times \cdots \times \nu^{n_2}\chi$. This representation is the character

$$\nu^{(n_1+n_2)/2}\chi \, \mathbf{1}_{GL(n_2-n_1+1, F)}. \tag{1-4}$$

We find it convenient to write it as follows:

$$\langle \Delta \rangle \qquad \text{or} \qquad \langle [n_1, n_2]^{(\chi)} \rangle. \tag{1-5}$$

We let

$$e(\Delta) = e([n_1, n_2]^{(\chi)}) = (n_1 + n_2)/2 + e(\chi). \tag{1-6}$$

Related to Theorem 1-2, we see

$$\langle \Delta \rangle = \langle [n_1, n_2]^{(\chi)} \rangle = \sigma^{GL(n_1+n_2+1, F)}(\nu^{n_1}\chi, \nu^{n_1+1}\chi, \dots, \nu^{n_2}\chi).$$

The segments Δ_1 and Δ_2 of unramified characters are called linked if and only $\Delta_1 \cup \Delta_2$ is a segment but $\Delta_1 \not\subset \Delta_2$ and $\Delta_2 \not\subset \Delta_1$. We consider the empty set as a segment of unramified characters. It is not linked to any other segment. We let

$$\langle \emptyset \rangle = \mathbf{1} \in \mathrm{Irr}\ \mathrm{GL}(0, F).$$

Now, we give the Zelevinsky classification.

Theorem 1-7. (i) *Let $\Delta_1, \dots, \Delta_k$ be a sequence of segments of unramified characters. Then $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle$ is reducible if and only there are indices i, j, such that the segments Δ_i and Δ_j are linked. Moreover, if $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle$ is irreducible, then it belongs to $\mathrm{Irr}^{unr}(\mathrm{GL})$.*

(ii) *Conversely, if* $\sigma \in \mathrm{Irr}^{unr}(\mathrm{GL})$*, then there is, up to a permutation, a unique sequence of segments of unramified characters* $\Delta_1, \dots, \Delta_k$ *such that* $\sigma \simeq \langle\Delta_1\rangle \times \cdots \times \langle\Delta_k\rangle$.

A similar classification exists in the case of the classical groups (see [**M4**]). We recall it in Sections 5 and 6. We end this section with the following remark:

REMARK 1-8. *It follows from Theorem 1-2 (ii) that every unramified representation* $\sigma \in \mathrm{Irr}^{unr}(S)$ *is self–dual. Also, if* $\pi \in \mathrm{Irr}^{unr}(\mathrm{GL})$*, then there exists a unique unramified irreducible subquotient, say* σ_1 *of* $\pi \rtimes \sigma$*. The representation* σ_1 *is self–dual, and it is also a subquotient of* $\widetilde{\pi} \rtimes \sigma$*. (See the basic properties for the induction for the split classical groups listed above.)*

2. Some General Results on Unitarizability

Let G be a connected reductive p–adic group or $\mathrm{O}(2n, F)$ $(n \geq 0)$. We recall that the contragredient representation π of G is denoted by $\widetilde{\pi}$. We write $\bar{\pi}$ for the complex conjugate representation of the representation π. We remind the reader that this means the following: In the representation space V_π we change the multiplication to $\alpha._{new} v := \bar{\alpha}._{old} v$, $\alpha \in \mathbb{C}$, $v \in V_\pi$. In this way we obtain $V_{\bar{\pi}}$. We let $\bar{\pi}(g)v = \pi(g)v$, $g \in G$, $v \in V_{\bar{\pi}} = V_\pi$. It is easy to see the following:

$$\widetilde{\bar{\pi}} \simeq \bar{\widetilde{\pi}}.$$

The Hermitian contragredient of the representation of π is defined as follows:

$$\pi^+ := \widetilde{\bar{\pi}}.$$

Let $P = MN$ be a parabolic subgroup of G. We have the following:

(H-IC) $\mathrm{Ind}_P^G(\sigma)^+ \simeq \mathrm{Ind}_P^G(\sigma^+)$.

A representation $\pi \in \mathrm{Irr}(G)$ is said to be Hermitian if there is a non–degenerate G–invariant Hermitian form $\langle\ ,\ \rangle$ on V_π. This means the following:

$$\langle \alpha v + \beta w, u\rangle = \alpha\langle v, u\rangle + \beta\langle w, u\rangle \tag{2-1}$$

$$\overline{\langle v, w\rangle} = \langle w, v\rangle \tag{2-2}$$

$$\langle \pi(g)v, \pi(g)v\rangle = \langle v, w\rangle, \tag{2-3}$$

$$\langle v, w\rangle = 0,\ \forall w \in V_\pi,\ \text{implies } v = 0. \tag{2-4}$$

Since π is irreducible, the Hermitian form $\langle\ ,\ \rangle$ is unique up to a non–zero real scalar. Let $\mathrm{Irr}^+(G)$ be the set of equivalence classes of irreducible Hermitian representations of G. Since we work with unramified representations, we let

$$\begin{cases} \mathrm{Irr}^{+,unr}(\mathrm{GL}) = \cup_{n\geq 0}\mathrm{Irr}^{+,unr}(GL(n,F)) \\ \mathrm{Irr}^{+,unr}(S) = \cup_{n\geq 0}\mathrm{Irr}^{+,unr}(S_n). \end{cases} \tag{2-5}$$

We list the following basic properties of Hermitian representations:

(H-Irr) If $\pi \in \mathrm{Irr}(G)$, then $\pi \in \mathrm{Irr}^+(G)$ if and only if $\pi \simeq \pi^+$

(H-Ind) Let $P = MN$ be a parabolic subgroup of G. Let $\sigma \in \mathrm{Irr}^+(M)$. Then there is a non–trivial G–invariant Hermitian form on $\mathrm{Ind}_P^G(\sigma)$. In particular, if $\mathrm{Ind}_P^G(\sigma)$ is irreducible, then it is Hermitian.

In addition, a Hermitian representation $\pi \in \mathrm{Irr}(G)$ is said to be unitarizable if the form $\langle\ ,\ \rangle$ is definite. Let $\mathrm{Irr}^u(G)$ be the set of equivalence classes of irreducible unitarizable representations of G. We have the following:

$$\mathrm{Irr}^u(G) \subset \mathrm{Irr}^+(G) \subset \mathrm{Irr}(G).$$

In this paper we classify unramified unitarizable representations (see (1-3))

$$\begin{cases} \mathrm{Irr}^{u,unr}(\mathrm{GL}) = \cup_{n\geq 0}\mathrm{Irr}^{u,unr}(GL(n,F)) \\ \mathrm{Irr}^{u,unr}(S) = \cup_{n\geq 0}\mathrm{Irr}^{u,unr}(S_n). \end{cases} \tag{2-6}$$

Now, we recall some principles used in the construction and classification of unitarizable unramified representations. Some of them are already well–known (see [**T8**]), but some of them are new (see (NU-RP)). Below, $P = MN$ denotes a parabolic subgroup of G and σ an irreducible representation of M.

(UI) **Unitary parabolic induction**: the unitarizability of σ implies that the parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ is unitarizable.

(UR) **Unitary parabolic reduction**: if σ is a Hermitian representation such that the parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ is irreducible and unitarizable, then σ is an (irreducible) unitarizable representation.

(D) **Deformation (or complementary series)**: let X be a connected set of characters of M such that each representation $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in X$, is Hermitian and irreducible. Now, if $\mathrm{Ind}_P^G(\chi_0\sigma)$ is unitarizable for some $\chi_0 \in X$, then the whole family $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in X$, consists of unitarizable representations.

(ED) **Ends of deformations**: suppose that Y is a set of characters of M, and X a dense subset of Y satisfying (D); then each irreducible subquotient of any $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in Y$, is unitarizable.

Sometimes we get important irreducible unitarizable representations in the following way. Let Z denote the center of G. Let k be a global field, P_k the set of places of k, k_v the completion of k at the place v, $\mathbb{A}_k$ the ring of adeles of k, ω a unitary character of $Z(\mathbb{A}_k)$ and $L^2(\omega, G(k)\backslash G(\mathbb{A}_k))$ the representation of $G(\mathbb{A}_k) \simeq \otimes_{v\in P_k} G(k_v)$ by right translations on the space of square integrable functions on $G(\mathbb{A}_k)$ which transform under the action of $Z(\mathbb{A}_k)$ according to ω. Suppose that π is an irreducible representation of $G = G(F)$.

(RS) Residual automorphic spectrum factors: if $F \simeq k_v$ for some global field k and $v \in P_k$, and there exists an irreducible (non-cuspidal) subrepresentation Π of $L^2(\omega, G(\mathbb{A}_k))$ such that π is isomorphic to a (corresponding) tensor factor of Π, then π is unitarizable.

It is evident that π as above is unitarizable. But this construction is technically much more complicated than the above four. It requires computation of residues of Eisenstein series.

The last principle is not necessary to use for the classification of $\mathrm{Irr}^{u,unr}(\mathrm{GL})$, but we use it in [**M6**] in order to prove the unitarity of "basic building blocks" of $\mathrm{Irr}^{u,unr}(S)$. (See Theorem 5-11 in Section 5.)

In addition, the following simple remark is useful for proving non–unitarity. Obviously, the Cauchy-Schwartz inequality implies that matrix coefficients of unitarizable representations are bounded. Now, (D) directly implies the following:

Remark 2-7. ***(unbounded matrix coefficients)*** *Let X be a connected set of characters of M such that each representation $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in X$, is Hermitian and irreducible and that $Ind_P^G(\chi_0\sigma)$ has an unbounded matrix coefficient for some $\chi_0 \in X$. Then all $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in X$, are non–unitarizable.*

In addition, we use the following two criteria for proving non–unitarity. The criteria are very technical. In a special case they were already applied in [**LMT**]. We present them here in a more general form. Let $P = MN$ be a maximal parabolic subgroup of G. Assume that the Weyl group $W(M) = N_G(M)/M$ has two elements. (It always has one or two elements.) We write w_0 for a representative of the nontrivial element in $W(M)$. Assume that $\sigma \in \mathrm{Irr}(M)$ is an irreducible unitarizable representation such that $w_0(\sigma) \simeq \sigma$. Then there is a standard normalized intertwining operator (at least in the cases that we need) $N(\delta_P^s\sigma) : \mathrm{Ind}_P^G(\delta_P^s\sigma) \to \mathrm{Ind}_P^G(\delta_P^{-s}\sigma)$. We have the following:

(N-1) $N(\delta_P^s\sigma)N(\delta_P^{-s}\sigma) = id$

(N-2) $N(\delta_P^s\sigma)$ is Hermitian, and therefore holomorphic, for $s \in \sqrt{-1}\mathbb{R}$.

Let $\langle\ ,\ \rangle_\sigma$ be M–invariant definite Hermitian form on V_σ. Then

$$\langle f_1, f_2\rangle_s = \int_K \langle f_1(k),\ N(\delta_P^s\sigma)f_2(k)\rangle_\sigma dk \tag{2-8}$$

is a Hermitian form on $\mathrm{Ind}_P^G(\delta_P^s\pi)$. It is non–degenerate whenever $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ is irreducible and $N(\delta_P^s\sigma)$ is holomorphic. Now, we make the following two assumptions:

(A-1) If $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ is reducible at $s = 0$, then $N(\sigma)$ is non–trivial.

(A-2) If $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ is irreducible at $s = 0$, let $s_1 > 0$ be the first point of reducibility (this must exist because of Remark 2-7). We assume that $N(\delta_P^s\sigma)$ is holomorphic and non–trivial for $s \in]0,\ s_1]$. (Then (N-1) implies that $N(\delta_P^{-s}\sigma)$ is holomorphic for $s \in]0,\ s_1[$). We assume that $N(\delta_P^{-s}\sigma)$ has a pole at $s = s_1$ of an odd order.

If (A-1) holds, then (N-1) implies that $\mathrm{Ind}_P^G(\sigma)$ is a direct sum of two non–trivial (perhaps reducible) representations on which $N(\sigma)$ acts as $-id$ and id, respectively. Now, since $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ is irreducible and $N(\delta_P^s\sigma)$ is holomorphic for $s > 0$, s close to 0, we conclude that $\langle\ ,\ \rangle_s$ is not definite. Hence $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ is not unitarizable for $s > 0$, s close to 0.

If (A-2) holds, then we write k for the order of pole of $N(\delta_P^{-s}\sigma)$ at $s = s_1$. We realize the family of representations $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ ($s \in \mathbb{C}$) in the compact picture, say with space X. Let $f \in X$ such that

$$F_s := (s - s_1)^k N(\delta_P^{-s}\sigma)f \text{ is holomorphic and non–zero at } s = s_1.$$

We see that F_s is real analytic near s_1. Let $h \in X$. Using (N-1), we compute:

$$\begin{aligned}\langle h, F_s\rangle_s &= \int_K \langle h(k),\ N(\delta_P^s\pi)F_s(k)\rangle_\sigma dk \\ &= (s - s')^k \int_K \langle h(k),\ f(k)\rangle_\sigma dk.\end{aligned} \tag{2-9}$$

Now, we apply some elementary results from linear algebra (see ([**Vo**], Theorem 3.2, Proposition 3.3)) to our situation. First, we may assume that f belongs to some fixed K–isotypic component, say E, of X. Since X is an admissible representation of K, we see $\dim E < \infty$. We consider the restriction of the family of Hermitian forms $\langle\ ,\ \rangle_s$ to E. We write this restriction as $(\ ,\ )_t$, where $t = s - s_1$. Let

$$E = E^0 \supset E^1 \supset \cdots \supset E^N = \{0\}$$

be the filtration of E defined as follows. The space E^n is the space of vectors $e \in E$ for which there is a neighborhood U of 0 and a (real) analytic function $f_e : U \longrightarrow E$ satisfying

(i) $f_e(0) = e$
(ii) $\forall e' \in E$ the function $s \longmapsto (f_e(s), e')_t$ vanishes at 0 to order at least n.

Let $F = F_{s_1}$. Since $t \longmapsto F_{t+s_1}$ is a real analytic function from a neighborhood U of 0 into E, (2-9) implies $F \in E^k$. Moreover, since also $f \in E$, we see that (2-9) applied to $f = h$ implies that $F \notin E^{k+1}$. We conclude

$$E^k/E^{k+1} \neq 0. \tag{2-10}$$

Next, we define a Hermitian form $(\ ,\ )^n$ on E^n by the formula

$$(e, e')^n = \lim_{t \longrightarrow 0} \frac{1}{t^n} (f_e(s), f_{e'}(s))_s.$$

(It is easy to see that this definition is independent of the choices of f_e and $f_{e'}$.) The radical of the form $(\ ,\ )^n$ is exactly E^{n+1}. We write (p_n, q_n), for the signature on E^n/E^{n+1}. It is proved in ([**Vo**], Proposition 3.3) that for t small positive, $(\ ,\ )_t$ has a signature

$$(\sum_n p_n,\ \sum_n q_n)$$

and for t small negative

$$(\sum_{n \text{ even}} p_n + \sum_{n \text{ odd}} q_n,\ \sum_{n \text{ odd}} p_n + \sum_{n \text{ even}} q_n).$$

Now, we are ready to show the the non–unitarity of $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ for $s - s_1$ small positive. It is enough to show the Hermitian form $\langle\ ,\ \rangle_s$ is not definite.

Without a loss of generality we may assume that $\langle\ ,\ \rangle_s$ $(s \in]0, s_1[)$ is positive definite. Then it is positive definite on $\mathrm{Ind}_P^G(\delta_P^{s_1}\sigma)/\ker N(\delta_P^{(s_1)}\sigma)$. Thus, if there is unitarity immediately after s_1, then the form $\langle\ ,\ \rangle_s$ is positive definite for $s > s_1$ close to s_1. In particular, $(\ ,\ )_t$ is positive definite for $t > 0$ close to 0. Hence

$$\sum_n q_n = \sum_{n \text{ odd}} p_n + \sum_{n \text{ even}} q_n = 0.$$

Since k is odd, we see that

$$p_k = q_k = 0.$$

This contradicts (2-10). We have proved the following non–unitarity criteria:

(RP) Let $P = MN$ be a self-dual maximal parabolic subgroup of G. We write w_0 for the representative of the nontrivial element in $W(M)$. Assume that $\sigma \in \mathrm{Irr}(M)$ is an irreducible unitarizable representation such that $w_0(\sigma) \simeq \sigma$. Then we have the following:

(i) If (A-1) holds (i.e., $\mathrm{Ind}_P^G(\sigma)$ is reducible and $N(\sigma)$ is non–trivial), then $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ is not unitarizable for $s > 0$, s close to 0.

(ii) If (A-2) holds (i.e., $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ is irreducible at $s=0$, $s_1>0$ is the first reducibility point, $N(\delta_P^s\sigma)$ is holomorphic and non–trivial for $s\in]0,\ s_1]$ and $N(\delta_P^{-s}\sigma)$ has a pole at $s=s_1$ of an odd order), then $\mathrm{Ind}_P^G(\delta_P^s\sigma)$ is not unitarizable for $s>s_1$, s close to s_1.

3. The Topology of the Unramified Dual

Let G be a connected reductive p–adic group or $\mathrm{O}(2n,F)$ ($n\geq 0$). The topology on the non–unitary dual $\mathrm{Irr}(G)$ is given by the uniform convergence of matrix coefficients on compact sets ([**T1**], [**T6**]; see also [**F**], [**Di**]). Then $\mathrm{Irr}^u(G)$ is closed subset in $\mathrm{Irr}(G)$. We supply $\mathrm{Irr}^u(G)$ with the relative topology.

Now, we assume that G is one of the groups $\mathrm{GL}(n,F)$, $\mathrm{O}(2n,F)$, $\mathrm{SO}(2n+1,F)$ or $\mathrm{Sp}(2n,F)$. Let K be the maximal compact subgroup introduced in the paragraph before Theorem 1-2. The Weyl group W of G acts naturally on the analytic manifold $D_n=(\mathbb{C}^\times)^n$. The space of W–orbits D_n^W has the structure of analytic manifold. The manifold D_n parameterizes unramified principal series of G as follows:

$$\mathrm{Ind}_{P_{\min}}^G(\chi_1\otimes\cdots\otimes\chi_n)\to(\chi_1(\varpi),\dots,\chi_n(\varpi)).$$

Therefore, the manifold D_n^W parameterizes unramified principal series of G, up to association. Let $\mathrm{Irr}^I(G)$ be the set of equivalence classes of irreducible representations σ of G for which there exists a representation in unramified principal series, say $\mathrm{Ind}_{P_{\min}}^G(\chi_1\otimes\cdots\otimes\chi_n)$, such that σ is an irreducible subquotient of $\mathrm{Ind}_{P_{\min}}^G(\chi_1\otimes\cdots\otimes\chi_n)$. The principal series is determined, up to association, uniquely by this condition. We have a well–defined map

$$\varphi_G:\mathrm{Irr}^I(G)\to D_n^W \tag{3-1}$$

defined by

$$\varphi_G(\sigma)=W\text{–orbit of a } n\text{–tuple } (\chi_1(\varpi),\dots,\chi_n(\varpi)).$$

We call $\varphi_G(\sigma)$ the infinitesimal character of σ. The fibers of φ_G are finite. Its restriction to $\mathrm{Irr}^{unr}(G)$ induces a bijection $\varphi_G:\mathrm{Irr}^{unr}(G)\to D_n^W$,

$$\varphi_G(\sigma^G(\chi_1,\dots,\chi_n))=W\text{–orbit of the } n\text{–tuple } (\chi_1(\varpi),\dots,\chi_n(\varpi)). \tag{3-2}$$

Now, we recall some results from [**T6**].

Lemma 3-3. *Suppose that G is connected (later we discuss the case of* $\mathrm{O}(2n,F)$*). Then the set* $\mathrm{Irr}^I(G)$ *is a connected component of* $\mathrm{Irr}(G)$*. Therefore it is open and closed there. The map φ_G given by (3-1) is continuous and closed.*

Next, ([**T6**], Lemma 5.8) implies the following:

Lemma 3-4. *Suppose that G is connected. Then* $\mathrm{Irr}^{unr}(G)$ *is an open subset of* $\mathrm{Irr}^I(G)$.

We have the following description of the topology on $\mathrm{Irr}^{unr}(G)$:

Theorem 3-5. *Suppose that G is connected. Then the map (3-2) is a homeomorphism.*

Proof. As it is continuous and bijective, it is enough to show that it is closed. So, let Z be a closed set in $\mathrm{Irr}^{unr}(G)$. We must show that $\varphi_G(Z)$ is closed. In order to prove that, let $Cl(\varphi_G(Z))$ be its closure. We must show that $Cl(\varphi_G(Z))=$

$\varphi_G(Z)$. Let $x \in Cl(\varphi_G(Z))$. Then there exists a sequence $(x_m)_{m\geq 1}$ in $\varphi_G(Z)$ such that $\lim_m x_n = x$. (We remark that D_n^W is a complex analytic manifold.) We write

$$x_m = W\text{–orbit of the } n\text{–tuple } (s_{m,1}, \dots, s_{m,n}) \in D_n \quad (m \geq 1);$$
$$x = W\text{–orbit of the } n\text{–tuple } (s_1, \dots, s_n) \in D_n.$$

After passing to a subsequence and making the appropriate identification, we may assume $\lim_m s_{m,i} = s_i$ for $i = 1, \dots, n$. We may take unramified characters $\chi_{m,i}$ and $\chi_1, \dots, \chi_n$, $m \geq 1, i = 1, \dots, n$, such that $\chi_{m,i}(\varpi) = s_{m,i}$ and $\chi_i(\varpi) = s_i$. Clearly, we have the following:

$$\varphi_G(\sigma^G(\chi_{m,1}, \dots, \chi_{m,n})) = x_m \quad (m \geq 1);$$
$$\varphi_G(\sigma^G(\chi_1, \dots, \chi_n)) = x.$$

Now, Proposition 5.2 of [**T6**] tells us that, passing to a subsequence, we may assume that the characters of $\sigma^G(\chi_{m,1}, \dots, \chi_{m,n})$ converge pointwise, and that there are irreducible subquotients $\sigma_1, \dots, \sigma_l$ of $\mathrm{Ind}_{P_{\min}}^G(\chi_1 \otimes \cdots \otimes \chi_n)$ and $k_1, \dots, k_l \in \mathbb{Z}_{>0}$ such that the pointwise limit is a character of $\sum_{i=1}^l k_i\sigma_i$. Since the representations $\sigma^G(\chi_{m,1}, \dots, \chi_{m,n})$ are unramified, among the representations $\sigma_1, \dots, \sigma_l$ is $\sigma^G(\chi_1, \dots, \chi_n)$. Therefore, one of the equivalent descriptions of the topology in [**T6**], implies $\sigma^G(\chi_1, \dots, \chi_n) \in Z$. Hence $x = \varphi_G(\sigma^G(\chi_1, \dots, \chi_n)) \in \varphi_G(Z)$. This shows that $\varphi_G(Z)$ is closed. □

REMARK 3-6. *Suppose that G is connected. Then the set* $\mathrm{Irr}^{u,unr}(G)$ *is a closed subset of* $\mathrm{Irr}^{unr}(G)$ *(see* [**T6**]*). Therefore, it can be identified via φ_G with a closed subset of D_n^W.*

In this paper we shall need only the topology of the unitary dual. The following theorem describes it.

THEOREM 3-7. *Let G be one of the groups* $\mathrm{GL}(n, F)$, $\mathrm{O}(2n, F)$, $\mathrm{SO}(2n + 1, F)$ *or* $\mathrm{Sp}(2n, F)$. *Then map (3-2) restricts to a homeomorphism*

$$\varphi_G : \mathrm{Irr}^{u,unr}(G) \to D_n^W \tag{3-8}$$

of $\mathrm{Irr}^{u,unr}(G)$ *onto a compact (closed) subset of D_n^W.*

PROOF. If G is connected, then the φ_G is homeomorphism on the image by Theorem 3-5. The image is compact by Theorem 3.1 of [**T1**] (this is also Theorem 2.5 of [**T6**]).

Now we briefly explain the proof in the case of $G = \mathrm{O}(2n, F)$ (below, sometimes we do not distinguish between elements in D_n^G and the W-orbits that they determine; one can easily complete details). The compactness for the case of $\mathrm{SO}(2n, F)$ implies that the image of φ_G has compact closure. Further, the topology can be described by characters (see [**Mi**]). Suppose that we have a convergent sequence $\psi_m \to \psi$ in D_n^W, such that the sequence ψ_m is contained in the image of φ_G. Suppose that ψ_m corresponds to unramified characters ψ'_m, and ψ to ψ'. Let π_m be such that $\varphi_G(\pi_m) = \psi_m$. Now, ([**T6**], Proposition 5.2) says that we can pass to a subsequence such that characters of π_m converge pointwise to the character of subquotient π of the representation induced by ψ'. It is obvious that π has an irreducible unramified subquotient, say π'. Clearly, $\varphi_G(\pi') = \psi$. Now, [**T7**] implies that all irreducible subquotients are unitarizable. So, π' is unitarizable. This implies that ψ is in the image of φ_G. Thus, the image is closed. Denote the image by X.

Let $Y \subset X$, and let ψ be a point in the closure of Y. Take a sequence ψ_m in Y converging to ψ. Let the ψ_m correspond to unramified characters ψ'_m, and ψ to ψ'. Take π_m such that $\varphi_G(\pi_m) = \psi_m$. This means $\pi_m \in \varphi_G^{-1}(Y)$. As above we can pass to a subsequence such that characters of π_m converge pointwise to the character of subquotient π of the representation induced by ψ'. Further, π has an irreducible unramified subquotient π' with $\varphi_G(\pi') = \psi$ and π' unitarizable. The description of the topology by characters implies that π' is a limit of the sequence π_m. This implies that π' is in the closure of $\varphi_G^{-1}(Y)$. This implies that $\varphi_G^{-1} : X \to \mathrm{Irr}^{u,unr}(G)$ is continuous.

Now, let $S \subset \mathrm{Irr}^{unr,u}(G)$. Take $\pi \in \mathrm{Irr}^{unr,u}(G)$ from the closure of S. Then we can find a sequence $\pi_m \in S$ converging to π. Let π_m and π be subquotients of representations induced by unramified characters ψ_m and ψ, respectively. Since X is compact, we can pass to a subsequence such that ψ_m converges (to some ψ_0). Next, arguing as above, we can pass to a subsequence of π_m such that all limits are subquotients of the representation induced by ψ_0. Now, the linear independence of characters of irreducible representations implies that $\psi = \psi_0$. Let $\psi'_m, \psi' \in D_n^W$ correspond to ψ_m, ψ, respectively. Observe that $\psi'_m = \varphi_G(\pi_m) \in \varphi_G(S)$, $\varphi_G(\pi) = \psi'$. Therefore, $\varphi_G(\pi)$ is in the closure of $\varphi_G(S)$. This ends the proof of continuity of φ_G. The proof of the theorem is now complete. $\square$

4. The Unramified Unitary Dual of $GL(n, F)$

The second named author classified unramified unitarizable representations $\mathrm{Irr}^{u,unr}(\mathrm{GL})$ in [**T4**]. The proof was based on Theorem 1-7 and a result of Bernstein on irreducibility of unitary parabolic induction proved in [**Be2**]. In this section we give the classification of $\mathrm{Irr}^{u,unr}(\mathrm{GL})$ without using the result of Bernstein. The main result of this section is the following theorem:

Theorem 4-1. (i) *Let $\phi_1, \ldots, \phi_a, \psi_1, \ldots, \psi_b \in \mathrm{Irr}^{unr}(GL)$ be a sequence of unramified unitary characters (one–dimensional unramified representations). Let $\alpha_1, \ldots, \alpha_b \in \,]0, \ \frac{1}{2}[$ be a sequence of real numbers. (The possibility $a = 0$ or $b = 0$ is not excluded here.) Then*

$$(4\text{-}2) \quad \phi_1 \times \cdots \times \phi_a \times (\nu^{\alpha_1}\psi_1 \times \nu^{-\alpha_1}\psi_1) \times \cdots \times (\nu^{\alpha_b}\psi_b \times \nu^{-\alpha_b}\psi_b) \in \mathrm{Irr}^{u,unr}(GL).$$

(ii) *Let $\pi \in \mathrm{Irr}^{u,unr}(GL)$. Then there exist $\phi_1, \ldots, \phi_a, \psi_1, \ldots, \psi_b$ and $\alpha_1, \ldots, \alpha_b$ as in (i) such that π is isomorphic to the induced representation given by (4-2). Each sequence $\phi_1, \ldots, \phi_a$ and $(\psi_1, \alpha_1), \ldots, (\psi_b, \alpha_b)$ is uniquely determined by π up to a permutation.*

Proof. Applying (H-IC) and (H-Irr), we see that a representation given by (4-2) is Hermitian. Next, fixing $\phi_1, \ldots, \phi_a, \psi_1, \ldots, \psi_b$ and letting $0 \le \alpha_1, \ldots, \alpha_b < 1/2$ vary, the representations in (4-2) form a continuous family of irreducible Hermitian representations with a unitarizable representation in it (namely, the one attached to $\alpha_1 = \cdots = \alpha_b = 0$). Thus, by (D), they are all unitarizable. The uniqueness in (ii) follows from the Zelevinsky classification (see Theorem 1-7).

Let $\pi \in \mathrm{Irr}^{u,unr}(GL)$. It remains to prove that π can be written in the form of (4-2). First, being unramified, the Zelevinsky classification (see Theorem 1-7) implies that π is fully–induced from (not–necessarily unitary) characters in $\mathrm{Irr}^{unr}(GL)$. Now, since $\pi \in \mathrm{Irr}^{+,unr}(GL)$, using (H-IC), (H-Irr), and (H-Ind), we obtain

$$\pi \simeq \phi_1 \times \cdots \times \phi_a \times (\nu^{\alpha_1}\psi_1 \times \nu^{-\alpha_1}\psi_1) \times \cdots \times (\nu^{\alpha_b}\psi_b \times \nu^{-\alpha_b}\psi_b), \tag{4-3}$$

where everything is as in (i) except that we have only $\alpha_1, \dots, \alpha_b > 0$. To prove the theorem, we need to prove that $\alpha_i < 1/2$, $i = 1, \dots, b$.

First, as all representations $\phi_1, \dots, \phi_a, \nu^{\alpha_1}\psi_1 \times \nu^{-\alpha_1}\psi_1, \dots, \nu^{\alpha_b}\psi_b \times \nu^{-\alpha_b}\psi_b$ are Hermitian and π is unitarizable, we conclude that $\phi_1 \otimes \cdots \otimes \phi_a \otimes (\nu^{\alpha_1}\psi_1 \times \nu^{-\alpha_1}\psi_1) \otimes \cdots \otimes (\nu^{\alpha_b}\psi_b \times \nu^{-\alpha_b}\psi_b)$ is unitarizable (see (UR)). In particular,

$$\nu^{\alpha_1}\psi_1 \times \nu^{-\alpha_1}\psi_1, \ \dots, \ \nu^{\alpha_b}\psi_b \times \nu^{-\alpha_b}\psi_b \ \text{ are unitarizable representations.} \tag{4-4}$$

Suppose that some $\alpha_i \geq 1/2$ for some i. Let $\alpha = \alpha_i$ and $\psi = \psi_i$. We can write $\psi = \langle [-x, \ x]^{(\chi)} \rangle$, where χ is a unitary unramified character of $F^\times$ and $x \in \mathbb{Z}_{\geq 0}$ (see (1-4) and (1-5)). We let

$$\pi_{\beta,x} = \nu^{\beta}\psi \times \nu^{-\beta}\psi = \langle [-x+\beta, \ x+\beta]^{(\chi)} \rangle \times \langle [-x-\beta, \ x-\beta]^{(\chi)} \rangle, \text{ where } \beta \in \mathbb{R}. \tag{4-5}$$

Note that

$$\text{if } \pi_{\beta,x} \ \text{ is irreducible, then } \pi_{\beta,x} \in \mathrm{Irr}^+(\mathrm{GL}) \tag{4-6}$$

$$\pi_{\beta,x} \ \text{ is reducible if and only if} [-x+\beta, \ x+\beta]^{(\chi)}, [-x-\beta, \ x-\beta]^{(\chi)} \text{ are linked.} \tag{4-7}$$

Now, we consider the two cases.

First, we assume that $\alpha - x > 1/2$. Then, (4-4), (4-6) and (4-7) imply that the continuous family of representations $\pi_{\beta,x}$ ($\beta \geq \alpha$) is irreducible, Hermitian and, at $\beta = \alpha$, unitarizable. Therefore it is unitarizable everywhere. But this contradicts Remark 2-7 since for large enough β, $\pi_{\beta,x}$ has unbounded matrix coefficients. (See [**T1**], [**T6**].)

Therefore $\alpha - x \leq 1/2$. Now, using the definition (4-6) and (4-7), the irreducibility of $\pi_{\alpha,x}$ implies

$$\alpha \notin (1/2)\,\mathbb{Z}. \tag{4-8}$$

Next, there exists $k \in \mathbb{Z}_{>0}$ such that

$$\left| \frac{(-x+\alpha-k) + (x+\alpha-1)}{2} \right| = |\alpha - k/2 - 1/2| < 1/2$$

(there are exactly two such k's). Now, the representation

$$\pi_{\alpha-(k+1)/2, x+(k-1)/2} = \langle [-x+\alpha-k, x+\alpha-1]^{(\chi)} \rangle \times \langle [-x-\alpha+1, x-\alpha+k]^{(\chi)} \rangle$$

is irreducible and unitarizable by (i). Hence

$$\pi := \pi_{\alpha,x} \times \pi_{\alpha-(k+1)/2, x+(k-1)/2} \tag{4-9}$$

is a unitarizable representation. Next, (4-8) implies that
(4-10)

$$a - b \notin \mathbb{Z}, \text{ where } a \text{ (resp., } b\text{) belongs to the first (resp., the last) two sequences:}$$

$$\begin{cases} -x+\alpha, \dots, x+\alpha, \\ -x+\alpha-k, \dots, x+\alpha-1, \\ -x-\alpha, \dots, x-\alpha, \\ -x-\alpha+1, \dots, x-\alpha+k. \end{cases}$$

In particular, this and [**Ze**] imply

$$\langle[-x-\alpha,\ x-\alpha]^{(\chi)}\rangle\times\langle[-x+\alpha-k,x+\alpha-1]^{(\chi)}\rangle\simeq \langle[-x+\alpha-k,x+\alpha-1]^{(\chi)}\rangle\times\langle[-x-\alpha,\ x-\alpha]^{(\chi)}\rangle.$$

Hence, $\pi=\pi_{\alpha,x}\times\pi_{\alpha-(k+1)/2,x+(k-1)/2}$ is isomorphic to

$$\Big(\langle[-x+\alpha,\ x+\alpha]^{(\chi)}\rangle\times\langle[-x+\alpha-k,x+\alpha-1]^{(\chi)}\rangle\Big)\times \Big(\langle[-x-\alpha,\ x-\alpha]^{(\chi)}\rangle\times\langle[-x-\alpha+1,x-\alpha+k]^{(\chi)}\rangle\Big). \tag{4-11}$$

Since, by the Zelevinsky classification, the induced representations in both parenthesis in (4-11) reduce, we conclude that π has at least four irreducible subrepresentations. Since, by definition,

$$\pi\simeq\langle[-x+\alpha,\ x+\alpha]^{(\chi)}\rangle\times\langle[-x-\alpha,\ x-\alpha]^{(\chi)}\rangle\times \langle[-x+\alpha-k,x+\alpha-1]^{(\chi)}\rangle\times\langle[-x-\alpha+1,x-\alpha+k]^{(\chi)}\rangle, \tag{4-12}$$

Frobenius reciprocity implies that the multiplicity of

$$\tau:=\langle[-x+\alpha,\ x+\alpha]^{(\chi)}\rangle\times\langle[-x-\alpha,\ x-\alpha]^{(\chi)}\rangle\otimes \langle[-x+\alpha-k,x+\alpha-1]^{(\chi)}\rangle\times\langle[-x-\alpha+1,x-\alpha+k]^{(\chi)}\rangle,$$

in the Jacquet module

$$r_{(4x+2,4x+2k),\ 8x+2k+2}(\pi)$$

must be at least four. This contradicts (the following) Lemma 4-13, and proves the theorem. □

It remains to prove the following lemma:

LEMMA 4-13. *The multiplicity of τ in $r_{(4x+2,4x+2k),\ 8x+2k+2}(\pi)$ is exactly two.*

PROOF. We begin by introducing some notation. If ρ is an admissible representation of $\mathrm{GL}(n,F)$, then we let

$$m^*(\rho)=\mathbf{1}\otimes\rho+\sum_{i=1}^{n-1}r_{(i,n-i),\ n}(\pi)+\rho\otimes\mathbf{1}$$

in $(\oplus_{n\geq0}\mathrm{R}(\mathrm{GL}(n,F)))\otimes(\oplus_{n\geq0}\mathrm{R}(\mathrm{GL}(n,F)))$. By [**Ze**], m^* is multiplicative:

$$m^*(\rho_1\times\rho_2)=m^*(\rho_1)\times m^*(\rho_2).$$

Also, we recall (see [**Ze**])

$$m^*(\langle[a,b]^{(\chi)}\rangle)=\sum_{k=a-1}^{b}\langle[a,k]^{(\chi)}\rangle\otimes\langle[k+1,b]^{(\chi)}\rangle.$$

Combining this with the expression for π given by (4-12), we compute $m^*(\pi)$ as follows:

$$\sum\langle[-x+\alpha,k_1]^{(\chi)}\rangle\times\langle[-x-\alpha,k_2]^{(\chi)}\rangle\times\langle[-x+\alpha-k,k_3]^{(\chi)}\rangle\times\langle[-x-\alpha+1,k_4]^{(\chi)}\rangle\otimes \langle[k_1+1\ x+\alpha]^{(\chi)}\rangle\times\langle[k_2+1,\ x-\alpha]^{(\chi)}\rangle\times\langle[k_3+1,x+\alpha-1]^{(\chi)}\rangle\times\langle[k_4+1,x-\alpha+k]^{(\chi)}\rangle$$

where the summation runs over

$$\begin{cases} -x+\alpha-1 \le k_1 \le x+\alpha \\ -x-\alpha-1 \le k_2 \le x-\alpha \\ -x+\alpha-k-1 \le k_3 \le x+\alpha-1 \\ -x-\alpha \le k_4 \le x-\alpha+k. \end{cases}$$

Now, we determine the multiplicity of τ in that expression. First, we find all possible terms where it occurs. Applying (4-10), we see $k_1 = x+\alpha$ and $k_3 = -x+\alpha-k-1$. The expression for τ shows that $k_3 \ge -x-\alpha$. There are two cases. First, if $k_3 = -x-\alpha$, then the expression for τ shows that $k_4 = x-\alpha$. The term contains τ with multiplicity one since it is the tensor product of two induced representations where τ is the unique unramified irreducible subquotient. If $k_3 > -x-\alpha$, then $k_4 = -x-\alpha$. Hence, the expression for τ shows $k_3 = x-\alpha$. The resulting term is τ itself. $\square$

Now, we turn our attention to the topological structure of $\mathrm{Irr}^{u,unr}(GL(n,F))$. The topology of the unitary dual of $GL(n,F)$ is described in [**T5**]. Here we recall a simple description in the unramified case which follows directly from the general and simple Theorem 3-5.

Let X be a subset of the unramified unitary dual of $GL(n,F)$. We describe its closure $Cl(X)$. We consider all sequences in X of the form:

$$\pi^{(k)} \simeq \phi_1^{(k)} \times \cdots \times \phi_a^{(k)} \times (\nu^{\alpha_1^{(k)}}\psi_1^{(k)} \times \nu^{-\alpha_1^{(k)}}\psi_1^{(k)}) \times \cdots \times (\nu^{\alpha_b^{(k)}}\psi_b^{(k)} \times \nu^{-\alpha_b^{(k)}}\psi_b^{(k)})$$

where $\phi_i^{(k)}$ (resp., $\psi_j^{(k)}$) is a convergent sequence (in the obvious natural topology) of unramified unitary characters of a fixed general linear group, converging to some ϕ_i (resp., ψ_j), and $0 < \alpha_j^{(k)} < 1/2$ converges to $0 \le \alpha_j \le 1/2$ (the possibility $a = 0$ or $b = 0$ is not excluded). Let

$$\pi \simeq \phi_1 \times \cdots \times \phi_a \times (\nu^{\alpha_1}\psi_1 \times \nu^{-\alpha_1}\psi_1) \times \cdots \times (\nu^{\alpha_b}\psi_b \times \nu^{-\alpha_b}\psi_b). \tag{4-14}$$

This representation might be reducible, but its unique irreducible unramified subquotient $\pi^{\#}$ is unitarizable. Then $Cl(X)$ is exactly the set all possible such $\pi^{\#}$. The representation $\pi^{\#}$ can be described in the form given by Theorem 4-1 (i) as follows. If $\alpha_j = 1/2$, for some j in (4-14), we write $\psi_j = \chi_j 1_{GL(h_j,F)}$, where χ_j is an unramified unitary character of $F^\times$, and in (4-14) change $\nu^{\alpha_j}\psi_j \times \nu^{-\alpha_j}\psi_j = \nu^{1/2}\chi_j 1_{GL(h_j,F)} \times \nu^{-1/2}\chi_j 1_{GL(h_j,F)}$ to $\chi_j 1_{GL(h_j+1,F)} \times \chi_j 1_{GL(h_j-1,F)}$.

5. The Unramified Unitary Dual $\mathrm{Irr}^{u,unr}(S)$

In this section we state the result on the classification of the unitary unramified dual $\mathrm{Irr}^{u,unr}(S)$. We begin by recalling some results of [**M4**].

Definition 5-1. *Let* $\mathbf{sgn}_u$ *be the unique unramified character of order two of* $F^\times$. *Let* $\mathbf{1}_{F^\times}$ *be the trivial character of* $F^\times$.

We remark that $\mathbf{sgn}_u(\varpi) = -1$.

The following definition is crucial for us:

DEFINITION 5-2. *Let $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. Then we define α_χ as follows:*

if $S_n = \mathrm{O}(2n, F)$ $(n \geq 0)$, then $\alpha_\chi = 0$

if $S_n = \mathrm{SO}(2n+1, F)$ $(n \geq 0)$, then $\alpha_\chi = \dfrac{1}{2}$

if $S_n = \mathrm{Sp}(2n, F)$ $(n \geq 0)$, then $\alpha_{\mathbf{sgn}_u} = 0$ and $\alpha_{\mathbf{1}_{F^\times}} = 1$.

Next, we recall the following well–known result that explains Definition 5-2:

REMARK 5-3. *For an unramified unitary character χ of $F^\times$ and $s \in \mathbb{R}$, we have the following:*

(i) *$\nu^s\chi \rtimes \mathbf{1}$ (a representation of S_1; see Section 1 for the notation), and $\nu^{-s}\chi^{-1} \rtimes \mathbf{1}$ have the same composition series (and therefore $\nu^s\chi \rtimes \mathbf{1}$ reduces if and only if $\nu^{-s}\chi^{-1} \rtimes \mathbf{1}$ reduces).*
(ii) *$\nu^s\chi \rtimes \mathbf{1}$ is irreducible if $\chi \notin \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$.*
(iii) *Suppose $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. Then $\nu^s\chi \rtimes \mathbf{1}$ reduces if and only if $s = \pm\alpha_\chi$.*

A pair (m, χ), where $m \in \mathbb{Z}_{>0}$ and χ is an unramified unitary character of $F^\times$ is called a Jordan block.

DEFINITION 5-4. *Let $n > 0$. We write $\mathrm{Jord}_{sn}(n)$ for the collection of all sets Jord of Jordan blocks such that the following holds:*

$$\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\} \text{ and } m - (2\alpha_\chi + 1) \in 2\mathbb{Z} \text{ for all } (m, \chi) \in \mathrm{Jord}$$

$$\sum_{(m,\chi)\in \mathrm{Jord}} m = \begin{cases} 2n & \text{if } S_n = \mathrm{SO}(2n+1, F) \text{ or } S_n = \mathrm{O}(2n, F); \\ 2n+1 & \text{if } S_n = \mathrm{Sp}(2n, F), \end{cases}$$

and, additionally, if $\alpha_\chi = 0$, then $\mathrm{card}\, \{k; (k, \chi) \in \mathrm{Jord}\} \in 2\mathbb{Z}$.

REMARK 5-5. *Let $(m, \chi) \in \mathrm{Jord} \in \mathrm{Jord}_{sn}(n)$ be a Jordan block. Then m is even if we are dealing with odd-orthogonal groups, and odd otherwise (i.e., if we are dealing with even-orthogonal or symplectic groups).*

Let $\mathrm{Jord} \in \mathrm{Jord}_{sn}(n)$. Then, for $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, we let

$$\mathrm{Jord}_\chi = \{k; (k, \chi) \in \mathrm{Jord}\}.$$

We let

$$\mathrm{Jord}'_\chi = \begin{cases} \mathrm{Jord}(\chi); \ \mathrm{card}(\mathrm{Jord}_\chi) \text{ is even}; \\ \mathrm{Jord}(\chi) \cup \{-2\alpha_\chi + 1\}; \ \mathrm{card}(\mathrm{Jord}_\chi) \text{ is odd}. \end{cases}$$

We write Jord'_χ according to the character χ (the case $l_{\mathbf{1}_{F^\times}} = 0$ or $l_{\mathbf{sgn}_u} = 0$ is not excluded):

$$\begin{cases} \chi = \mathbf{1}_{F^\times}: & a_1 < a_2 < \cdots < a_{2l_{\mathbf{1}_{F^\times}}} \\ \chi = \mathbf{sgn}_u: & b_1 < b_2 < \cdots < b_{2l_{\mathbf{sgn}_u}} \end{cases} \tag{5-6}$$

(here $a_i, b_j \in 1 + 2\mathbb{Z}_{\geq 0}$ if $S_n = \mathrm{Sp}(2n, F)$ or $S_n = \mathrm{O}(2n, F)$, and $a_i, b_j \in 2\mathbb{Z}_{>0}$ if $S_n = \mathrm{SO}(2n+1, F)$).

Next, we associate to $\mathrm{Jord} \in \mathrm{Jord}_{sn}(n)$, the unramified representation $\sigma(\mathrm{Jord})$ of S_n defined as the unique irreducible unramified subquotient of the induced representation

(5-7)
$$\left(\times_{i=1}^{l_{\mathbf{1}_{F^\times}}} \langle [-\frac{a_{2i}-1}{2}, \frac{a_{2i-1}-1}{2}]^{(\mathbf{1}_{F^\times})} \rangle\right) \times \left(\times_{j=1}^{l_{\mathbf{sgn}_u}} \langle [-\frac{b_{2j}-1}{2}, \frac{b_{2j-1}-1}{2}]^{(\mathbf{sgn}_u)} \rangle\right) \rtimes \mathbf{1}.$$

In fact, $\sigma(\mathrm{Jord})$ is a subrepresentation of the induced representation in (5-7).

We have the following result (see [**M4**]):

THEOREM 5-8. *Let $n \in \mathbb{Z}_{>0}$. The map* Jord $\mapsto \sigma(\mathrm{Jord})$ *defines a one–to–one correspondence between the set* $\mathrm{Jord}_{sn}(n)$ *and the set of all strongly negative unramified representations of* S_n. *(An unramified representation is strongly negative if its Aubert dual is in the discrete series.)*

The inverse mapping to Jord $\mapsto \sigma(\mathrm{Jord})$ is denoted by $\sigma \mapsto \mathrm{Jord}(\sigma)$.

For technical reasons, we consider the trivial representation of the trivial group S_0 to be strongly negative. We associate the set of Jordan blocks (depending on the tower S_n $(n \geq 0)$) as follows

$$\mathrm{Jord}(\mathbf{1}) = \begin{cases} \{(1, \mathbf{1}_{F^\times})\}; & \text{if } S_n = \mathrm{Sp}(2n, F) \ (n \geq 0), \\ \emptyset; & \text{otherwise.} \end{cases} \tag{5-9}$$

If we let $\mathrm{Jord}_{sn}(0) = \{\mathrm{Jord}(\mathbf{1})\}$ and $\mathbf{1} = \sigma(\mathrm{Jord})$, $\mathrm{Jord} \in \mathrm{Jord}_{sn}(0)$, then Theorem 5-8 holds for $n = 0$. (We remark that Definition 5-4 and (5-7) hold for $\mathrm{Jord} \in \mathrm{Jord}_{sn}(0)$.)

An unramified representation is negative if its Aubert dual is tempered. Negative representations are classified in terms of strongly negative ones as follows:

THEOREM 5-10. *Let $\sigma_{neg} \in \mathrm{Irr}^{unr}(S)$ be a negative representation. Then there exists a sequence of pairs $(l_1, \chi_1), \ldots, (l_k, \chi_k)$ ($l_i \in \mathbb{Z}_{\geq 1}$, χ_i an unramified unitary character of $F^\times$), unique up to a permutation and taking inverses of characters, and unique strongly negative representation σ_{sn} such that*

$$\sigma_{neg} \hookrightarrow \quad \langle [-\frac{l_1 - 1}{2}, \frac{l_1 - 1}{2}]^{(\chi_1)} \rangle \times \cdots \times \langle [-\frac{l_k - 1}{2}, \frac{l_k - 1}{2}]^{(\chi_k)} \rangle \rtimes \sigma_{sn}.$$

Conversely, for a sequence of the pairs $(l_1, \chi_1), \ldots, (l_k, \chi_k)$ ($l_i \in \mathbb{Z}_{>0}$, χ_i is an unramified unitary character of $F^\times$) and a strongly negative representation σ_{sn}, the unique irreducible unramified subquotient of

$$\langle [-\frac{l_1 - 1}{2}, \frac{l_1 - 1}{2}]^{(\chi_1)} \rangle \times \cdots \times \langle [-\frac{l_k - 1}{2}, \frac{l_k - 1}{2}]^{(\chi_k)} \rangle \rtimes \sigma_{sn}$$

is negative and it is a subrepresentation.

For the irreducible negative unramified representation $\sigma_{neg} \in \mathrm{Irr}^{unr}(S)$ given by above Theorem 5-10, one defines $\mathrm{Jord}(\sigma_{neg})$ to be the multiset

$$\mathrm{Jord}(\sigma_{sn}) + \sum_{i=1}^{k} \{(l_i, \chi_i), (l_i, \chi_i^{-1})\}$$

(multisets are sets where multiplicities are allowed). For a unitary unramified character χ of $F^\times$, we let $\mathrm{Jord}(\sigma_{neg})_\chi$ to be the multiset consisting of all l (counted with multiplicity) such that $(l, \chi) \in \mathrm{Jord}(\sigma_{neg})$.

Now, we turn our attention to $\mathrm{Irr}^{u,unr}(S)$. First, we have the following particular case of ([**M6**]):

THEOREM 5-11. *Let $\sigma \in \mathrm{Irr}^{unr}(S)$ be a negative representation. Then σ is unitarizable.*

In order to describe the whole $\mathrm{Irr}^{u,unr}(S)$ we need to introduce more notation. We write $\mathcal{M}^{unr}(S)$ for the set of pairs $(\mathbf{e}, \sigma_{neg})$, where $\mathbf{e}$ is a (perhaps empty) multiset consisting of a finite number of triples (l, χ, α) where $l \in \mathbb{Z}_{>0}$, χ is an unramified unitary character of $F^\times$, and $\alpha \in \mathbb{R}_{>0}$. For $l \in \mathbb{Z}_{>0}$ and an unramified unitary character χ of $F^\times$, we let $\mathbf{e}(l, \chi)$ to be the submultiset of $\mathbf{e}$ consisting of all positive real numbers α (counted with multiplicity) such that $(l, \chi, \alpha) \in \mathbf{e}$. We have the following:

$$\mathbf{e} = \sum_{(l,\chi)} \sum_{\alpha \in \mathbf{e}(l,\chi)} \{(l, \chi, \alpha)\}.$$

We define the map $n : \mathcal{M}^{unr}(S) \to \mathbb{Z}$ as follows:

$$n(\mathbf{e}, \sigma_{neg}) = \sum_{(l,\chi)} l \cdot \mathrm{card}\ \mathbf{e}(l, \chi) + n_{neg}$$

where n_{neg} is defined by $\sigma_{neg} \in \mathrm{Irr}\ S_{n_{neg}}$.

We attach $\sigma \in \mathrm{Irr}^{unr}(S)$ to $(\mathbf{e}, \sigma_{neg})$ in a canonical way. By definition, σ is the unique irreducible unramified subquotient of the following induced representation:

$$\left(\times_{(l,\chi,\alpha)\in\mathbf{e}} \ \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)} \rangle \right) \rtimes \sigma_{neg}. \tag{5-12}$$

It is a representation of $S_{n(\mathbf{e},\sigma_{neg})}$.

We remark that the definition of σ does not depend on the choice of ordering of characters in (5-12). Next, the results of [**M4**] (see Lemma 6-2 in Section 6) imply that the constructed map $\mathcal{M}^{unr}(S) \to \mathrm{Irr}^{unr}(S)$ is surjective but not injective.

In order to obtain unitary representations, we impose further conditions on $\mathbf{e}$ in the following definition:

Definition 5-13. *Let $\mathcal{M}^{u,unr}(S)$ be the subset of $\mathcal{M}^{unr}(S)$ consisting of the pairs $(\mathbf{e}, \sigma_{neg})$ satisfying the following conditions:*

1. *If $\chi \notin \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, then $\mathbf{e}(l, \chi) = \mathbf{e}(l, \chi^{-1})$ and $0 < \alpha < \frac{1}{2}$ for all $\alpha \in \mathbf{e}(l, \chi)$.*
2. *If $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$, then $0 < \alpha < \frac{1}{2}$ for all $\alpha \in \mathbf{e}(l, \chi)$.*
3. *If $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l - (2\alpha_\chi + 1) \in 2\mathbb{Z}$, then $0 < \alpha < 1$ for all $\alpha \in \mathbf{e}(l, \chi)$. Moreover, if we write the exponents that belong to $\mathbf{e}(l, \chi)$ as follows:*

$$0 < \alpha_1 \le \cdots \le \alpha_u \le \frac{1}{2} < \beta_1 \le \cdots \le \beta_v < 1.$$

(We allow $u = 0$ or $v = 0$.) Then we also require the following:
   (a) *If $(l, \chi) \notin \mathrm{Jord}(\sigma_{neg})$, then $u + v$ is even.*
   (b) *If $u > 1$, then $\alpha_{u-1} \neq \frac{1}{2}$.*
   (c) *If $v \ge 2$, then $\beta_1 < \cdots < \beta_v$.*
   (d) *$\alpha_i \notin \{1 - \beta_1, \ldots, 1 - \beta_v\}$ for all i.*
   (e) *If $v \ge 1$, then the number of indices i such that $\alpha_i \in]1 - \beta_1, \frac{1}{2}]$ is even.*
   (f) *If $v \ge 2$, then the number of indices i such that $\alpha_i \in]1 - \beta_{j+1}, 1 - \beta_j[$ is odd.*

We advise the reader to construct some pairs $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S)$. This should be done in the following way. One first chooses an arbitrary σ_{neg}. Then one adds multisets $\mathbf{e}(l, \chi)$ to $\mathbf{e}$ following (1)–(3) and (a)–(g), in that order.

The following theorem gives an explicit classification (with explicit parameters) of unramified unitary duals of classical groups S_n, i.e., of $\mathrm{Irr}^{u,unr}(S)$. The proof of the classification theorem is in Sections 7, 8, and 9. At no point in the proof, does the explicit internal structure of representations play a role. This is the reason that this can be considered as an external approach to the unramified unitary duals (of classical groups), along the lines of such approaches in [**T3**], [**T2**], [**LMT**] etc.

THEOREM 5-14. *Let $n \in \mathbb{Z}_{\geq 0}$. We write $\mathcal{M}^{u,unr}(S_n)$ for the set of all $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S)$ such that $n(\mathbf{e}, \sigma_{neg}) = n$. Then, for $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$, the induced representation (5-12) is an irreducible unramified representation of S_n. Moreover, the map $(\mathbf{e}, \sigma_{neg}) \longmapsto \times_{(l,\chi,\alpha)\in\mathbf{e}} \ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \rtimes \sigma_{neg}$ is a one–to–one correspondence between $\mathcal{M}^{u,unr}(S_n)$ and $\mathrm{Irr}^{u,unr}(S_n)$.*

This result, along with Theorem 5-11, was partly obtained by Barbasch and Moy (see [**Bb**], [**BbMo**], [**BbMo1**], [**BbMo2**]). See the introduction for more explanation.

REMARK 5-15. *We separate the conditions of Definition 5-13 into three groups:*
Irreducibility conditions: (b), (d) in (3).
Hermicity condition: (1).
Unitarizability conditions: (2) and (a), (c), (e), (f) in (3), and also the condition $0 < \alpha < 1$ in (3).

6. Some Technical Results

In this section we recall some results from [**M4**] and prove some results about reducibility and subquotients of certain induced representations needed in the proof of Theorem 5-14. The reader should skip this section at the first reading. We begin with the following lemma:

LEMMA 6-1. *Let $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{unr}(S)$. Then the induced representation (5-12) is reducible if and only if one of the following holds:*

(1) *there exist $(l, \chi, \alpha), (l', \chi', \alpha') \in \mathbf{e}$ such that the segments $[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}$ and $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}$ are linked*
(2) *there exist $(l, \chi, \alpha), (l', \chi', \alpha') \in \mathbf{e}$ such that the segments $[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}$ and $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{-\alpha'}\chi')}$ are linked*
(3) *there exist $(l, \chi, \alpha) \in \mathbf{e}$ such that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \rtimes \sigma_{neg}$ reduces*

Further, let $(l, \chi, \alpha) \in \mathbf{e}$ and consider the following statements:

(4) *there exists $(l', \chi') \in \mathrm{Jord}(\sigma_{neg})$ such that the segments $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle$ and $\langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\chi')}\rangle$ are linked*
(5) *$\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l - (2|\alpha - \alpha_\chi| + 1) \in 2\mathbb{Z}_{\geq 0}$*

Then we have the following:

- *If $\chi = \mathbf{1}_{F^\times}$, $\alpha_\chi = 1$, $\mathrm{card}\ \mathrm{Jord}(\sigma_{neg})_{\mathbf{1}_{F^\times}}$ is odd, and $-\frac{l-1}{2} + \alpha = 1$, then the induced representation in (3) is reducible if and only if (4) holds.*

- *Otherwise, the induced representation in (3) is reducible if and only if (4) or (5) holds.*

PROOF. First, ([**M4**], Lemma 4.8) implies that the induced representation (5-12) is reducible if and only if (1) or (2) holds or $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle \rtimes \sigma_{neg}$ is reducible. We will describe the reducibility of $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle \rtimes \sigma_{neg}$, and this will conclude the proof. We write σ_{neg} as in Theorem 5-10. Then ([**M4**], Corollary 4.2) implies that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle \rtimes \sigma_{neg}$ reduces if and only if one of the following holds:

(a) $[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}$ is linked with $[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\chi_i)}$ for some i
(b) $[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}$ is linked with $[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\chi_i^{-1})}$ for some i
(c) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle \rtimes \sigma_{sn}$ reduces.

Using ([**M4**], Lemma 5.6), we see that (c) holds if and only if $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, $l + 2\alpha - (2\alpha_\chi + 1) \in 2\mathbb{Z}$ and one of the following holds:

(d) card $\mathrm{Jord}(\sigma_{sn})_\chi$ is even.
 (d-1) there exists $(l', \chi') \in \mathrm{Jord}(\sigma_{sn})$ such that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle$ and $\langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\chi')}\rangle$ are linked
 (d-2) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle \rtimes \mathbf{1}$ reduces (which is equivalent to $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l - (2|\alpha - \alpha_\chi| + 1) \in 2\mathbb{Z}_{\geq 0}$ by ([**M4**], Lemma 5.6 (i)).
(e) card $\mathrm{Jord}(\sigma_{sn})_\chi$ is odd; $\alpha_\chi = 1/2$, or $\alpha_\chi = 1$ and $-\frac{l-1}{2} + \alpha \neq 1$. Let $l_{min} = \min \mathrm{Jord}(\sigma_{sn})_\chi$.
 (e-1) there exists $l' \in \mathrm{Jord}(\sigma_{sn}) - \{l_{min}\}$ such that $\langle[-\frac{l-1}{2}, \frac{e-1}{2}]^{(\nu^\alpha \chi)}\rangle$ and $\langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\chi)}\rangle$ are linked
 (e-2) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle$ and $\langle[\alpha_\chi, \frac{l_{min}-1}{2}]^{(\chi)}\rangle$ are linked
 (e-3) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle$ and $\langle[-\frac{l_{min}-1}{2}, -\alpha_\chi]^{(\chi)}\rangle$ are linked
 (e-4) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle \rtimes \mathbf{1}$ reduces (which is equivalent to $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l - (2|\alpha - \alpha_\chi| + 1) \in 2\mathbb{Z}_{\geq 0}$ by ([**M4**], Lemma 5.6 (i))
(f) card $\mathrm{Jord}(\sigma_{sn})_\chi$ is odd; $\alpha_\chi = 1$ and $-\frac{l-1}{2} + \alpha = 1$. Then $\chi = \mathbf{1}_{F^\times}$. Let $l_{min} = \min \mathrm{Jord}(\sigma_{sn})_{\mathbf{1}_{F^\times}}$.
 (f-1) there exists $l' \in \mathrm{Jord}(\sigma_{sn})_{\mathbf{1}_{F^\times}} - \{l_{min}\}$ such that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \mathbf{1}_{F^\times})}\rangle$ and $\langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle$ are linked
 (f-2) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \mathbf{1}_{F^\times})}\rangle$ and $\langle[1, \frac{l_{min}-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle$ are linked
 (f-3) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \mathbf{1}_{F^\times})}\rangle$ and $\langle[-\frac{l_{min}-1}{2}, -1]^{(\mathbf{1}_{F^\times})}\rangle$ are linked
 (f-4) $l > l_{min}$.

It is easy to check that (e-2), (e-3) or (e-4) holds if and only if one of the following holds:

(e'-2) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle$ and $\langle[-\frac{l_{min}-1}{2}, \frac{l_{min}-1}{2}]^{(\chi)}\rangle$ are linked
(e'-4) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)}\rangle \rtimes \mathbf{1}$ reduces.

It is easy to check that (f-2), (f-3) or (f-4) holds if and only if the following holds:

(f'-2) $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \mathbf{1}_{F^\times})}\rangle$ and $\langle[-\frac{l_{min}-1}{2}, \frac{l_{min}-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle$ are linked.

Clearly, this analysis completes the proof of the lemma. □

The next lemma will play a crucial role in determining surjectivity of the map in Theorem 5-14 (see [**M4**], Theorem 4.3):

LEMMA 6-2. *Let $\sigma \in \mathrm{Irr}^{unr}(S)$. Then there exists a unique $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{unr}(S)$ such that σ is isomorphic to the induced representation given by (5-12).*

Next, we determine when the representation $\sigma \in \mathrm{Irr}^{unr}(S)$, given by Lemma 6-2, is Hermitian. We compute using (H-IC) and Theorem 5-11:

$$
\begin{aligned}
\sigma^+ &\simeq \left(\times_{(l,\chi,\alpha)\in\mathbf{e}} \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \rtimes \sigma_{neg}\right)^+ \\
&\simeq \times_{(l,\chi,\alpha)\in\mathbf{e}} \left(\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle\right)^+ \rtimes (\sigma_{neg})^+ \\
&\simeq \times_{(l,\chi)} \times_{\alpha\in\mathbf{e}(l,\chi)} \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\alpha}\chi)}\rangle \rtimes \sigma_{neg} \\
&\simeq \times_{(l,\chi)} \times_{\alpha\in\mathbf{e}(l,\chi)} \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi^{-1})}\rangle \rtimes \sigma_{neg}.
\end{aligned}
\tag{6-3}
$$

(The last isomorphism follows from the fact that every representation in $\mathrm{Irr}^{unr}(S)$ is self-dual. See Remark 1-8.) Therefore, (H-Irr), (6-3), and

$$
\sigma \simeq \times_{(l,\chi)} \times_{\alpha\in\mathbf{e}(\chi,l)} \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle \rtimes \sigma_{neg}
$$

imply the following result:

LEMMA 6-4. *Let $\sigma \in \mathrm{Irr}^{unr}(S)$ be given by Lemma 6-2. Then $\sigma \in \mathrm{Irr}^{+,unr}(S)$ if and only if $\mathbf{e}(l,\chi) = \mathbf{e}(l,\chi^{-1})$ for all $l \in \mathbb{Z}_{\geq 1}$ and all unitary unramified characters χ of $F^\times$.*

LEMMA 6-5. *Let $\sigma_{neg} \in \mathrm{Irr}^{unr}(S)$ be a negative representation. Let χ be a unitary unramified character of $F^\times$ and $l \in \mathbb{Z}_{\geq 1}$. Then $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma_{neg}$ is reducible if and only if $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, $l-(2\alpha_\chi+1) \in 2\mathbb{Z}$ and $(l,\chi) \notin \mathrm{Jord}(\sigma_{neg})$. If $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma_{neg}$ is reducible, then it is the direct sum of two non–equivalent representations (one of them is negative).*

PROOF. The proof of this result is standard and in the dual picture well–known (see [**MœT**]). We indicate the steps to explain why the result holds for local fields F of all characteristics. First, we apply the Aubert's involution, extended to orthogonal groups by C. Jantzen [**Jn**], to reduce to the tempered case. Then we use the results of Goldberg [**G**] [1], extended to orthogonal groups using simple Mackey machinery (see [**LMT**], Section 2), and some general algebraic considerations based on them (see [**LMT**], Lemma 2.2 and Corollary 2.3), to reduce the claim to the case when the image $\hat{\sigma}_{neg}$ of σ_{neg} under Aubert's involution is in the discrete series. As σ_{neg} and $\hat{\sigma}_{neg}$ have the same supercuspidal support which is explicitly known by Theorem 5-8, we can easily compute the Plancherel measure attached to the induced representation $\chi Steinberg_{GL(l,F)} \rtimes \hat{\sigma}_{neg}$ (which has the same reducibility as $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma_{neg}$). The computation of the Plancherel measure is done by using the factorization (see [**W**]) and reduction to the split rank–one case. Now, we use the usual theory developed by Harish–Chandra to decompose $\chi Steinberg_{GL(l,F)} \rtimes \hat{\sigma}_{neg}$.

[1] Goldberg stated his results in the characteristic zero, but this assumption is not necessary. In fact, all fundamental results of Harish–Chandra used there follow from [W2] as it was explained to the first named author by V. Heiermann.

Finally, assume that $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\chi)}\rangle\rtimes\sigma_{neg}$ is reducible. Then $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\chi)}\rangle\rtimes\hat{\sigma}_{neg}$ is a direct sum of two non–equivalent tempered representations (see [**LMT**], Lemma 2.2 and Corollary 2.3). Hence the composition series of $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\chi)}\rangle\rtimes\sigma_{neg}$ of two non–equivalent irreducible representations. Hence the last claim of the lemma follows from Theorems 5-10 and 5-11. □

Next, we prove the following lemma:

LEMMA 6-6. *Let $\chi\in\{\mathbf{1}_{F^\times},\mathbf{sgn}_u\}$ and let $l\in\mathbb{Z}_{\geq 1}$ such that $l-(2\alpha_\chi+1)\notin 2\mathbb{Z}$. Let σ_{neg} be a negative representation. Then the induced representation $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle\rtimes\sigma_{neg}$, where $\alpha\in\mathbb{R}_{>0}$, reduces at $\alpha=1/2$ and its unique unramified irreducible subquotient σ'_{neg} is a negative representation. Further, we have the following:*

$$\mathrm{Jord}(\sigma'_{neg})=\mathrm{Jord}(\sigma_{neg})+\{(l-1,\chi),(l+1,\chi)\}$$

(If $l=1$, then we omit $(l-1,\chi)$.)

PROOF. First, Lemma 6-1 implies that $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle\rtimes\sigma_{neg}$ is reducible at $\alpha=1/2$. We need to show that the unique unramified irreducible subquotient there is negative. First, applying Theorem 5-10, we find a a sequence of the pairs $(l_1,\chi_1),\ldots,(l_k,\chi_k)$ ($l_i\in\mathbb{Z}_{>0}$, χ_i is an unramified unitary character of $F^\times$), unique up to a permutation and taking inverses of characters, and the unique strongly negative representation σ_{sn} such that

$$\sigma_{neg}\hookrightarrow\quad\langle[-\frac{l_1-1}{2},\frac{l_1-1}{2}]^{(\chi_1)}\rangle\times\cdots\times\langle[-\frac{l_k-1}{2},\frac{l_k-1}{2}]^{(\chi_k)}\rangle\rtimes\sigma_{sn}\tag{6-7}$$

and

$$\mathrm{Jord}(\sigma_{neg})=\mathrm{Jord}(\sigma_{sn})+\sum_{i=1}^{k}\{(l_i,\chi_i),(l_i,\chi_i^{-1})\}.\tag{6-8}$$

Now, using Theorem 5-8 and the explicit description of strongly negative representations (see (5-6) and (5-7)), it is easy to check the following:

- If $(l-1,\chi),(l+1,\chi)\notin\mathrm{Jord}(\sigma_{sn})$, then there is a strongly negative representation σ'_{sn} such that $\mathrm{Jord}(\sigma'_{sn})=\mathrm{Jord}(\sigma_{sn})+\{(l-1,\chi),(l+1,\chi)\}$. We let ${\sigma''}_{neg}=\sigma'_{sn}$.
- If $(l-1,\chi)\in\mathrm{Jord}(\sigma_{sn})$, $(l+1,\chi)\notin\mathrm{Jord}(\sigma_{sn})$, then there is a unique strongly negative representation σ'_{sn} such that

  $$\mathrm{Jord}(\sigma'_{sn})=\mathrm{Jord}(\sigma_{sn})-\{(l-1,\chi)\}+\{(l+1,\chi)\}.$$

  Let ${\sigma''}_{neg}$ be the unique irreducible unramified subrepresentation of $\langle[-\frac{l-2}{2},\frac{l-2}{2}]^{(\chi)}\rangle\rtimes\sigma_{sn}$. Then

  $$\mathrm{Jord}({\sigma''}_{neg})=\mathrm{Jord}(\sigma'_{sn})+2\cdot\{(l-1,\chi)\}.$$

- If $(l-1,\chi)\notin\mathrm{Jord}(\sigma_{sn})$, $(l+1,\chi)\in\mathrm{Jord}(\sigma_{sn})$, then there is a unique strongly negative representation σ'_{sn} such that

  $$\mathrm{Jord}(\sigma'_{sn})=\mathrm{Jord}(\sigma_{sn})+\{(l-1,\chi)\}-\{(l+1,\chi)\}.$$

  Let ${\sigma''}_{neg}$ be the unique irreducible unramified subrepresentation of $\langle[-\frac{l}{2},\frac{l}{2}]^{(\chi)}\rangle\rtimes\sigma'_{sn}$. Then

  $$\mathrm{Jord}({\sigma''}_{neg})=\mathrm{Jord}(\sigma'_{sn})+2\cdot\{(l+1,\chi)\}.$$

- If $(l-1,\chi),(l+1,\chi)\in \mathrm{Jord}(\sigma_{sn})$, then there is a unique strongly negative representation σ'_{sn} such that
$$\mathrm{Jord}(\sigma'_{sn})=\mathrm{Jord}(\sigma_{sn})-\{(l-1,\chi),(l+1,\chi)\}.$$
Let σ''_{neg} be the unique irreducible unramified subrepresentation of $\langle[-\frac{l-2}{2},\frac{l-2}{2}]^{(\chi)}\rangle\times\langle[-\frac{l}{2},\frac{l}{2}]^{(\chi)}\rangle\rtimes\sigma'_{sn}$. Then
$$\mathrm{Jord}(\sigma''_{neg})=\mathrm{Jord}(\sigma'_{sn})+2\cdot\{(l-1,\chi),(l+1,\chi)\}.$$

Now, the unique irreducible unramified subquotient of $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\frac{1}{2}}\chi)}\rangle\rtimes\sigma_{sn}$ is σ''_{neg}, which is described above. Combining this with (6-7) and (6-8), we find that the unique irreducible unramified subquotient σ'_{neg} of $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\frac{1}{2}}\chi)}\rangle\rtimes\sigma_{neg}$ is a subrepresentation of $\langle[-\frac{l_1-1}{2},\frac{l_1-1}{2}]^{(\chi_1)}\rangle\times\cdots\times\langle[-\frac{l_k-1}{2},\frac{l_k-1}{2}]^{(\chi_k)}\rangle\rtimes\sigma''_{neg}$. Clearly, it is negative and $\mathrm{Jord}(\sigma'_{neg})=\mathrm{Jord}(\sigma_{neg})+\{(l-1,\chi),(l+1,\chi)\}$. $\square$

We end this section by proving the following lemma:

LEMMA 6-9. *Assume that χ,χ' are unitary unramified characters of $F^\times$, $l,l'\in\mathbb{Z}_{\geq 1}$, and $\alpha,\alpha'\in\mathbb{R}_{>0}$. Then we have the following:*

(i) *If $\alpha\in]0,\ 1[$ and $\alpha'\in]0,\ \frac{1}{2}]$, then the segments $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle$ and $\langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\nu^{-\alpha'}\chi')}\rangle$ are linked if and only if $\alpha+\alpha'=1$, $\chi'=\chi$, $l'=l$.*

(ii) *If $\alpha,\alpha'\in]0,\ 1[$, then $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle$ and $\langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle$ are not linked.*

(iii) *If $\alpha,\alpha'\in]\frac{1}{2},\ 1[$, then $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle$ and $\langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\nu^{-\alpha'}\chi')}\rangle$ are linked if and only if $\alpha+\alpha'=3/2$, $\chi'=\chi$, and $l'=l\pm 1$.*

(iv) *If $\alpha\in]1,\ \frac{3}{2}[$ and $\alpha'\in]0,\ \frac{1}{2}[$, then the segments $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle$ and $\langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\nu^{-\alpha'}\chi')}\rangle$ are linked if and only if $\alpha+\alpha'=3/2$, $\chi'=\chi$, and $l'=l\pm 1$.*

(v) *If $\alpha\in]1,\ \frac{3}{2}[$ and $\alpha'\in]0,\ \frac{1}{2}[$, then the segments $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle$ and $\langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle$ are linked if and only if $\alpha=1+\alpha'$, $\chi'=\chi$, and $l'=l$.*

PROOF. We prove (i). The segments are linked if and only if $\chi'=\chi$, and there exist $m,m'\in\mathbb{Z}_{\geq 1}$ such that

$$\begin{cases}\frac{l'-1}{2}-\alpha'=-m+\frac{l-1}{2}+\alpha\geq-\frac{l-1}{2}+\alpha-1\\ -\frac{l'-1}{2}-\alpha'=-m'-\frac{l-1}{2}+\alpha.\end{cases}\tag{6-10}$$

Adding the equalities, we obtain $(m+m')/2=\alpha+\alpha'$. Since $\alpha+\alpha'\in]0,\ \frac{3}{2}[$, we obtain $\alpha+\alpha'=1$ and $m=m'=1$. This proves one direction in (i). The opposite direction is obvious. We prove (ii). We may assume $\alpha'\leq\alpha$. If the segments are linked, then $\chi'=\chi$, and there exist $m,m'\in\mathbb{Z}_{\geq 1}$ such that

$$\begin{cases}\frac{l'-1}{2}+\alpha'=-m+\frac{l-1}{2}+\alpha\geq-\frac{l-1}{2}+\alpha-1\\ -\frac{l'-1}{2}+\alpha'=-m'-\frac{l-1}{2}+\alpha.\end{cases}\tag{6-11}$$

Adding the equalities, we obtain $\alpha=(m+m')/2+\alpha'\geq 1$. This is a contradiction. We prove (iii). If the segments are linked, then $\chi'=\chi$, and there exist $m,m'\in\mathbb{Z}_{\geq 1}$ such that (6-10) holds. Adding the equalities in (6-10), we obtain $(m+m')/2=$

$\alpha+\alpha'$. Since $\alpha+\alpha' \in]1,\ 2[$, we obtain $\alpha+\alpha'=\frac{3}{2}$, and $m=1, m'=2$ or $m=2, m'=1$. If $m=1, m'=2$, then $l'=l+1$. Otherwise, $l'=l-1$. The converse is obvious. The proof of (iv) is similar to that of (iii). We prove (v). Adding the equalities in (6-11), we obtain $\alpha=(m+m')/2+\alpha'$. Since $\alpha \in]1,\ \frac{3}{2}[$, $\alpha' \in]0,\ \frac{1}{2}[$, and $(m+m')/2 \in \frac{1}{2}\mathbb{Z}$, we find $m=m'=1$ and $\alpha=1+\alpha'$.

□

7. A Result on Non–Unitarity

In this section we use analytic techniques from [**M6**] to prove the non–unitarity of certain representations. The non–trivial part is an application of (RP) (see Section 2). The proof of the surjectivity of the map from Theorem 5-14 given in Section 9 depends critically on that result. We advise the reader to skip this section on the first reading.

The main result is the following theorem:

THEOREM 7-1. *Let $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and let $l \in \mathbb{Z}_{\geq 1}$. Let σ_{neg} be a negative representation. Then the induced representation $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \rtimes \sigma_{neg}$ is not unitarizable in the following two cases:*

- *for $\alpha \in]0,\ 1[$ if $l-(2\alpha_\chi+1) \in 2\mathbb{Z}$ and $(l,\chi) \notin \mathrm{Jord}(\sigma_{neg})$*
- *for $\alpha \in]\frac{1}{2},\ 1[$ if $l-(2\alpha_\chi+1) \notin 2\mathbb{Z}$*

The remainder of this section is devoted to the proof of Theorem 7-1. We freely use the notation and results of [**M6**]. We consider a continuous family of representations:

$$\sigma_s = \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^s\chi)}\rangle \rtimes \sigma_{neg} \quad (s \in \mathbb{C})$$

of S_n. Let K be the maximal compact subgroup of S_n fixed in Section 1. Restricting to K, we may realize all representations σ_s on the same space X.

Let w_0 be the non–trivial element of the Weyl group $W(M)$, where M is the Levi subgroup of the standard maximal parabolic subgroup $P=MN$ of S_n such that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^s\chi)}\rangle \otimes \sigma_{neg}$ is a representation of M. Hence $\sigma_s = \mathrm{Ind}_P^{S_n}(\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^s\chi)}\rangle \otimes \sigma_{neg})$. We fix (and denote by the same letter) the representative of w_0 as explained in ([**M6**], Section 2). Next, let $N(s,w_0)$ be the standard normalized intertwining operator

$$N(s,w_0): \sigma_s \to \sigma_{-s}$$

as explained in ([**M6**], Section 2) (see also [**Sh2**]). The geometric construction is given in [**M7**]. We consider it realized in the compact picture. We list its basic properties.

(norm-1) $N(s,w_0) \neq 0$ since it takes a suitable normalized K–invariant vector $0 \neq f_0 \in X$ onto itself.
(norm-2) $N(s,w_0)N(-s,w_0)=N(-s,w_0)N(s,w_0)=id_X$
(norm-3) $N(s,w_0)$ is Hermitian for $s \in \sqrt{-1}\mathbb{R}$, and therefore holomorphic there.

Now, we begin the proof of Theorem 7-1. We consider the family of Hermitian forms introduced in (2-8). We remark that σ_0 reduces if and only if $l-(2\alpha_\chi+1) \in 2\mathbb{Z}$ and $(l,\chi) \notin \mathrm{Jord}(\sigma_{neg})$ (see Lemma 6-5) while σ_s is irreducible for $s \in]0,\ 1[-\{\frac{1}{2}\}$ by Lemma 6-1. Next, $\sigma_{1/2}$ is reducible if and only if $l-(2\alpha_\chi+1) \notin 2\mathbb{Z}$.

Assume that $l-(2\alpha_\chi+1)\in 2\mathbb{Z}$ and $(l,\chi)\notin \mathrm{Jord}(\sigma_{neg})$. Then, by Lemma 6-5, σ_0 is a direct sum of two irreducible representations. Then using standard properties of normalized intertwining operators, we reduce the proof of non-triviality of $N(0,w_0)$ to the case when σ_{neg} is strongly negative. Then we may apply [**Bn**]. Therefore, $N(0,w_0)$ acts on one of the representations as $+id$ while on the other it acts as $-id$. Therefore the Hermitian form defined by (2-8) is not definite for $s\in]0,\ 1[$, proving the first claim.

Assume that $l-(2\alpha_\chi+1)\notin 2\mathbb{Z}$. We apply our general principle (RP) (see Section 2) to prove the second claim. We must check its assumption. We prove the following result which completes the proof of Theorem 7-1:

Lemma 7-2. *Maintaining the above assumptions, $N(s,w_0)$ is holomorphic and $N(-s,w_0)$ has a simple pole at $s=1/2$.*

First, let π be the unique irreducible subrepresentation of $\sigma_{1/2}$ (which is equivalent to the unique irreducible quotient of $\sigma_{-1/2}$) (see [**M6**]). By the classification of irreducible unramified representations [**M6**] and Lemma 6-6, π is not unramified. Therefore, we have the following:

$$N(-s,w_0^{-1}) \text{ has a pole at } s=1/2. \tag{7-3}$$

From this point, the argument is standard and it follows the lines of the proof of ([**M6**], Lemma 3.5). First, we reduce to the case where σ_{neg} is strongly negative. Applying Theorem 5-10, we can find $l'\in\mathbb{Z}_{\geq 1}$, a unitary unramified character χ', and a negative representation σ'_{neg} such that

$$\sigma_{neg}\hookrightarrow \langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\chi')}\rangle\rtimes\sigma'_{neg}.$$

This implies the following commutative diagram (all involved intertwining operators are standard normalized operators; i_s is an embedding depending holomorphically on s):

$$\begin{array}{ccc}
\sigma_s & \xrightarrow{i_s} & \langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^s\chi)}\rangle\times\langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\chi')}\rangle\rtimes\sigma'_{neg}\\
 & & \Big\downarrow{\scriptstyle N_1(s)}\\
 & & \langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\chi')}\rangle\times\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^s\chi)}\rangle\rtimes\sigma'_{neg}\\
{\scriptstyle N(s,w_0)}\Big\downarrow & & \Big\downarrow{\scriptstyle N_2(s)}\\
 & & \langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\chi')}\rangle\times\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{-s}\chi)}\rangle\rtimes\sigma'_{neg}\\
 & & \Big\downarrow{\scriptstyle N_3(s)}\\
\sigma_{-s} & \xrightarrow{i_{-s}} & \langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{-s}\chi)}\rangle\times\langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\chi')}\rangle\rtimes\sigma'_{neg}.
\end{array}$$

At $s=\pm 1/2$, $\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\pm s}\chi)}\rangle\times\langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\chi')}$ is irreducible, and therefore $N_1(s)$ and $N_3(s)$ are holomorphic (by the clear analogies of (norm-1) and (norm-2) for them). This proves the first step of the reduction; we assume that σ_{neg} is strongly negative. To avoid any confusion we write σ_{sn} instead of σ_{neg}.

Next, we consider the following diagram:

$$\begin{array}{cccl} & \sigma_s & \xrightarrow{i_s} & \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\nu^{s-\frac{1}{2}}\chi)}\rangle \times \nu^{\frac{l-1}{2}+s}\chi \rtimes \sigma_{sn} \\ & & & \Big\downarrow{\scriptstyle N_1(s)} \\ & & & \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\nu^{s-\frac{1}{2}}\chi)}\rangle \times \nu^{-\frac{l-1}{2}-s}\chi \rtimes \sigma_{sn} \\ \text{(7-4)} & \Big\downarrow{\scriptstyle N(s,w_0)} & & \Big\downarrow{\scriptstyle N_2(s)} \\ & & & \nu^{-\frac{l-1}{2}-s}\chi \times \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\nu^{s-\frac{1}{2}}\chi)}\rangle \rtimes \sigma_{sn} \\ & & & \Big\downarrow{\scriptstyle N_3(s)} \\ & \sigma_{-s} & \xrightarrow{i_{-s}} & \nu^{-\frac{l-1}{2}-s}\chi \times \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\nu^{-s+\frac{1}{2}}\chi)}\rangle \rtimes \sigma_{sn}. \end{array}$$

(If $l = 1$, then $N_2(s)$ and $N_3(s)$ are not present.)

Now, as in the proof of ([**M6**], Lemma 3.5) (or adapting the argument for the normalized intertwining operator (7-8) below), it follows that $N_1(s)$ is holomorphic at $s = 1/2$. Next, $N_2(s)$ is holomorphic at $s = 1/2$ since we have the following diagram (j_s is an embedding depending holomorphically on s):

$$\begin{array}{cccc} & \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\nu^{s-\frac{1}{2}}\chi)}\rangle \times \nu^{-\frac{l-1}{2}-s}\chi & \xrightarrow{j_s} & Ind(s) \\ \text{(7-5)} & \Big\downarrow{\scriptstyle N_2(s)} & & \Big\downarrow{\scriptstyle N_4(s)} \\ & \nu^{-\frac{l-1}{2}-s}\chi \times \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\nu^{s-\frac{1}{2}}\chi)}\rangle & \xrightarrow{j_{-s}} & Ind_1(s), \end{array}$$

where

$$\begin{cases} Ind(s) = \nu^{-\frac{l-1}{2}+s}\chi \times \cdots \times \nu^{\frac{l-1}{2}-1+s}\chi \times \nu^{-\frac{l-1}{2}-s}\chi \\ Ind_1(s) = \nu^{-\frac{l-1}{2}-s}\chi \times \nu^{-\frac{l-1}{2}+s}\chi \times \cdots \times \nu^{\frac{l-1}{2}-1+s}\chi, \end{cases}$$

and the normalized operator $N_4(s)$ is a composition of normalized operators induced from the rank–one operators:

$$Q_i(s) : \nu^{-\frac{l-1}{2}+s+i}\chi \times \nu^{-\frac{l-1}{2}-s}\chi \to \nu^{-\frac{l-1}{2}-s}\chi \times \nu^{-\frac{l-1}{2}+s+i}\chi,$$

where $i = 0, \ldots l-2$. The normalized intertwining operators $Q_i(s)$ are holomorphic at $s = 1/2$ by the basic property of the normalization (see for example ([**M6**], Theorem 2.1)). Finally, by the analogue of (norm-3), $N_3(s)$ is holomorphic. This proves that $N(s, w_0)$ is holomorphic at $s = 1/2$ (see (7-4)). It remains to prove that $N(-s, w_0)$ has a simple pole at $s = 1/2$. To accomplish this we reverse the vertical arrows in (7-4) and change s into $-s$ in the arguments of all $N_i(\cdot)$ and $N(\cdot, w_0)$. We remind the reader that $N_2(-s)$ and $N_3(-s)$ are present if and only if $l > 1$. We assume $l > 1$. Now, arguing as above, we see that $N_3(-s)$ is holomorphic. Similarly, arguing as in (7-5), we see that $N_2(-s)$ has at most a simple pole at $s = 1/2$. The pole must be present since, by [**Ze**], the unique irreducible quotient of $\nu^{-\frac{l}{2}}\chi \times \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\chi)}\rangle$ is the unique irreducible subrepresentation of $\langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\chi)}\rangle \times \nu^{-\frac{l}{2}}\chi$ and it is different than $\langle[-\frac{l}{2}, \frac{l}{2} - 1]^{(\chi)}\rangle$. Thus, we obtain

(7-6) $\qquad N_2(-s)$ has a simple pole at $s = 1/2$.

We investigate the influence of that pole on the image of $N_3(-1/2)$. Then the discussion above shows that $N_3(-1/2)$ is not an isomorphism if and only if $\langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\chi)}\rangle \rtimes \sigma_{sn}$ reduces. Applying Lemma 6-5 and our assumption $l-(2\alpha_\chi +$

$1) \notin 2\mathbb{Z}$, we see that if $l > 1$, then $N_3(-1/2)$ is not an isomorphism if and only if $(l-1,\chi) \notin \mathrm{Jord}(\sigma_{sn})$. In particular, if $N_3(-1/2)$ is an isomorphism (hence, a scalar multiple of the identity), then the image of $\sigma_{-1/2}$ under $i_{-1/2}$ is the same as its image under $N_3(-1/2)i_{-1/2}$; $N_2(-1/2)$ is holomorphic on that image. Thus, we summarize the discussion as follows:

(7-7) If $l > 1$ and $(l-1,\chi) \in \mathrm{Jord}(\sigma_{sn})$, then $N_2(-s)N_3(-s)i_{-s}$ is holomorphic at $s = 1/2$.

Next, we consider

$$N_1(-s) : \nu^{-\frac{l-1}{2}-s}\chi \rtimes \sigma_{sn} \to \nu^{\frac{l-1}{2}+s}\chi \rtimes \sigma_{sn} \tag{7-8}$$

at $s = 1/2$. First, we have the following:

(7-9) if $\nu^{\frac{l}{2}}\chi \rtimes \sigma_{sn}$ is irreducible, then $N_1(-s)$ is holomorphic at $s = 1/2$.

We describe the reducibility of $\nu^{\frac{l}{2}}\chi \rtimes \sigma_{sn}$ using Lemma 6-1:

- (red-1) Assume $l > 1$. Then $\nu^{\frac{l}{2}}\chi \rtimes \sigma_{sn}$ is reducible if and only if $(l-1,\chi) \in \mathrm{Jord}(\sigma_{sn})$.
- (red-2) Assume $l = 1$. Then the assumption $l - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$ implies $\alpha_\chi = 1/2$. In this case we have the reducibility.

We analyze $N_1(-s)$ at $s = 1/2$. First, we apply Theorem 5-8 and (5-7) to obtain:

$$\sigma_{sn} \hookrightarrow \tag{7-10}$$
$$\times_{i=1}^{l_{\mathbf{1}_{F^\times}}} \langle [-\frac{a_{2i}-1}{2}, \frac{a_{2i-1}-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle \times_{j=1}^{l_{\mathbf{sgn}_u}} \langle [-\frac{b_{2j}-1}{2}, \frac{b_{2j-1}-1}{2}]^{(\mathbf{sgn}_u)}\rangle \rtimes \mathbf{1}.$$

Hence we can write the following commutative diagram (where the vertical arrows are normalized intertwining operators; j_s is an embedding depending holomorphically on s):

$$\begin{array}{ccc}
\nu^{-\frac{l-1}{2}-s}\chi \rtimes \sigma_{sn} & \xrightarrow{j_{-s}} & \nu^{-\frac{l-1}{2}-s}\chi \rtimes Ind \\
{\scriptstyle N_1(-s)}\big\downarrow & & {\scriptstyle N0_1(s)}\big\downarrow \\
\nu^{\frac{l-1}{2}+s}\chi \rtimes \sigma_{sn} & \xrightarrow{j_s} & \nu^{\frac{l-1}{2}+s}\chi \rtimes Ind,
\end{array}$$

where

$$Ind = \times_{i=1}^{l_{\mathbf{1}_{F^\times}}} \langle [-\frac{a_{2i}-1}{2}, \frac{a_{2i-1}-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle \times_{j=1}^{l_{\mathbf{sgn}_u}} \langle [-\frac{b_{2j}-1}{2}, \frac{b_{2j-1}-1}{2}]^{(\mathbf{sgn}_u)}\rangle \rtimes \mathbf{1}.$$

Next, the normalized operator $N0_1(s)$ can be factorized into the product of the following normalized operators:

$$\nu^{-\frac{l-1}{2}-s}\chi \times \langle [-\frac{a_{2i}-1}{2}, \frac{a_{2i-1}-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle \xrightarrow{Q_{i,\mathbf{1}_{F^\times}}}$$
$$\langle [-\frac{a_{2i}-1}{2}, \frac{a_{2i-1}-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle \times \nu^{-\frac{l-1}{2}-s}\chi,$$

$$\nu^{-\frac{l-1}{2}-s}\chi \times \langle [-\frac{b_{2i}-1}{2}, \frac{b_{2i-1}-1}{2}]^{(\mathbf{sgn}_u)}\rangle \xrightarrow{Q_{i,\mathbf{sgn}_u}}$$
$$\langle [-\frac{b_{2i}-1}{2}, \frac{b_{2i-1}-1}{2}]^{(\mathbf{sgn}_u)}\rangle \times \nu^{-\frac{l-1}{2}-s}\chi,$$

$$\nu^{-\frac{l-1}{2}-s}\chi \rtimes \mathbf{1} \xrightarrow{P(s)} \nu^{\frac{l-1}{2}+s}\chi \rtimes \mathbf{1},$$

$$\langle [-\frac{a_{2i}-1}{2}, \frac{a_{2i-1}-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle \times \nu^{\frac{l-1}{2}+s}\chi \xrightarrow{R_{i,\mathbf{1}_{F^\times}}} \nu^{\frac{l-1}{2}+s}\chi \times \langle [-\frac{a_{2i}-1}{2}, \frac{a_{2i-1}-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle,$$

and

$$\langle [-\frac{b_{2i}-1}{2}, \frac{b_{2i-1}-1}{2}]^{(\mathbf{sgn}_u)}\rangle \times \nu^{\frac{l-1}{2}+s}\chi \xrightarrow{R_{i,\mathbf{sgn}_u}} \nu^{\frac{l-1}{2}+s}\chi \times \langle [-\frac{b_{2i}-1}{2}, \frac{b_{2i-1}-1}{2}]^{(\mathbf{sgn}_u)}\rangle$$

We analyze them at $s=-1/2$. First, if $l=1$, then all of them except $P(s)$ are holomorphic. By the rank–one theory $P(s)$ has a simple pole at $s=-1/2$. This proves that $N(-s, w_0)$ has a simple pole in this case. We assume $l>1$. Now, if $(l-1,\chi) \not\in \mathrm{Jord}(\sigma_{sn})$, then (red-1) and (7-9) imply that $N_1(-s)$ is holomorphic at $s=-1/2$; combining (7-3) and (7-6), $N(-s, w_0)$ has a simple pole at $s=-1/2$. From now on, we assume $(l-1,\chi) \in \mathrm{Jord}(\sigma_{sn})$. Then we need to prove that $N_1(-s)$ has at most a simple pole at $s=1/2$ since then (7-3) and (7-7) imply $N(-s, w_0)$ has a simple pole at $s=-1/2$.

Assume that $l>2$ or $l=2$ and $\alpha_\chi \neq 1$. Then $Q_{i,\mathbf{1}_{F^\times}}(s)$ (resp., $Q_{i,\mathbf{sgn}_u}(s)$) has at most a simple pole at $s=-1/2$ if and only if $\chi = \mathbf{1}_{F^\times}$ (resp., $\chi=\mathbf{sgn}_u$) (noting $(l-1,\chi)\in \mathrm{Jord}(\sigma_{sn})$). If this is so, $R_{i,\mathbf{1}_{F^\times}}(s)$ and $R_{i,\mathbf{sgn}_u}(s)$ are ismorphisms (hence holomorphic) at $s=-1/2$. The converse statement is also true. Thus, the contribution of all the normalized operators $Q_{i,\mathbf{1}_{F^\times}}(s), Q_{i,\mathbf{sgn}_u}(s), R_{i,\mathbf{1}_{F^\times}}(s), R_{i,\mathbf{1}_{F^\times}}(s)$ is just at a simple pole at $s=-1/2$. Further, since $l>2$ or $l=2$ and $\alpha_\chi \neq 1$, $R(s)$ is holomorphic at $s=-1/2$. This proves that $N(-s, w_0)$ has a simple pole at $s=-1/2$ in this case. Finally, we assume $l=2$ and $\alpha_\chi=1$. Then $S_n = \mathrm{Sp}(2n, F)$ and card $\mathrm{Jord}(\sigma_{sn})_{\mathbf{1}_{F^\times}}$ is odd (see Definition 5-4). Applying (5-6), we obtain $a_1=-1<0<a_2<\cdots$. Since $(1,\mathbf{1}_{F^\times}) \in \mathrm{Jord}(\sigma_{sn})$, $a_2=1$. Thus, $\langle [-\frac{a_2-1}{2}, \frac{a_1-1}{2}]^{(\mathbf{1}_{F^\times})}\rangle = \langle [0,-1]^{(\mathbf{1}_{F^\times})}\rangle$ is empty. In particular, $Q_{1,\mathbf{1}_{F^\times}}(s)$ and $R_{1,\mathbf{1}_{F^\times}}(s)$ are not present. Thus, all normalized operators $Q_{i,\mathbf{1}_{F^\times}}(s), Q_{i,\mathbf{sgn}_u}(s), R_{i,\mathbf{1}_{F^\times}}(s), R_{i,\mathbf{sgn}_u}(s)$ are holomorphic at $s=-1/2$. Since $R(s)$ has a simple pole at $s=-1/2$, the proof is complete.

8. The Injectivity of $\mathcal{M}^{u,unr}(S_n) \to \mathrm{Irr}^{u,unr}(S_n)$

In this section we show the map $\mathcal{M}^{u,unr}(S_n) \to \mathrm{Irr}^{u,unr}(S_n)$ given by

$$(\mathbf{e}, \sigma_{neg}) \longmapsto \times_{(l,\chi,\alpha)\in\mathbf{e}} \ \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \rtimes \sigma_{neg}$$

(see Theorem 5-14) is well–defined and injective. Lemmas 8-1 and 8-2 show that the map is well-defined; injectivity then follows.

Lemma 8-1. *Let $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S)$. Then the induced representation $\left(\times_{(l,\chi,\alpha)\in\mathbf{e}} \ \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle\right) \rtimes \sigma_{neg}$ is irreducible.*

Proof. This follows from Lemma 6-1 and Lemma 6-9 (i), (ii) and (iii). □

LEMMA 8-2. *Let* $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S)$. *Then the induced representation* $\left(\times_{(l,\chi,\alpha)\in\mathbf{e}} \ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle\right) \rtimes \sigma_{neg}$ *is unitarizable (and irreducible).*

PROOF. We let σ be that induced (irreducible) representation. We prove the unitarity of σ by induction on $m := \text{card } \mathbf{e}$. If $m = 0$, then $\mathbf{e} = \emptyset$. Therefore $\sigma = \sigma_{neg}$ is unitarizable by Theorem 5-11. Assume that the claim is true for all $(\mathbf{e}', \sigma'_{neg}) \in \mathcal{M}^{u,unr}(S)$ with card $\mathbf{e}' < m$. Now, we proceed according to Definition 5-13 (1)–(3) as follows.

$(Def-1)$ Assume that there exists $\chi \notin \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l \in \mathbb{Z}_{\geq 1}$ such that $\mathbf{e}(l,\chi) \neq \emptyset$. Then we pick some $\alpha \in \mathbf{e}(l,\chi)$. Applying Definition 5-13 (1), $\alpha \in \mathbf{e}(l,\chi^{-1})$. We let $\mathbf{e}' = \mathbf{e} - \{(l,\chi,\alpha),(l,\chi^{-1},\alpha)\}$. Then it is easy to see that $(\mathbf{e}', \sigma_{neg}) \in \mathcal{M}^{u,unr}(S)$. Since card $\mathbf{e}' <$ card $\mathbf{e} =: m$, we apply the inductive assumption to obtain the unitarity of σ' defined by

$$\sigma' = \times_{(l',\chi',\alpha')\in\mathbf{e}'} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg}.$$

Next, we have

$$\begin{aligned}\sigma &\simeq \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \times \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi^{-1})} \rtimes \sigma' \\ &\simeq \left(\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \times \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\alpha}\chi)}\rangle\right) \rtimes \sigma'.\end{aligned}$$

Since, by Definition 5-13 (1), $\alpha \in]0, \ \frac{1}{2}[$, the unitarity of σ follows from Theorem 4-1 and (UI).

$(Def-2)$ Assume that there exists $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l \in \mathbb{Z}_{\geq 1}$, such that $l - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$ and $\mathbf{e}(l,\chi) \neq \emptyset$. Then we pick some $\alpha \in \mathbf{e}(l,\chi)$. We let $\mathbf{e}' = \mathbf{e} - \{(l,\chi,\alpha)\}$. Then it is obvious that $(\mathbf{e}', \sigma_{neg}) \in \mathcal{M}^{u,unr}(S)$. Since card $\mathbf{e}' <$ card $\mathbf{e} =: m$, we apply the inductive assumption to obtain the unitarity of σ' defined by $\sigma' = \times_{(l',\chi',\alpha')\in\mathbf{e}'} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg}$. We claim that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma'$ is irreducible. Namely, since $l - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$, Lemma 6-5 implies that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma_{neg}$ is irreducible. Clearly, this representation is negative and we denote it by σ'_{neg}. Using this it is easy to show $(\mathbf{e}', \sigma'_{neg}) \in \mathcal{M}^{u,unr}(S)$. Next, the attached induced representation

$$\begin{aligned}&\times_{(l',\chi',\alpha')\in\mathbf{e}'} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma'_{neg} \simeq \\ &\quad \times_{(l',\chi',\alpha')\in\mathbf{e}'} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \left(\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma_{neg}\right) \simeq \\ &\quad \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \times \left(\times_{(l',\chi',\alpha')\in\mathbf{e}'} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg}\right) \simeq \\ &\qquad\qquad\qquad\qquad \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma'\end{aligned}$$

is irreducible by Lemma 8-1. Similarly, using induction in stages, Lemma 6-1 implies the irreducibility of $\sigma_s = \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^s\chi)}\rangle \rtimes \sigma'$ for $s \in]0, \ \frac{1}{2}[$. Now, (D) implies the unitarity of σ_s. Since $\sigma \simeq \sigma_\alpha$, we have proved its unitarity.

$(Def-3)$ Assume that there exists $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l \in \mathbb{Z}_{\geq 1}$, such that $l - (2\alpha_\chi + 1) \in 2\mathbb{Z}$ and $\mathbf{e}(l,\chi) \neq \emptyset$. We use the notation introduced in Definition 5-13 (3). If there are indices $i_1 \neq i_2$ such that α_{i_1} and α_{i_2} both belong to one of the segments $]1-\beta_1, \frac{1}{2}]$, $]0, 1-\beta_v[$, or $]1-\beta_{j+1}, 1-\beta_j[$ (for some j) or simply if $v = 0$

but $u \geq 2$ (so we can arbitrarily pick the two different indices $i_1 \neq i_2$), then we let $\mathbf{e}' = \mathbf{e} - \{(l, \chi, \alpha_{i_1}), (l, \chi, \alpha_{i_2})\}$. Then it is obvious that $(\mathbf{e}', \sigma_{neg}) \in \mathcal{M}^{u,unr}(S)$. Since card $\mathbf{e}' <$ card $\mathbf{e} =: m$, we apply the inductive assumption to obtain the unitarity of σ' defined by $\sigma' = \times_{(l',\chi',\alpha')\in \mathbf{e}'} \ \langle [-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg}$. It is easy to see that we can write σ as follows:

$$\begin{aligned}\sigma \simeq \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_{i_1}}\chi)}\rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_{i_2}}\chi)} \rtimes \sigma' \\ \simeq \left(\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_{i_1}}\chi)}\rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\alpha_{i_2}}\chi)}\rangle \right) \rtimes \sigma'.\end{aligned}$$

The inductive assumption applied to σ' and Lemma 6-4 implies that σ is Hermitian. Now, since

$$\left(\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_{i_2}}\chi)}\rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\alpha_{i_2}}\chi)}\rangle \right) \rtimes \sigma' \tag{8-3}$$

is unitarizable by (UI) (applying $\alpha_{i_2} \in]0, \ \frac{1}{2}[$ and Theorem 4-1), the way we have chosen α_{i_1} and α_{i_2} enable us to deform the first exponent α_{i_2} (see (8-3)) to α_{i_1} proving the unitarity of σ by (D). Thus, if $v = 0$, we may assume that $u \in \{0, 1\}$. If $v = 0, u = 1$, then $u + v = 1$ is odd. Hence $(l, \chi) \in \mathrm{Jord}(\sigma_{neg})$ (see Definition 5-13 (3) (a)). Therefore, by Lemma 6-1, $\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma_{neg}$ is irreducible, and we proceed as in the case (Def-2) with $\alpha = \alpha_1$.

Now, we assume $v = 1$. Then, by our reduction, we may assume that $]1-\beta_1, \frac{1}{2}]$ does not contain any α_i, while $]0, \ 1-\beta_1[$ contains all. Therefore we may assume $u \in \{0, 1\}$. If $u = 0$, then $u+v = 1$ is odd. Hence $(l, \chi) \in \mathrm{Jord}(\sigma_{neg})$ (see Definition 5-13 (3) (a)). Therefore, by Lemma 6-1, $\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma_{neg}$ is irreducible, and we proceed as in the case $(Def-2)$ with $\alpha = \beta_1$. If $u = v = 1$, then we need to prove the unitarity of

$$\sigma \simeq \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_1}\chi)}\rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\beta_1}\chi)}\rangle \rtimes \sigma',$$

where σ' is attached to $(\mathbf{e}', \sigma_{neg})$ with

$$\mathbf{e}' = \mathbf{e} - \{(l, \chi, \alpha_1), (l, \chi, \beta_1)\}.$$

(Clearly, by induction, σ' is unitarizable.) We start from the following family of induced representations:

$$\begin{aligned}\sigma_s := \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_1}\chi)}\rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{s}\chi)}\rangle \rtimes \sigma' \simeq \\ \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_1}\chi)}\rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-s}\chi)}\rangle \rtimes \sigma',\end{aligned}$$

where $s \in [\alpha_1, 1-\alpha_1[$. Lemma 8-1 implies that every representation σ_s ($s \in [\alpha_1, 1-\alpha_1[$) is irreducible. Since it is unitarizable for $s = \alpha_1$ by the above isomorphism and Theorem 4-1, (D) implies the unitarizability of σ_s for every $s \in [\alpha_1, 1-\alpha_1[$. Since $\beta_1 \in [\alpha_1, 1-\alpha_1[$, we see that $\sigma = \sigma_{\beta_1}$ is unitarizable.

Finally, we assume $v \geq 2$. Then, by our reduction, we may assume that $]1 - \beta_1, \frac{1}{2}]$ does not contain any α_i while $]1-\beta_2, 1-\beta_1[$ must contain a unique α_i. Hence $u \geq 1$ and $\alpha_u \in]1-\beta_2, 1-\beta_1[$. We need to prove the unitarity of

$$\sigma \simeq \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_u}\chi)}\rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\beta_1}\chi)}\rangle \rtimes \sigma',$$

where σ' is attached to $(\mathbf{e}', \sigma_{neg})$ with

$$\mathbf{e}' = \mathbf{e} - \{(l, \chi, \alpha_u), (l, \chi, \beta_1)\}.$$

(Clearly, by induction, σ' is unitarizable.) We start from the following family of induced representations:

$$\sigma_s := \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_1}\chi)}\rangle \times \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{s}\chi)}\rangle \rtimes \sigma' \simeq$$
$$\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha_u}\chi)}\rangle \times \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-s}\chi)}\rangle \rtimes \sigma',$$

where $s \in [\alpha_u, 1-\alpha_u[$. Lemma 8-1 implies that every representation σ_s ($s \in [\alpha_u, 1-\alpha_u[$) is irreducible. Since it is unitarizable for $s = \alpha_u$ by the above isomorphism and Theorem 4-1, (D) implies the unitarizability of σ_s for every $s \in [\alpha_u, 1-\alpha_u[$. Since $\beta_1 \in [\alpha_u, 1-\alpha_u[$, we see that $\sigma = \sigma_{\beta_1}$ is unitarizable. This completes the proof of the lemma. □

9. The Surjectivity of $\mathcal{M}^{u,unr}(S_n) \to \mathrm{Irr}^{u,unr}(S_n)$

In this section we prove the surjectivity of the map $\mathcal{M}^{u,unr}(S_n) \to \mathrm{Irr}^{u,unr}(S_n)$ given by $(\mathbf{e}, \sigma_{neg}) \longmapsto \times_{(l,\chi,\alpha)\in\mathbf{e}}\ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle \rtimes \sigma_{neg}$. This completes the proof of Theorem 5-14.

The proof will be done by induction on n. If $n = 0$ then $\mathcal{M}^{u,unr}(S_n) = \{(\emptyset, \mathbf{1})\}$, $\mathrm{Irr}^{u,unr}(S_n) = \{\mathbf{1}\}$, and above map is just $(\emptyset, \mathbf{1}) \longmapsto \mathbf{1}$. Therefore, the theorem is obvious in this case. Assume the surjectivity of the maps for all non–negative integers $< n$. Then we prove the surjectivity of the map for n. More precisely, for

$$\sigma \in \mathrm{Irr}^{u,unr}(S_n) \tag{9-1}$$

we need to produce the datum $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$ such that

$$\sigma \simeq \times_{(l,\chi,\alpha)\in\mathbf{e}}\ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle \rtimes \sigma_{neg}. \tag{9-2}$$

First, by Lemma 6-2, there is a unique

$$(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{unr}(S_n)$$

such that (9-2) holds. Therefore it remains to prove the following theorem:

Theorem 9-3. *$(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$ (that is, $(\mathbf{e}, \sigma_{neg})$ satisfies Definition 5-13).*

The proof of this result (that is, the proof of the inductive step) will occupy the remainder of this section. It is done by (another) induction on $m = \mathrm{card}\ \mathbf{e}$, $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$. If $m = 0$, then the representation is $\sigma \simeq \sigma_{neg}$, and, clearly, $(\emptyset, \sigma_{neg})$ satisfies Definition 5-13. Next, we state the following useful observation that will be used several times in the proof below:

Remark 9-4. *Lemma 6-1 and (D) imply that "being in complementary series" is an "open condition". This means, for every $(l, \chi, \alpha) \in \mathbf{e}$ we may choose ϵ having small absolute value such that $\times_{(l,\chi,\alpha)\in\mathbf{e}}\ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha+\epsilon}\chi)}\rangle \rtimes \sigma_{neg}$ is irreducible and unitarizable, and $\alpha + \epsilon \notin (1/2)\mathbb{Z}$, $(\alpha+\epsilon) \pm (\alpha' + \epsilon') \notin (1/2)\mathbb{Z}$ for all $(l, \chi, \alpha) \neq (l', \chi', \alpha') \in \mathbf{e}$. We refer to this perturbation of exponents as bringing σ into a general position.*

The appropriate definition is the following:

DEFINITION 9-5. *We say that σ is in general position if $\alpha \notin (1/2)\mathbb{Z}$, $\alpha \pm \alpha' \notin (1/2)\mathbb{Z}$ for all $(l, \chi, \alpha) \neq (l', \chi', \alpha') \in \mathbf{e}$.*

The first step in the proof is easy:

LEMMA 9-6. *If there exist $\chi \notin \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l \in \mathbb{Z}_{\geq 1}$ such that $\mathbf{e}(l, \chi) \neq \emptyset$, then $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$.*

PROOF. By our assumption, σ is unitarizable. Therefore, σ is Hermitian. Now, Lemma 6-4 implies $\mathbf{e}(l', \chi') = \mathbf{e}(l', (\chi')^{-1})$ for $\chi' \notin \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l' \in \mathbb{Z}_{\geq 1}$. If $\mathbf{e}(l, \chi) \neq \emptyset$, then let $\alpha \in \mathbf{e}(l, \chi)$. Then $\alpha \in \mathbf{e}(l, \chi^{-1})$. We let $\mathbf{e}' = \mathbf{e} - \{(l, \chi, \alpha), (l, \chi^{-1}, \alpha)\}$. Then $(\mathbf{e}', \sigma_{neg}) \in \mathcal{M}^{unr}(S_n)$. Let σ' be the irreducible unramified representation attached to $(\mathbf{e}', \sigma_{neg})$. By definition, it is an irreducible subquotient of $\times_{(l',\chi',\alpha') \in \mathbf{e}'} \ \langle [-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')} \rangle \rtimes \sigma_{neg}$. As we can permute the characters in (9-2), we may write

$$\sigma \simeq \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)} \rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi^{-1})} \rangle \\ \times_{(l',\chi',\alpha') \in \mathbf{e}'} \ \langle [-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')} \rangle \rtimes \sigma_{neg}.$$

This shows that

$$\sigma' \simeq \times_{(l',\chi',\alpha') \in \mathbf{e}'} \ \langle [-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')} \rangle \rtimes \sigma_{neg} \tag{9-7}$$

and

$$\sigma \simeq \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)} \rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi^{-1})} \rangle \rtimes \sigma'. \tag{9-8}$$

Since σ is Hermitian, Lemma 6-4, the definition of $\mathbf{e}'$ and (9-7) imply that σ' is also Hermitian. Next, the isomorphism (9-8) implies

$$\sigma \simeq \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)} \rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\alpha}\chi)} \rangle \rtimes \sigma'.$$

Therefore, σ is fully–induced from the tensor product of two irreducible Hermitian representations: $\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)} \rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\alpha}\chi)} \rangle$ and σ'. Since σ is unitarizable, (UR) implies that $\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)} \rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\alpha}\chi)} \rangle$ and σ' are unitarizable. By induction, this means that $(\mathbf{e}', \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_{n'})$, where $n' < n$ is defined by $\sigma' \in \mathrm{Irr}^{unr}(S_{n'})$. Now, by induction, we have the following:

$$(\mathbf{e}', \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_{n'}). \tag{9-9}$$

Also, since $\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)} \rangle \times \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\alpha}\chi)} \rangle$ is irreducible and unitarizable, Theorem 4-1 implies that $\alpha]\frac{-1}{2}, \ \frac{1}{2}[$. Since by definition of $(\mathbf{e}, \sigma_{neg})$ we have $\alpha > 0$, we obtain $0 < \alpha < \frac{1}{2}$. Now, since $\mathbf{e}' = \mathbf{e} - \{(l, \chi, \alpha), (l, \chi^{-1}, \alpha)\}$ and (9-9) holds, it is easy to check that $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$ (see Definition 5-13). □

In the remainder of the proof of Theorem 9-3, Lemma 9-6 enables us to assume that $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ whenever $e(l, \chi) \neq \emptyset$ for some l. (See Definition 5-13 (1).) Next, we prove the following lemma:

Lemma 9-10. *Assume that there exist* $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ *and* $l \in \mathbb{Z}_{\geq 1}$ *such that* $\mathbf{e}(l,\chi)$ *contains* α *and* β, $\alpha \leq \beta$, *(if* $\alpha = \beta$, *then we assume* α *is contained with multiplicity at least two) such that the following hold:*

(1) $]\alpha,\ \beta[\ \cap (\frac{1}{2})\mathbb{Z} = \emptyset$

(2) *there is no* $\gamma \in]\alpha,\ \beta[$ *such that* $[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\pm\gamma}\chi)}$ *is linked with a segment* $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}$ *for* $(l',\chi',\alpha') \in \mathbf{e}$.

Then $\alpha, \beta \in]0,\ \frac{1}{2}[$, *and* $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$.

Proof. We consider the following family of induced representations:

$$\text{(9-11)}\quad \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\gamma}\chi)}\rangle \times \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\beta}\chi)}\rangle \times \times_{(l',\chi',\alpha')\in\mathbf{e}-\{(l,\chi,\alpha),\ (l,\chi,\beta)\}} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg},$$

where $\gamma \in [\alpha,\ \beta]$. We prove the irreducibility of the induced representation in (9-11). First, applying Lemma 6-1 (see the beginning of the proof of Lemma 6-1) we list necessary and sufficient conditions for the irreducibility:

(i) for $(l_1,\chi_1,\alpha_1), (l_2,\chi_2,\alpha_2) \in \{(l,\chi,\gamma)\} + \mathbf{e} - \{(l,\chi,\alpha)\}$, the segments $[-\frac{l_1-1}{2}, \frac{l_1-1}{2}]^{(\nu^{\pm\alpha_1}\chi_1)}$ and $[-\frac{l_2-1}{2}, \frac{l_2-1}{2}]^{(\nu^{\alpha_2}\chi_2)}$ are not linked

(ii) for $(l',\chi',\alpha') \in \{(l,\chi,\gamma)\} + \mathbf{e} - \{(l,\chi,\alpha)\}$, the induced representation $\langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg}$ is irreducible.

Now, since for $\gamma = \alpha$ the induced representation is isomorphic to σ, it is irreducible. Thus, (i) and (ii) hold for $\gamma = \alpha$. Combining this with (2) shows (i) holds for any $\gamma \in [\alpha,\ \beta]$. If $\gamma = \beta$, (ii) is obviously satisfied, proving the irreducibility for $\gamma = \beta$. Let $\gamma \in]\alpha,\ \beta[$. Then (1) implies $\gamma \notin (\frac{1}{2})\mathbb{Z}$. Hence, Lemma 6-1 implies that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\gamma}\chi)}\rangle \rtimes \sigma_{neg}$ is irreducible. Thus, (ii) always holds. Thus, the induced representation (9-11) is irreducible for all $\gamma \in [\alpha,\ \beta]$.

Since we assume $\chi' \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ when $e(l',\chi') \neq \emptyset$ for some l', the family of representations (9-11) is Hermitian (see Lemma 6-4). Finally, it is unitarizable for $\gamma = \alpha$ (since it is isomorphic to σ), and therefore for all $\gamma \in [\alpha,\ \beta]$ (see (D)). In particular, it is irreducible and unitarizable for $\gamma = \beta$. Since in that case we can write (9-11) as follows:

$$\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\beta}\chi)}\rangle \times \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\beta}\chi)}\rangle \times_{(l',\chi',\alpha')\in\mathbf{e}-\{(l,\chi,\alpha),\ (l,\chi,\beta)\}} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg},$$

we conclude that the following two induced representations are irreducible and Hermitian:

$$\text{(9-12)}\qquad \begin{cases} \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\beta}\chi)}\rangle \times \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{-\beta}\chi)}\rangle \\ \times_{(l',\chi',\alpha')\in\mathbf{e}-\{(l,\chi,\alpha),\ (l,\chi,\beta)\}} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg}. \end{cases}$$

Therefore they are unitarizable by (UR). Now, Theorem 4-1 implies $\beta \in]0,\ \frac{1}{2}[$. Since $0 < \alpha \leq \beta$. We conclude $\alpha \in]0,\ \frac{1}{2}[$. Now, since $\alpha, \beta \in]0,\ \frac{1}{2}[$ and the other representation in 9-12 is unitarizable, we conclude by induction that $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$. (Lemma 6-1 needs to be applied for the irreducibility conditions.) □

LEMMA 9-13. *Assume that there exist $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l \in \mathbb{Z}_{\geq 1}$ such that $\mathbf{e}(l, \chi)$ contains $\alpha \in]0, \ \frac{1}{2}[$ satisfying the following:*

> *there is no $\beta \in]0, \ \alpha[$ such that $[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\pm\beta}\chi)}$ is linked with a segment $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}$ for $(l', \chi', \alpha') \in \mathbf{e}$.*

If $\chi' \neq \chi$ or $l' \neq l$, then $\mathbf{e}(l', \chi')$ satisfies Definition 5-13 (2) and (3).

PROOF. We may assume that σ is in a general position. We consider the following family of induced representations:

$$\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\beta \chi)} \rangle \times_{(l', \chi', \alpha') \in \mathbf{e} - \{(l, \chi, \alpha)\}} \langle [-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'} \chi')} \rangle \rtimes \sigma_{neg}, \tag{9-14}$$

where $\beta \in]0, \ \alpha]$. As in the proof of Lemma 9-10, we conclude the irreducibility of the induced representation given by (9-14).

Next, since we assume $\chi' \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ when $e(l', \chi') \neq \emptyset$ for some l', the family of representation (9-14) is Hermitian (see Lemma 6-4). Finally, it is unitarizable for $\beta = \alpha$ (since it is isomorphic to σ), and therefore for all $\beta \in]0, \ \alpha]$ (see (D)). Applying (ED), we conclude that all irreducible subquotients of

$$\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)} \rangle \times_{(l', \chi', \alpha') \in \mathbf{e} - \{(l, \chi, \alpha)\}} \langle [-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'} \chi')} \rangle \rtimes \sigma_{neg} \tag{9-15}$$

are unitarizable. In particular, its unique irreducible unramified subquotient is unitarizable. We determine this subquotient. First, let σ'_{neg} be the unique irreducible unramified subquotient of $\langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)} \rtimes \sigma_{neg}$. Since σ_{neg} is unitarizable (see Theorem 5-11), we see that

$$\sigma'_{neg} \hookrightarrow \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)} \rangle \rtimes \sigma_{neg}. \tag{9-16}$$

Now, the classification of negative representations (see Theorem 5-10) implies that σ'_{neg} is negative. and

$$\mathrm{Jord}(\sigma'_{neg}) = \mathrm{Jord}(\sigma_{neg}) + \{2 \cdot (l, \chi)\}. \tag{9-17}$$

Next, since σ is in a general position, we easily see that the induced representation

$$\times_{(l', \chi', \alpha') \in \mathbf{e} - \{(l, \chi, \alpha)\}} \langle [-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'} \chi')} \rangle \rtimes \sigma'_{neg}$$

is irreducible; it is the unique irreducible unramified subquotient of (9-15). Since it is unitarizable and $\mathrm{card}(\mathbf{e} - \{(l, \chi, \alpha)\}) < \mathrm{card}\ \mathbf{e}$, by induction we conclude that

$$(\mathbf{e} - \{(l, \chi, \alpha)\}, \sigma'_{neg}) \in \mathcal{M}^{u,unr}(S_n) \tag{9-18}$$

Since $\chi' \neq \chi$ or $l' \neq l$, we see $\mathbf{e}(l', \chi') \subset \mathbf{e} - \{(l, \chi, \alpha)\}$. Thus, (9-18) implies that $\mathbf{e}(l', \chi')$ satisfies Definition 5-13 (2) and (3). □

We record the following corollary to the proof of Lemma 9-13:

COROLLARY 9-19. *Assume that there exist $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l \in \mathbb{Z}_{\geq 1}$ such that $\mathbf{e}(l, \chi)$ contains $\alpha \in]0, \ \frac{1}{2}[$ satisfying the following:*

> *there is no $\beta \in]0, \ \alpha[$ such that $[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\pm\beta}\chi)}$ is linked with a segment $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}$ for $(l', \chi', \alpha') \in \mathbf{e}$.*

We define σ'_{neg} using (9-16) (or, equivalently, using (9-17)). Then

$$(\mathbf{e} - \{(l, \chi, \alpha)\}, \sigma'_{neg}) \in \mathcal{M}^{u,unr}(S_n).$$

Proof. The claim follows from (9-18). □

Similarly we prove the following result:

Lemma 9-20. *Assume that there exist $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l \in \mathbb{Z}_{\geq 1}$ such that $l - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$ and $\mathbf{e}(l, \chi)$ contains $\alpha \in]0,\ \frac{1}{2}[$ satisfying the following:*

> *there is no $\beta \in]\alpha,\ \frac{1}{2}[$ such that $[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\pm\beta}\chi)}$ is linked with a segment $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}$ for $(l', \chi', \alpha') \in \mathbf{e}$.*

We define σ'_{neg} using Lemma 6-6. Then $(\mathbf{e} - \{(l, \chi, \alpha)\}, \sigma'_{neg}) \in \mathcal{M}^{u,unr}(S_n)$.

The next step is a technical result used several times in the proof below.

Lemma 9-21. *Assume that σ is in a general position. Then, for every submultiset $\mathbf{e}_0$ of $\mathbf{e}$, there exists a multiset $\mathbf{e}_1$, consisting of triples of the form (l, χ, α) ($\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, $l \in \mathbb{Z}_{\geq 1}$, $\alpha \in\ ]0,\ \frac{1}{2}[$), such that the induced representation $\times_{(l,\chi,\alpha)\in \mathbf{e}_0+\mathbf{e}_1}\ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \rtimes \sigma_{neg}$ is an irreducible unitarizable representation of S_n. Moreover, we may choose $\mathbf{e}_1$ such that $\operatorname{card}(\ \mathbf{e} - \mathbf{e}_0) \leq \operatorname{card}\ \mathbf{e}_1$, and if all $(l, \chi, \alpha) \in \mathbf{e} - \mathbf{e}_0$ have the same $\chi = \chi_0$, then all $(l, \chi, \alpha) \in \mathbf{e}_1$ satisfy $\chi = \chi_0$.*

Proof. If $\mathbf{e}_0 = \mathbf{e}$, then there is nothing to be proved; we may take $\mathbf{e}_1 = \emptyset$. Therefore, we may assume $\mathbf{e}_0 \neq \mathbf{e}$. If for all $(l, \chi, \alpha) \in \mathbf{e} - \mathbf{e}_0$ we have $\alpha \in\ ]0,\ \frac{1}{2}[$, we are done; we may take $\mathbf{e}_1 = \mathbf{e} - \mathbf{e}_0$. Therefore, let $(l, \chi, \alpha) \in \mathbf{e} - \mathbf{e}_0$ such that $\alpha \geq \frac{1}{2}$. Since σ is in a general position, we must have $\alpha \notin \frac{1}{2}\mathbb{Z}$. Then there is a unique $k \in \mathbb{Z}_{\geq 1}$ such that $\alpha \in\ ]\frac{k}{2},\ \frac{k+1}{2}[$. Then $\frac{k+1}{2} - \alpha \in\ ]0,\ \frac{1}{2}[$, and the following induced representation is in a GL–complementary series (see Theorem 4-1):
(9-22)

$$\pi = \langle[-\frac{l+k-2}{2}, \frac{l+k-2}{2}]^{(\nu^{-\alpha+\frac{k+1}{2}}\chi)}\rangle \times \langle[-\frac{l+k-2}{2}, \frac{l+k-2}{2}]^{(\nu^{\alpha-\frac{k+1}{2}}\chi)}\rangle.$$

Therefore, the representation $\pi \rtimes \sigma$ is unitarizable, but reducible. We determine its unique irreducible unramified subquotient. Since

$$\begin{aligned}\langle[-\frac{l+k-2}{2}, \frac{l+k-2}{2}]^{(\nu^{\alpha-\frac{k+1}{2}}\chi)}\rangle \times \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle \\ = \langle[\alpha - \frac{l-1}{2} - k, \alpha + \frac{l-1}{2} - 1]^{(\chi)}\rangle \times \langle[\alpha - \frac{l-1}{2}, \alpha + \frac{l-1}{2}]^{(\chi)}\rangle,\end{aligned}$$

and $\alpha \notin \frac{1}{2}\mathbb{Z}$, Zelevinsky theory implies that the unique irreducible unramified subquotient of

$$\begin{aligned}\langle[-\frac{l+k-2}{2}, \frac{l+k-2}{2}]^{(\nu^{-\alpha+\frac{k+1}{2}}\chi)}\rangle \times \langle[-\frac{l+k-2}{2}, \frac{l+k-2}{2}]^{(\nu^{\alpha-\frac{k+1}{2}}\chi)}\rangle \times \\ \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle\end{aligned}$$

is exactly

$$\langle[-\frac{l+k-2}{2},\frac{l+k-2}{2}]^{(\nu^{-\alpha+\frac{k+1}{2}}\chi)}\rangle\times\langle[\alpha-\frac{l-1}{2}-k,\alpha+\frac{l-1}{2}]^{(\chi)}\rangle\times$$
$$\langle[\alpha-\frac{l-1}{2},\alpha+\frac{l-1}{2}-1]^{(\chi)}\rangle,$$

or written differently,

$$\langle[-\frac{l+k-2}{2},\frac{l+k-2}{2}]^{(\nu^{-\alpha+\frac{k+1}{2}}\chi)}\rangle\times$$
$$\langle[-\frac{l+k-1}{2},\frac{l+k-1}{2}]^{(\nu^{\alpha-\frac{k}{2}}\chi)}\rangle\times\langle[\frac{-(l-2)}{2},\frac{(l-2)}{2}]^{(\nu^{\alpha-\frac{1}{2}}\chi)}\rangle.$$

(We remark that the segment $[\frac{-(l-2)}{2},\frac{(l-2)}{2}]^{(\nu^{\alpha-\frac{1}{2}}\chi)}$ is empty if $l=1$, and should be omitted.) Since σ is in a general position, and the segments

$$\begin{cases}[-\frac{l+k-2}{2},\frac{l+k-2}{2}]^{(\nu^{\alpha-\frac{k+1}{2}}\chi)},\ [-\frac{l+k-1}{2},\frac{l+k-1}{2}]^{(\nu^{\alpha-\frac{k}{2}}\chi)}\\ [-\frac{l+k-2}{2},\frac{l+k-2}{2}]^{(\nu^{\alpha-\frac{k+1}{2}}\chi)},\ [\frac{-(l-2)}{2},\frac{(l-2)}{2}]^{(\nu^{\alpha-\frac{1}{2}}\chi)}\end{cases}$$

are not linked, Lemma 6-1 implies that the unique irreducible unramified subquotient of $\pi\rtimes\sigma$ is

$$\langle[-\frac{l+k-2}{2},\frac{l+k-2}{2}]^{(\nu^{-\alpha+\frac{k+1}{2}}\chi)}\rangle\times\langle[-\frac{l+k-1}{2},\frac{l+k-1}{2}]^{(\nu^{\alpha-\frac{k}{2}}\chi)}\rangle\times$$
$$\langle[\frac{-(l-2)}{2},\frac{(l-2)}{2}]^{(\nu^{\alpha-\frac{1}{2}}\chi)}\rangle\times_{(l',\chi',\alpha')\in\mathbf{e}-\{(l,\chi,\alpha)\}}\ \langle[-\frac{l'-1}{2},\frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle\rtimes\sigma_{neg}. \tag{9-23}$$

Now, since $-\alpha+\frac{k+1}{2},\alpha-\frac{k}{2}\in]0,\ \frac{1}{2}[$, and $\alpha-\frac{1}{2}<\alpha$, we may iterate this procedure until we obtain what we want. □

Next, we prove the following lemma:

Lemma 9-24. *Assume that σ is in a general position. Assume that there exist $l_1,k_1\in\mathbb{Z}_{\geq1}$, such that $\mathbf{e}(l_1,\mathbf{1}_{F^\times})\neq\emptyset$ and $\mathbf{e}(k_1,\mathbf{sgn}_u)\neq\emptyset$. Then $(\mathbf{e},\sigma_{neg})\in\mathcal{M}^{u,unr}(S_n)$.*

Proof. Let $\chi_0\in\{\mathbf{1}_{F^\times},\mathbf{sgn}_u\}$ and $l_0\in\mathbb{Z}_{\geq1}$. We need to show that that the exponents from $\mathbf{e}(l_0,\chi_0)$ satisfy Definition 5-13 (2) or (3). By the assumption of the lemma, we may find $\chi'\in\{\mathbf{1}_{F^\times},\mathbf{sgn}_u\}$ and $l'\in\mathbb{Z}_{\geq1}$ such that $\chi'\neq\chi_0$ and $\mathbf{e}(l',\chi')\neq\emptyset$. Letting $\mathbf{e}_0=\sum_{l'_0}\mathbf{e}(l'_0,\chi_0)$ in Lemma 9-21, we may assume that $\alpha<1/2$ for all $(l,\chi,\alpha)\in\mathbf{e}-\mathbf{e}_0$. Now, we take some (l,χ), $\chi\neq\chi_0$, such that $\mathbf{e}(l,\chi)\neq\emptyset$. If card $\mathbf{e}(l,\chi)>1$, then we apply Lemma 9-10 to complete the proof of the lemma. Otherwise, we use Lemma 9-13. □

In the remainder of the proof we assume that there is a unique $\chi\in\{\mathbf{1}_{F^\times},\mathbf{sgn}_u\}$ such that if $\mathbf{e}(l',\chi')\neq\emptyset$, then $\chi'=\chi$. We prove the following lemma:

Lemma 9-25. *Assume that σ is in a general position. Assume that $\mathbf{e}(1,\chi)\neq\emptyset$. Then, if $\alpha>1$, for some $\alpha\in\mathbf{e}(1,\chi)$, then*

$$k-(2\alpha_\chi+1)\in2\mathbb{Z}, \tag{9-26}$$

where $k\in\mathbb{Z}_{\geq2}$ is defined by $\alpha\in]\frac{k}{2},\ \frac{k+1}{2}[$.

Proof. Applying Lemma 9-21, we may assume that every $(l', \chi, \alpha') \in \mathbf{e} - \{(1, \chi, \alpha)\}$ satisfies $\alpha' \in]0, \frac{1}{2}[$. We let $(l', \chi, \alpha') \in \mathbf{e} - \{(1, \chi, \alpha)\}$. Then the segment $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi)}$ is linked with the segment $\{\nu^\alpha\chi\}$ if and only if $\alpha = \frac{l'-1}{2} + \alpha' + 1$. Since $\alpha \in]\frac{k}{2}, \frac{k+1}{2}[$ and $\alpha' \in]0, \frac{1}{2}[$, we see that this is equivalent to $l' = k-1$ and $\alpha' = \alpha - k/2$. Similarly, $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{-\alpha'}\chi)}$ is linked with the segment $\{\nu^\alpha\chi\}$ if and only if $l' = k$ and $\alpha' = \frac{k+1}{2} - \alpha$. Therefore, since the induced representation (9-2) is irreducible, we see that for $(l', \chi, \alpha') \in \mathbf{e} - \{(1, \chi, \alpha)\}$ we must have

$$(l', \alpha') \neq (k-1, \alpha - \frac{k}{2}),\ (k, \frac{k+1}{2} - \alpha).$$

We remark that the segments $[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{x'}\chi)}$ and $[-\frac{l''-1}{2}, \frac{l''-1}{2}]^{(\nu^{\pm x''}\chi)}$, where $x', x'' \in]0, \frac{1}{2}[$, are never linked.

Those observations enable us to assume that there are no triples $(l', \chi, \alpha') \in \mathbf{e} - \{(1, \chi, \alpha)\}$ such that one of the following holds:

$$\begin{cases} l' \notin \{k-1, k\} \\ l' = k-1 \text{ and } \alpha' < \alpha - \frac{k}{2} \\ l' = k \text{ and } \alpha' < \frac{k+1}{2} - \alpha \end{cases}$$

applying Corollary 9-19 several times.

Next, applying Lemma 9-10, we may assume that $\mathbf{e} - \{(1, \chi, \alpha)\}$ contains at most two elements (each with multiplicity at most one) which are necessary of the form:

$$\begin{cases} (k-1, \chi, \beta), \text{ where } \beta \in]\alpha - \frac{k}{2}, \frac{1}{2}[; \\ (k, \chi, \gamma), \text{ where } \gamma \in]\frac{k+1}{2} - \alpha, \frac{1}{2}[. \end{cases}$$

Thus, we may assume the following:

$$\mathbf{e} = \{(1, \chi, \alpha),\ n_\beta \cdot (k-1, \chi, \beta),\ n_\gamma \cdot (k, \chi, \gamma)\}. \tag{9-27}$$

(Here $n_\beta, n_\gamma \in \{0, 1\}$ are the multiplicities.)

Now, proceed as follows: We use the complementary series ($l = 1$ in our case)

$$\pi = \langle[-\frac{l+k-3}{2}, \frac{l+k-3}{2}]^{(\nu^{-\alpha+\frac{k}{2}}\chi)}\rangle \times \langle[-\frac{l+k-3}{2}, \frac{l+k-3}{2}]^{(\nu^{\alpha-\frac{k}{2}}\chi)}\rangle, \tag{9-28}$$

and repeat the steps of the proof of Lemma 9-21 from the point (9-22) up to (9-23) where instead of (9-23) we obtain a new irreducible unitarizable unramified representation σ' which is isomorphic to ($l = 1$ in our case)

$$\langle[-\frac{l+k-3}{2}, \frac{l+k-3}{2}]^{(\nu^{\alpha-\frac{k}{2}}\chi)}\rangle \times \langle[-\frac{l+k-2}{2}, \frac{l+k-2}{2}]^{(\nu^{\alpha-\frac{k-1}{2}}\chi)}\rangle \times \tag{9-29}$$
$$\langle[\frac{-(l-2)}{2}, \frac{(l-2)}{2}]^{(\nu^{\alpha-\frac{1}{2}}\chi)}\rangle \times_{(l', \chi', \alpha') \in \mathbf{e} - \{(l, \chi, \alpha)\}} \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\nu^{\alpha'}\chi')}\rangle \rtimes \sigma_{neg}.$$

Thus, σ' is attached to $(\mathbf{e}', \sigma_{neg})$ where

$$\mathbf{e}' = \{(k-1, \chi, \alpha - \frac{k}{2}),\ (k, \chi, \alpha - \frac{k-1}{2}),\ n_\beta \cdot (k-1, \chi, \beta),\ n_\gamma \cdot (k, \chi, \gamma)\}.$$

We remark that $\alpha - \frac{k-1}{2} \in]\frac{1}{2}, 1[$.

Now, using Corollary 9-19 and Lemma 6-9 (i), (ii), we obtain a new unitarizable unramified representation σ'' attached to $(\mathbf{e}'', \sigma'_{neg})$ where

$$\mathbf{e}'' = \{(k, \chi, \alpha - \frac{k-1}{2}),\ n_\beta \cdot (k-1, \chi, \beta),\ n_\gamma \cdot (k, \chi, \gamma)\}$$

and

$$\mathrm{Jord}(\sigma'_{neg}) = \mathrm{Jord}(\sigma_{neg}) + 2 \cdot \{(k-1, \chi)\}.$$

Therefore, either by the inductive assumption (that is, in the case $n_\beta > 0$ or $n_\gamma > 0$) or by Theorem 7-1 (if $n_\beta = n_\gamma = 0$) we obtain (9-26). $\square$

Lemma 9-30. *Assume that σ is in a general position. Assume that $\mathbf{e}(1, \chi) \neq \emptyset$. Then $\alpha < 3/2$ for $\alpha \in \mathbf{e}(1, \chi)$.*

Proof. Assume to the contrary that there exists $\alpha \in \mathbf{e}(1, \chi)$ such that $\alpha \geq \frac{3}{2}$. Then since σ is in a general position, there exists $k \in \mathbb{Z}_{\geq 3}$ such that $\alpha \in]\frac{k}{2},\ \frac{k+1}{2}[$. Then Lemma 9-25 implies that

$$k - (2\alpha_\chi + 1) \in 2\mathbb{Z}. \tag{9-31}$$

Next, applying Lemma 9-21, we may assume that every $(l', \chi, \alpha') \in \mathbf{e} - \{(1, \chi, \alpha)\}$ satisfies $\alpha' \in]0,\ \frac{1}{2}[$. Equipped with this, we may assume the reduction (9-27).

We would like to "move" α from $]\frac{k}{2},\ \frac{k+1}{2}[$ into $]\frac{k-1}{2},\ \frac{k}{2}[$. We have the two cases. First, we assume that there exists $\epsilon > 0$ such that the induced representation

$$\nu^x\chi \times \langle[-\frac{k-2}{2}, \frac{k-2}{2}]^{(\nu^y\chi)}\rangle \times \langle[-\frac{k-1}{2}, \frac{k-1}{2}]^{(\nu^z\chi)}\rangle \rtimes \sigma_{neg}$$

is irreducible and unitarizable for $x \in [\frac{k}{2} - \epsilon, \alpha]$, $y \in]\alpha - \frac{k}{2},\ \frac{1}{2} + \epsilon[$, and $z \in]\frac{k+1}{2} - \alpha,\ \frac{1}{2} + \epsilon[$. We explain this assumption. The irreducibility is an easy consequence of Lemma 6-9, and the unitarity follows from (D) since at $(x, y, z) = (\alpha, \beta, \gamma)$ is unitarizable, except that reducibility might occur for $x = \frac{k}{2}$ (y, z are arbitrary). Now, the induced representation $\nu^\alpha\chi \times \langle[-\frac{k-2}{2}, \frac{k-2}{2}]^{(\nu^\beta\chi)}\rangle \times \langle[-\frac{k-1}{2}, \frac{k-1}{2}]^{(\nu^\gamma\chi)}\rangle \rtimes \sigma_{neg}$ is irreducible and unitarizable for some $\alpha \in]\frac{k-1}{2},\ \frac{k}{2}[$. Since $k - 1 \geq 2$, then Lemma 9-25 shows that $k - 1 - (2\alpha_\chi + 1) \in 2\mathbb{Z}$. This is a contradiction since we already have $k - (2\alpha_\chi + 1) \in 2\mathbb{Z}$. (See (9-31).)

If we have reducibility at $x = \frac{k}{2}$ for some y and z, then Lemma 6-1 and $k - (2\alpha_\chi + 1) \in 2\mathbb{Z}$ would imply that $(k-1, \chi) \in \mathrm{Jord}(\sigma_{neg})$. Then, applying Lemma 6-5, since $k - 1 - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$, $(k-1, \chi)$ appears at least twice in $\mathrm{Jord}(\sigma_{neg})$, and there exists a negative representation σ'_{neg} such that

$$\sigma_{neg} \simeq \langle[-\frac{k-2}{2}, \frac{k-2}{2}]^{(\chi)}\rangle \rtimes \sigma'_{neg}.$$

Then σ is of the form

$$\begin{aligned} \nu^\alpha\chi \times \langle[-\frac{k-2}{2}, \frac{k-2}{2}]^{(\nu^\beta\chi)}\rangle \times \langle[-\frac{k-1}{2}, \frac{k-1}{2}]^{(\nu^\gamma\chi)}\rangle \rtimes \sigma_{neg} \simeq \\ \simeq \langle[-\frac{k-2}{2}, \frac{k-2}{2}]^{(\chi)}\rangle \times \\ \left(\nu^\alpha\chi \times \langle[-\frac{k-2}{2}, \frac{k-2}{2}]^{(\nu^\beta\chi)}\rangle \times \langle[-\frac{k-1}{2}, \frac{k-1}{2}]^{(\nu^\gamma\chi)}\rangle \rtimes \sigma'_{neg}\right) \end{aligned}$$

Now, applying (UR) we obtain the unitarity of

$$\nu^{\alpha}\chi \times \langle[-\frac{k-2}{2}, \frac{k-2}{2}]^{(\nu^{\beta}\chi)}\rangle \times \langle[-\frac{k-1}{2}, \frac{k-1}{2}]^{(\nu^{\gamma}\chi)}\rangle \rtimes \sigma'_{neg}$$

and this contradicts the general inductive assumption. □

LEMMA 9-32. *Let $l \in \mathbb{Z}_{\geq 1}$ such that $\mathbf{e}(l,\chi) \neq \emptyset$. Then, if $l-(2\alpha_{\chi}+1) \notin 2\mathbb{Z}$ (resp., $l-(2\alpha_{\chi}+1) \in 2\mathbb{Z}$), then $\alpha < 1/2$ (resp., $\alpha < 1$) for $\alpha \in \mathbf{e}(l,\chi)$.*

PROOF. We may assume that σ is in a general position. (See Remark 9-4.) Assume that the claim is not true for $\alpha \in \mathbf{e}(l,\chi)$. Applying Lemma 9-21, we may assume that every $(l',\chi,\alpha') \in \mathbf{e} - \{(l,\chi,\alpha)\}$ satisfies $\alpha' \in]0,\ \frac{1}{2}[$.

Now, we proceed as in the proof of Lemma 9-21 (that is, we imitate that proof "multiplying" σ by π given by (9-22) and repeating the steps done there to obtain (9-23)), keep replacing (l,χ,α) by $(l-1,\chi,\alpha-1/2)$ while $l \geq 2$. (This keeps all other exponents in $]0,\ \frac{1}{2}[$.) As a result, we may assume that one of the following holds:

(a) $l-(2\alpha_{\chi}+1) \notin 2\mathbb{Z}$ and $\alpha \in]\frac{1}{2},\ 1[$
(b) $l-(2\alpha_{\chi}+1) \notin 2\mathbb{Z}$ and $\alpha \in]1,\ \frac{3}{2}[$
(c) $l=1$ and there exists $k \in \mathbb{Z}_{\geq 2}$ such that $\alpha \in]\frac{k}{2},\ \frac{k+1}{2}[$.

Next, Lemma 9-30 implies that $k=2$ in (c). Now, we reduce that case to the previous two. Since $k-(2\alpha_{\chi}+1) \in 2\mathbb{Z}$ for $k=2$ (see (9-26)), we see that $l-(2\alpha_{\chi}+1) \notin 2\mathbb{Z}$ for $l=1$. Hence this is just the case (b). It remains to consider the cases (a) and (b). The case (a) is easy. We apply Corollary 9-19 several times in order to reduce to the case $\mathbf{e} = \{(l,\chi,\alpha)\}$. Then we apply Theorem 7-1 to obtain a contradiction. We consider the case (b). Arguing as in the proof of Lemma 9-25, we may assume the following:

$$\mathbf{e} = \{(l,\chi,\alpha),\ n_{\beta}\cdot(l-1,\chi,\beta),\ n_{\gamma}\cdot(l+1,\chi,\gamma)\}.$$

where $\beta \in]\alpha-1,\ \frac{1}{2}[$ and $\gamma \in]\frac{3}{2}-\alpha,\ \frac{1}{2}[$. (Here $n_{\beta}, n_{\gamma} \in \{0,1\}$ are the multiplicities.)

Now, we "move" α into $]\frac{1}{2},\ 1[$ arguing as in the last part of the proof of Lemma 9-30 reducing (b) to (a). In more detail, there exists $\epsilon > 0$ such that the induced representation

$$\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{x}\chi)}\rangle \times \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\nu^{y}\chi)}\rangle \times \langle[-\frac{l}{2}, \frac{l}{2}]^{(\nu^{z}\chi)}\rangle \rtimes \sigma_{neg}$$

is irreducible and unitarizable for $x \in [1-\epsilon, \alpha]$, $y \in]\alpha-1,\ \frac{1}{2}+\epsilon[$, and $z \in]\frac{3}{2}-\alpha,\ \frac{1}{2}+\epsilon[$. Hence $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\alpha}\chi)}\rangle \times \langle[-\frac{l-2}{2}, \frac{l-2}{2}]^{(\nu^{\beta}\chi)}\rangle \times \langle[-\frac{l}{2}, \frac{l}{2}]^{(\nu^{\gamma}\chi)}\rangle \rtimes \sigma_{neg}$ is unitarizable for some (new) $\alpha \in]\frac{1}{2},\ 1[$ where β and γ are less than but close to $\frac{1}{2}$. We are now in case (a). □

Let us summarize what we have done so far. We have reduced the proof that $(\mathbf{e},\sigma_{neg}) \in \mathcal{M}^{unr}(S_n)$ attached to $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ (see (9-1)) satisfies Definition 5-13 to the following. We may assume that there is a unique $\chi \in \{\mathbf{1}_{F^{\times}}, \mathbf{sgn}_u\}$ such that if $\mathbf{e}(l',\chi') \neq \emptyset$, then $\chi' = \chi$. In the glance of Definition 5-13 and since Lemma 9-32 holds, we may apply Lemma 6-9 and (several times) Corollary 9-19 to assume that there is also the unique l such that if $\mathbf{e}(l',\chi) \neq \emptyset$, then $l = l'$. Now, if $l-(2\alpha_{\chi}+1) \notin 2\mathbb{Z}$, we see that $(\mathbf{e},\sigma_{neg}) \in \mathcal{M}^{unr}(S_n)$ satisfies Definition 5-13 (applying Lemma 9-32), and Theorem 9-3 is proved. Thus, it remains to consider the case $l-(2\alpha_\chi+1)\in 2\mathbb{Z}$. We need to check Definition 5-13 (3). First, Lemma 9-32 implies $0<\alpha<1$ for all $\alpha\in \mathbf{e}(l,\chi)$. Then we write this multiset as in Definition 5-13 (3). Hence (9-2) can be written as follows:

$$\sigma \simeq \times_{i=1}^{u}\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\alpha_i}\chi)} \times_{j=1}^{v}\langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{\beta_j}\chi)}\rangle \rtimes \sigma_{neg}. \tag{9-33}$$

Now, we check (a)–(f) in Definition 5-13 (3). First, since the induced representation (9-33) is irreducible, (b) and (d) hold. (See Lemma 6-9.) Next, Lemma 9-10 implies (c). It remains to prove (a), (e), and (f).

Now, if there exist indices $i_1\neq i_2$ such that $\alpha_{i_1},\alpha_{i_1}\in]1-\beta_1,\frac{1}{2}]$ or $\alpha_{i_1},\alpha_{i_1}\in]1-\beta_{j+1},1-\beta_j[$ for some j, then we apply Lemma 9-10 and we are done. Otherwise, we may assume that the number of indices i such that $\alpha_i\in]1-\beta_1,\frac{1}{2}]$ (resp., $\alpha_i\in]1-\beta_{j+1},1-\beta_j[$ for some fixed j) is 0 or 1. We must show that this number is one $]1-\beta_{j+1},1-\beta_j[$ for every j, and zero for $]1-\beta_1,\frac{1}{2}]$.

If we can find j such that the number is zero for $]1-\beta_{j+1},1-\beta_j[$, then we apply Lemma 9-10 to deform β_j into β_{j+1}, and obtain $\beta_j,\beta_{j+1}<\frac{1}{2}$ which is a contradiction. If on the other hand $]1-\beta_1,\frac{1}{2}]$ contains the unique α_i, then we may deform it to β_1 and Lemma 9-10 would imply $\beta_1<\frac{1}{2}$ which is a contradiction. This proves (e) and (f).

In the same reduction (that is, no α_i's in $]1-\beta_{j+1},1-\beta_j[$ for every j, and there is the unique α_i in $]1-\beta_1,\frac{1}{2}]$) we prove (a).

If $v>0$, we may also assume that the number of indices i such that $\alpha_i\in]0,\ 1-\beta_v[$ is either 0 or 1. We must show if $(l,\chi)\notin \mathrm{Jord}(\sigma_{neg})$, then $u+v$ is even. We accomplish this as follows.

First, if $v=0$ (that is, no β_i's), then then there is no i such that $\alpha_i\in]0,\ 1-\beta_v[$ and the claim follows from from Theorem 7-1 ($u=v=0$ here). Next, we assume $v\geq 1$.

We reduce this case to the case $v=0$ as follows. We apply the complementary series (9-22) with $\alpha=\beta_1$ and $k=1$. We obtain a new unitary representation σ_1 attached to $(\mathbf{e}_1,\sigma_{neg})$, where

$$\mathbf{e}_1=\mathbf{e}-\{(l,\chi,\beta_1)\}+\{(l,\chi,1-\beta_1),(l+1,\chi,\beta_1-\frac{1}{2}),(l-1,\chi,\beta_1-\frac{1}{2})\}.$$

We apply Lemma 6-9 and Corollary 9-19 to obtain a new unitary representation σ' attached to $(\mathbf{e}',\sigma'_{neg})$, where

$$\begin{cases}\mathbf{e}'=\mathbf{e}_1-\{(l+1,\chi,\beta_1-\frac{1}{2}),(l-1,\chi,\beta_1-\frac{1}{2})\}\\ \mathrm{Jord}(\sigma'_{neg})=\mathrm{Jord}(\sigma_{neg})+\{(l-1,\chi),(l+1,\chi\}.\end{cases}$$

Clearly, $(l,\chi)\notin \mathrm{Jord}(\sigma'_{neg})$. Then by induction, we have $u'+v'$ is even. Since $v'=v-1$ and $u'=u-1$, we obtain the claim. This proves that all conditions (a)–(f) hold. This completes the proof of Theorem 5-14, and therefore the proof of the surjectivity of the map in Theorem 5-14.

10. Functoriality, Satake Parameters and an Algorithm for Testing Unitarity

In this section we present an algorithm that describes an effective and easy way of testing unitarity of an unramified representation given by its Satake parameter (see Theorem 1-2). We introduce the Langlands dual groups as follows:

$$\begin{aligned} &G = S_n = \mathrm{SO}(2n+1, F) && \hat{G}(\mathbb{C}) = \mathrm{Sp}(2n, \mathbb{C}) \subset GL(N, \mathbb{C}); N = 2n \\ &G = S_n = \mathrm{O}(2n, F) && \hat{G}(\mathbb{C}) = \mathrm{O}(2n, \mathbb{C}) \subset GL(N, \mathbb{C}); N = 2n \\ &G = S_n = \mathrm{Sp}(2n, F) && \hat{G}(\mathbb{C}) = \mathrm{SO}(2n+1, \mathbb{C}) \subset GL(N, \mathbb{C}); N = 2n+1. \end{aligned}$$

Assume that $(\chi_1, \dots, \chi_n)$ is a sequence of unramified characters of $F^\times$. Then the induced representation

$$\chi_1 \times \cdots \times \chi_n \rtimes \mathbf{1} = \mathrm{Ind}_{P_{\min}}^G (\chi_1 \otimes \cdots \otimes \chi_n)$$

contains the unique unramified irreducible subquotient,

$$\sigma^G := \sigma^G(\chi_1, \dots, \chi_n).$$

(See Theorem 1-2.) Its Langlands lift to $GL(N, F)$ is an unramified representation given by

$$\sigma^{GL(N,F)} := \sigma^{GL(N,F)}(\chi_1, \dots, \chi_n, \chi_1^{-1}, \dots, \chi_n^{-1}, \star), \tag{10-1}$$

where

$$\star = \begin{cases} \text{should be omitted if } G = \mathrm{O}(2n, F),\ \mathrm{SO}(2n+1, F) \\ \mathbf{1}_{F^\times} \text{ if } G = \mathrm{Sp}(2n, F). \end{cases}$$

In particular, the lift is an irreducible subquotient of

$$\chi_1 \times \cdots \times \chi_n \times \chi_1^{-1} \times \cdots \times \chi_n^{-1} \times \star. \tag{10-2}$$

Obviously, we have the following:

$$\sigma^{GL(N,F)} \text{ is self–dual: } \widetilde{\sigma}^{GL(N,F)} \simeq \sigma^{GL(N,F)}, \tag{10-3}$$

and

$$\sigma^{GL(N,F)} \text{ has a trivial central character.} \tag{10-4}$$

Since $\sigma^{GL(N,F)}$ is an irreducible subquotient of (10-2), its description in the Zelevinsky classification can be obtain by the well–known process of "linking" (see [**Ze**]):

$$\sigma^{GL(N,F)} \simeq \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle, \tag{10-5}$$

where $\Delta_1, \dots, \Delta_k$ is up to a permutation, the unique sequence of segments of unramified characters characterized by the following two conditions:

- There is an equality of the multisets:
$$\Delta_1 + \cdots + \Delta_k = \{\chi_1, \dots, \chi_n, \chi_1^{-1}, \dots, \chi_n^{-1}, \star\}.$$
- There are no indices i, j, such that the segments Δ_i and Δ_j are linked.

The expression (10-5) is easy to find for an unitary representation σ^G, and this is the basis for our algorithm. Assume that σ^G is unitarizable. Then we apply Theorem 5-14 to find $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S)$ such that

$$\sigma^G \simeq \left(\times_{(l,\chi,\alpha)\in \mathbf{e}} \ \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)} \rangle \right) \rtimes \sigma_{neg}. \tag{10-6}$$

We have the following lemma:

LEMMA 10-7. *Assume that σ^G is unitarizable and given by (10-6). Then the representation $\sigma^{GL(N,F)}$ is isomorphic to the following induced representation:*

$$\times_{(l,\chi,\alpha)\in\mathbf{e};\ \alpha\neq\frac{1}{2}} \quad \langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle \times \ \langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^{-\alpha}\chi^{-1})}\rangle$$
$$\times_{(l,\chi,\alpha)\in\mathbf{e};\ \alpha=\frac{1}{2}} \langle[-\frac{l-2}{2},\frac{l-2}{2}]^{(\chi)}\rangle \quad \textit{(we omit the segment if } l=1\textit{)}$$
$$\times_{(l,\chi,\alpha)\in\mathbf{e};\ \alpha=\frac{1}{2}} \langle[-\frac{l}{2},\frac{l}{2}]^{(\chi)}\rangle$$
$$\times_{(l,\chi)\in\mathrm{Jord}(\sigma_{neg})} \langle[-\frac{l-1}{2},\frac{l-1}{2}]^{(\chi)}\rangle$$

PROOF. Since $(\mathbf{e},\sigma_{neg})$ satisfies Definition 5-13, the claim easily follows from Lemma 6-1. (To compute the lift of σ_{neg} one applies Theorems 5-8 and 5-10; if $\sigma_{neg}=\mathbf{1}\in\mathrm{Irr}\ S_0$ we apply the definition (5-9).) We leave the simple verification to the reader. The reader should realize that the lemma does not hold if σ^G is not unitarizable. □

Now, we present the following:

Algorithm for testing the unitarity of $\sigma^G(\chi_1,\ldots,\chi_n)$.

It has the following steps:

(1) Introduce the multiset $\{\chi_1,\ldots,\chi_n,\chi_1^{-1},\ldots,\chi_n^{-1},\star\}$.
(2) Among the characters in (1), perform the (maximal) linking, to get the multisegment $\{\Delta_1,\ldots,\Delta_k\}$ which satisfies:
   – There is an equality of the multisets:
   $$\Delta_1+\cdots+\Delta_k=\{\chi_1,\ldots,\chi_n,\chi_1^{-1},\ldots,\chi_n^{-1},\star\}.$$
   – There are no indices i,j, such that the segments Δ_i and Δ_j are linked. [2]

We begin our second stage of the algorithm (where we apply Lemma 10-7). We recursively construct the multisets Jord and $\mathbf{e}$ that must be $\mathrm{Jord}(\sigma_{neg})$ and $\mathbf{e}$ for $\sigma^G(\chi_1,\ldots,\chi_n)$ if this representation is unitarizable. We start with Jord $=\emptyset$, $\mathbf{e}=\emptyset$ and the multiset $\eta=\{\Delta_1,\ldots,\Delta_k\}$, and modify them recursively. We execute the algorithm until $\eta=\emptyset$.

It is easy to show that $\tilde{\eta}=\eta$.

(3) Denote by $\eta_{nsd,unit}$ the multiset of all $\Delta\in\eta$; $\Delta=[-\frac{l-1}{2},\frac{l-1}{2}]^{(\nu^\alpha\chi)}$ (where as usual $l\in\mathbb{Z}_{\geq 1}$, χ is an unitary unramified character of $F^\times$, and $\alpha\in\mathbb{R}$) such that $\chi\notin\{\mathbf{1}_{F^\times},\mathbf{sgn}_u\}$ and $\alpha=0$. Add to Jord the multiset of all (l,χ) when Δ runs over $\eta_{nsd,unit}$, replace η by $\eta-\eta_{nsd,unit}$, and keep $\mathbf{e}$ unchanged. It is easy to see that for Δ from $\eta_{nsd,unit}$, the segments Δ and $\widetilde{\Delta}$ shows up in $\eta_{nsd,unit}$ (and η)) with the same multiplicity.

[2]The multisegment $\{\Delta_1,\ldots,\Delta_k\}$ is the result of the following simple algorithm: Let $\Delta_1=\{\chi_1\},\ldots,\Delta_n=\{\chi_n\},\Delta_{n+1}=\{\chi_1^{-1}\},\ldots,\Delta_{2n}=\{\chi_n^{-1}\},\Delta_N=\{\star\}$ and $k=N$ be the starting sequence $\Delta_1,\ldots,\Delta_k$ of the segments. Repeat the following recursive step until it is not possible: find two indices $i<j$ such that Δ_i and Δ_j are linked and replace

$$\begin{cases}\Delta_i\leftrightarrow\Delta_i\cup\Delta_j\\ \Delta_j\leftrightarrow\Delta_i\cap\Delta_j \text{ (omit this segment if the intersection is empty)}\end{cases}.$$

(4) Denote by $\eta_{nsd,+}$ the multiset of all $\Delta \in \eta$; $\Delta = [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}$ such that $\chi \notin \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $\alpha > 0$. **If some** $\alpha \geq 1/2$, then the **algorithm stops** and the representation is not unitarizable. Further, **if** $\eta_{nsd,+} \neq \bar{\eta}_{nsd,+}$ (this is a Hermitian condition, which is equivalent to $\mathbf{e}(l, \chi) \neq \mathbf{e}(l, \chi^{-1})$ for some χ and l as above), then the **algorithm stops** and the representation is not unitarizable. If neither of these happen, then add to $\mathbf{e}$ the multiset of all (l, χ, α) when Δ runs over $\eta_{nsd,+}$, replace η by $\eta - \eta_{nsd,+} - \tilde{\eta}_{nsd,+}$, and keep Jord unchanged.

(5) Denote by $\eta_{sd,\{\frac{1}{2}\},+}$ the multiset of all $\Delta \in \eta$; $\Delta = [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}$ such that $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, $l-(2\alpha_\chi+1) \notin 2\mathbb{Z}$ and $\alpha > 0$. **If some** $\alpha \geq 1/2$ (in which case $\alpha > 1/2$), then **the algorithm stops** and the representation is not unitarizable. If not, then add to $\mathbf{e}$ the multiset of all (l, χ, α) when Δ runs over $\eta_{sd,\{\frac{1}{2}\},+}$, replace η by $\eta - \eta_{sd,\{\frac{1}{2}\},+} - \tilde{\eta}_{sd,\{\frac{1}{2}\},+}$, and keep Jord unchanged.

(6) Denote by $\eta_{sd,\{0,1\},+}$ the multiset of all $\Delta \in \eta$; $\Delta = [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}$ such that $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, $l - (2\alpha_\chi + 1) \in 2\mathbb{Z}$ and $\alpha > 0$. Observe that we cannot have $\alpha = 1/2$ (in η we do not have linked segments). **If some** $\alpha \geq 1$, then **the algorithm stops** and the representation is not unitarizable. If not (i.e., if all $\alpha < 1$), then for all (l, χ) coming from Δ's in $\eta_{sd,\{0,1\},+}$, check if the multiset $\mathbf{e}(l, \chi)$ of all α such that $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}\rangle$ is in $\eta_{sd,\{0,1\},+}$ satisfies condition (c) of (3) in Definition 5-13 (observe that condition (d) of (3) in Definition 5-13 is satisfied, since $\tilde{\eta} = \eta$ and in η we do not have linked segments). **If all these conditions are not satisfied**, then **the algorithm stops** and the representation is not unitarizable. If not, then add to $\mathbf{e}$ the multiset of all (l, χ, α) when Δ runs over $\eta_{sd,\{0,1\},+}$, replace η by $\eta - \eta_{sd,\{0,1\},+} - \tilde{\eta}_{sd,\{0,1\},+}$, and keep Jord unchanged.

(7) Denote by $\eta_{sd,unit,red}$ the multiset of all $\Delta \in \eta$; $\Delta = [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}$ such that $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$, $\alpha = 0$ and $l - (2\alpha_\chi + 1) \in 2\mathbb{Z}$. [3] Add to Jord the multiset of all (l, χ) when Δ runs over $\eta_{sd,unit,red}$, replace η by $\eta - \eta_{sd,unit,red}$, and keep $\mathbf{e}$ unchanged.

(8) Take $\Delta \in \eta$; $\Delta = [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha\chi)}$ with <u>the largest</u> possible l (as usual, $l \in \mathbb{Z}_{\geq 1}$, χ is an unitary unramified character of $F^\times$, and $\alpha \in \mathbb{R}$). Then $\alpha = 0$, $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $l - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$. Form the multiset η_Δ consisting of all $\Psi \in \eta$ such that $\Psi = \Delta$.
   (i) If card η_Δ is even, say $2m$, we replace Jord by Jord $+ 2m\{(l, \chi)\}$, remove η_Δ from η and keep $\mathbf{e}$ unchanged.
   (ii) If card η_Δ is odd, say $2m + 1$, then we perform the following steps (see the second line in the displayed formula in Lemma (10-7)):
      (a) **If** $l = 1$, then **the algorithm stops** and the representation $\sigma^G(\chi_1, \ldots, \chi_n)$ is not unitarizable.
      (b) If $l = 2$, then we replace $\mathbf{e}$ by $\mathbf{e} + \{(1, \chi, \frac{1}{2})\}$, Jord by Jord $+ 2m\{(l, \chi)\}$ and η by $\eta - \eta_\Delta$.
      (c) If $l \geq 3$, then we let $\eta_{[-\frac{l-3}{2}, \frac{l-3}{2}]^{(\chi)}}$ to be the sub–multiset of η corresponding to $\Psi = [-\frac{l-3}{2}, \frac{l-3}{2}]^{(\chi)}$. **If** card $\eta_{[-\frac{l-3}{2}, \frac{l-3}{2}]^{(\chi)}}$ **is even**, then the **algorithm stops** and the representation

[3] Fix $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. One sees directly using (10-4) that the sum of multiplicities of all $\Delta = [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}$ in $\eta_{sd,unit,red}$ is even.

$\sigma^G(\chi_1, \ldots, \chi_n)$ is not unitary. If it is odd, say $2m' + 1$, we replace $\mathbf{e}$ by $\mathbf{e} + \{(l-1, \chi, \frac{1}{2})\}$, Jord by Jord $+ 2m\{(l, \chi)\} + 2m'\{(l-2)\}$ and η by $\eta - \eta_\Delta - \eta_\Psi$.

(9) If $\eta = \emptyset$, we go to the following step. Otherwise, we go back to the Step 8. The above procedure provides that (b) in (3) of Definition 5-13 is satisfied.

(10) One easily sees that exists σ_{neg} such that $\mathrm{Jord}(\sigma_{neg}) = \mathrm{Jord}$ (we can construct the representation σ_{neg} attached to Jord following the steps described in Section 5 (see Theorems 5-8 and 5-10), but we can finish the algorithm without constructing σ_{neg}). Check if for all (l, χ) from steps (6) and (8), the corresponding multiset $\mathbf{e}(l, \chi)$ satisfies conditions (a), (e) and (f) of (3) in Definition 5-13 with respect to the Jord that we have obtained. **If not,** $\sigma^G(\chi_1, \ldots, \chi_n)$ **is not unitarizable**. Otherwise, $\sigma^G(\chi_1, \ldots, \chi_n)$ is unitarizable.

This terminates the algorithm.

Observe that in the case of unitarizability of $\sigma^G(\chi_1, \ldots, \chi_n)$, the multisets $\mathbf{e}$ and Jord that we have obtained at the end of algorithm determine the parameters $(\mathbf{e}, \sigma(\mathrm{Jord}))$ of $\sigma^G(\chi_1, \ldots, \chi_n)$ from Theorem 5-14.

11. Isolated Points in $\mathrm{Irr}^{u,unr}(S_n)$

We equip $\mathrm{Irr}^{u,unr}(S_n)$ with the topology described in Section 3. In this section we determine all isolated representations in $\mathrm{Irr}^{u,unr}(S_n)$. It is based on our classification result Theorem 5-14 as well as the description of topology given by Theorem 3-7. In more detail, since $\mathrm{Irr}^{u,unr}(S_n)$ is a closed subset of $\mathrm{Irr}^{unr}(S_n)$, it is homeomorphic (via φ_{S_n}) to a closed subset of a complex manifold having a countable base of topology. Therefore, we have the following trivially: $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ is not an isolated point if and only if there is a sequence $(\sigma_m)_{m \in \mathbb{Z}_{>0}}$ in $\mathrm{Irr}^{unr}(S_n) \setminus \{\sigma\}$ such that

$$\lim_m \sigma_m = \sigma \quad (\text{equivalently, } \lim_m \varphi_{S_n}(\sigma_m) = \varphi_{S_n}(\sigma)). \tag{11-1}$$

We begin with the following lemma:

Lemma 11-2. *Assume that $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ is isolated. Then it must be strongly negative.*

Proof. Let $(\mathbf{e}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$ such that

$$\sigma \simeq \times_{(l,\chi,\alpha) \in \mathbf{e}} \ \langle [-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^\alpha \chi)} \rangle \rtimes \sigma_{neg}.$$

If σ is not strongly negative, then either $\mathbf{e} \neq \emptyset$, or $\mathbf{e} = \emptyset$ and $\sigma \simeq \sigma_{neg}$ is negative but not strongly negative.

If $\mathbf{e} \neq \emptyset$, then pick some $(l_0, \chi_0, \alpha_0) \in \mathbf{e}$. Applying Theorem 5-14 and Definition 5-13, we choose $\epsilon > 0$ small enough and a sequence $(\alpha_m)_{m \in \mathbb{Z}_{>0}}$ in $]\alpha_0 - \epsilon, \ \alpha_0 + \epsilon[\setminus \{\alpha_0\}$ converging to α_0 such that $(\mathbf{e}^{(m)}, \sigma_{neg}) \in \mathcal{M}^{u,unr}(S_n)$ where

$$\mathbf{e}^{(m)} = \mathbf{e} - \{(l_0, \chi_0, \alpha_0)\} + \{(l_0, \chi_0, \alpha_m)\} \ \text{ for all } m \in \mathbb{Z}_{>0}.$$

Now, we define a sequence of unramified unitary representations $(\sigma_m)_{m \in \mathbb{Z}_{>0}} \in \mathrm{Irr}^{u,unr}(S_n)$ by $\sigma_m \simeq \times_{(l^{(m)}, \chi^{(m)}, \alpha^{(m)}) \in \mathbf{e}^{(m)}} \ \langle [-\frac{l^{(m)}-1}{2}, \frac{l^{(m)}-1}{2}]^{(\nu^{\alpha^{(m)}} \chi^{(m)})} \rangle \rtimes \sigma_{neg}$. Obviously, $\sigma_m \not\simeq \sigma$ for all m and $\lim_m \varphi_{S_n}(\sigma_m) = \varphi_{S_n}(\sigma)$. Hence σ is not isolated.

If σ is negative but not strongly negative representation, then there exists $l \in \mathbb{Z}_{>0}$, an unramified unitary character χ of $F^\times$, and a negative representation σ'_{neg} such that $\sigma \hookrightarrow \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle \rtimes \sigma'_{neg}$. Now, $\chi(\varpi)$ is a complex number of absolute value one. We choose a sequence $(\alpha_m)_{m\in\mathbb{Z}_{>0}}$ of complex numbers of absolute value one converging to $\chi(\varpi)$ such that $\alpha_m \neq \chi(\varpi)$ for all m. Then we define a sequence $(\chi_m)_{m\in\mathbb{Z}_{>0}}$ of unramified unitary characters of $F^\times$ by $\chi_m(\varpi) = \alpha_m$ and a sequence of unramified (unitary) negative representations $(\sigma_m)_{m\in\mathbb{Z}_{>0}}$ in $\mathrm{Irr}^{u,unr}(S_n)$ by $\sigma_m \hookrightarrow \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi_m)}\rangle \rtimes \sigma'_{neg}$. Obviously, $\sigma_m \not\simeq \sigma$ for all m and $\lim_m \varphi_{S_n}(\sigma_m) = \varphi_{S_n}(\sigma)$. Hence σ is not isolated. $\square$

Now, we assume that σ is strongly negative. We write $\mathrm{Jord} = \mathrm{Jord}(\sigma)$ for the set of its Jordan blocks (See Theorem 5-8 and the notation introduced before that theorem.) If $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ and $a \in \mathrm{Jord}_\chi$ which is not the minimum, then we write a_- for the greatest $b \in \mathrm{Jord}_\chi$ such that $b < a$. We have

$$a - a_- \text{ is even (whenever } a_- \text{ is defined).}$$

This follows from the fact that $a - (2\alpha_\chi + 1) \in 2\mathbb{Z}$ for all $a \in \mathrm{Jord}_\chi$. (See Definition 5-4.)

The main result of this section is the following theorem:

THEOREM 11-3. *Let $n > 0$. Then $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ is isolated if and only if σ is strongly negative, and for every $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ such that $\mathrm{Jord}_\chi \neq \emptyset$, the following holds:*

(1) *$a - a_- \geq 4$, for all $a \in \mathrm{Jord}_\chi$ whenever a_- is defined.*
(2) *If $\mathrm{Jord}_\chi \neq \{1\}$, then $\min \mathrm{Jord}_\chi \setminus \{1\} \geq 4$.*

We do not claim that $1 \in \mathrm{Jord}_\chi$ in (2). If $1 \notin \mathrm{Jord}_\chi$, then (2) claims that $\min \mathrm{Jord}_\chi \geq 4$.

We start the proof of Theorem 11-3 with the following lemma:

LEMMA 11-4. *Assume that $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ is strongly negative and isolated. Assume that $\mathrm{Jord}_\chi \neq \emptyset$ for some $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. Then $a - a_- \geq 4$ for all $a \in \mathrm{Jord}_\chi$ whenever a_- is defined.*

PROOF. Assume that there exists $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ such that $\mathrm{Jord}_\chi \neq \emptyset$ and there is a gap in Jord_χ of 2, say $a - a_- = 2$ for $a, a_- \in \mathrm{Jord}_\chi$. Then the construction of strongly negative representations (see the text before Theorem 5-8) shows that

$$\mathrm{Jord}' := \mathrm{Jord} - \{(a_-, \chi), (a, \chi)\}$$

is set of Jordan blocks for some strongly negative representation $\sigma' \in \mathrm{Irr}^{u,unr}(S_{n'})$ (See Definition 5-4.) Moreover, Theorem 5-8 and Remark 1-8 imply that σ is a subquotient of

$$\langle[-\frac{a-1}{2}, \frac{a_- - 1}{2}]^{(\chi)}\rangle \rtimes \sigma' = \langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^{\frac{1}{2}}\chi)}\rangle \rtimes \sigma', \quad l = a_- + 1. \tag{11-5}$$

Now, we look at the family of induced representations $\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\nu^s\chi)} \rtimes \sigma'$ ($s \in [0, \frac{1}{2}]$). Since $l - (2\alpha_\chi + 1) = a_- + 1 - (2\alpha_\chi + 1) \notin 2\mathbb{Z}$, Lemma 6-5 implies reducibility at $s = 0$. Therefore we have unitarity and irreducibility for $s \in [0, \frac{1}{2}[$. At $s = 1/2$ we have reducibility, and σ appears as a subquotient (see (11-5)). Hence σ cannot be isolated arguing as in the proof of Lemma 11-2. $\square$

LEMMA 11-6. *Assume that* $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ *is strongly negative and isolated. Assume that* $\mathrm{Jord}_\chi \setminus \{1\} \neq \emptyset$ *for* $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. *Then* $\min \mathrm{Jord}_\chi \setminus \{1\} \geq 4$.

PROOF. We consider several cases.

First, we assume that $\alpha_\chi = 1$ (hence, $\chi = \mathbf{1}_{F^\times}$ and $S_n = \mathrm{Sp}(2n, F)$; see Lemma 5-2) and $1 \in \mathrm{Jord}_\chi$. Then the claim follows from the previous lemma. Assume that $\alpha_\chi = 1$ and $1 \notin \mathrm{Jord}_\chi$. Then the elements of Jord_χ are odd integers (since $a - (2\alpha_\chi + 1) \in 2\mathbb{Z}$). We need to show $3 \notin \mathrm{Jord}_\chi$. Assume that $3 \in \mathrm{Jord}_\chi$. Then we define a strongly negative representation σ' using

$$\mathrm{Jord}' := \mathrm{Jord} - \{(3, \chi)\} + \{(1, \chi)\}.$$

Since in this case $S_n = \mathrm{Sp}(2n, F)$, Definition 5-4 implies that card Jord_χ is odd. Hence, by Theorem 5-8 and Remark 1-8, we obtain that σ is an irreducible subquotient of $\nu\chi \rtimes \sigma'$. Now, we consider the family of representations $\nu^s\chi \rtimes \sigma'$ ($s \in [0,\ 1]$). Lemma 6-5 implies the irreducibility at $s = 0$ (since $(1, \chi) \in \mathrm{Jord}'$). Lemma 6-1 implies its irreducibility for $s \in]0,\ 1[$. Hence σ is not isolated.

Assume $\alpha_\chi = 0$. Then card Jord_χ is even by the very last property stated in Definition 5-4. Now, if $1 \in \mathrm{Jord}_\chi$, then $3 \notin \mathrm{Jord}_\chi$ by the previous lemma. If $1 \notin \mathrm{Jord}_\chi$, then we need to show that $3 \notin \mathrm{Jord}_\chi$. To prove this, assume contrary that $1 \notin \mathrm{Jord}_\chi$ and $3 \in \mathrm{Jord}_\chi$. Then we can arrive at the contradiction as in the case $\alpha_\chi = 1$ and $1 \notin \mathrm{Jord}_\chi$ above.

Assume $\alpha_\chi = \frac{1}{2}$. Then, by Definition 5-4, $S_n = \mathrm{SO}(2n+1, F)$ and Jord_χ consists of even integers. In this case we need to show $2 \notin \mathrm{Jord}_\chi$. Assume that $2 \in \mathrm{Jord}_\chi$. Then we define a strongly negative representation σ' using

$$\mathrm{Jord}' := \mathrm{Jord} - \{(2, \chi)\}.$$

Then, by Theorem 5-8 and Remark 1-8, we obtain that σ is an irreducible subquotient of $\nu^{1/2}\chi \rtimes \sigma'$. Now, we consider the family of representations $\nu^s\chi \rtimes \sigma'$ ($s \in [0,\ 1]$). Lemma 6-5 implies the irreducibility at $s = 0$. Lemma 6-1 implies its irreducibility for $s \in]0,\ \frac{1}{2}[$. Hence σ is not isolated. $\square$

Lemmas 11-4 and 11-6 prove that the conditions imposed upon σ in Theorem 11-3 are necessary. We need to show that they are sufficient. We start by constructing for an arbitrary sequence in $\mathrm{Irr}^{u,unr}(S_n)$ a convergent subsequence. Let $(\sigma_m)_{m \in \mathbb{Z}_{>0}}$ be a sequence in $\mathrm{Irr}^{u,unr}(S_n)$. The classification of unramified unitarizable representations (see Theorem 5-14) implies that there exists a unique sequence $(\mathbf{e}^{(m)}, \sigma_{neg}^{(m)}) \in \mathcal{M}^{u,unr}(S_n)$, $m \in \mathbb{Z}_{>0}$, such that

$$\sigma_m \simeq \times_{(l^{(m)}, \chi^{(m)}, \alpha^{(m)}) \in \mathbf{e}^{(m)}} \ \langle [-\frac{l^{(m)} - 1}{2}, \frac{l^{(m)} - 1}{2}]^{(\nu^{\alpha^{(m)}} \chi^{(m)})} \rangle \rtimes \sigma_{neg}^{(m)}.$$

Since σ_m is a representation of S_n, we have

$$\sum_{(l^{(m)}, \chi^{(m)}, \alpha^{(m)}) \in \mathbf{e}^{(m)}} l^{(m)} \leq n$$

for all $m \in \mathbb{Z}_{>0}$. Therefore, if we choose some enumeration writing elements of every $\mathbf{e}^{(m)}$ as a sequence:

$$\mathbf{e}^{(m)} \dots (l_1^{(m)}, \chi_1^{(m)}, \alpha_1^{(m)}), \dots, (l_{a^{(m)}}^{(m)}, \chi_{a^{(m)}}^{(m)}, \alpha_{a^{(m)}}^{(m)}),$$

then, passing to a subsequence, we may assume that the following is independent of m:

$$\begin{cases} a^{(m)} = a \\ l_i^{(m)} = l_i, \ i = 1, \ldots, a. \end{cases}$$

Next, the complex absolute value of $\chi_i^{(m)}(\varpi)$ is equal to 1. Hence, passing to subsequences we may assume that every sequence $(\chi_i^{(m)}(\varpi))_{m \in \mathbb{Z}_{>0}}$, $i = 1, \ldots, a$, converges. We define a sequence of unramified unitary characters $\chi_1, \ldots, \chi_a$ of $F^\times$ by

$$\lim_m \chi_i^{(m)}(\varpi) = \chi_i(\varpi).$$

Since every sequence $(\alpha_i^{(m)})$ is bounded (see Definition 5-13), we see that we may assume it converges:

$$\lim_m \alpha_i^{(m)} = \alpha_i.$$

Next, we apply Theorem 5-10 to the sequence $(\sigma_{neg}^{(m)})_{m \in \mathbb{Z}_{>0}}$. For every m, we find a sequence of the pairs $(k_1^{(m)}, \mu_1^{(m)}), \ldots, (k_{b^{(m)}}^{(m)}, \mu_{b^{(m)}}^{(m)})$ ($k_i \in \mathbb{Z}_{\geq 1}$, $\mu_i^{(m)}$ is an unramified unitary character of $F^\times$), and a strongly negative representation $\sigma_{sn}^{(m)}$ such that

$$\sigma_{neg}^{(m)} \hookrightarrow \langle [-\frac{k_1^{(m)} - 1}{2}, \frac{k_1^{(m)} - 1}{2}]^{(\mu_1^{(m)})} \rangle \times \cdots \times \langle [-\frac{k_{b^{(m)}}^{(m)} - 1}{2}, \frac{k_{b^{(m)}}^{(m)} - 1}{2}]^{(\mu_{b^{(m)}}^{(m)})} \rangle \rtimes \sigma_{sn}^{(m)}.$$

As above, passing to a subsequence, we may assume that the following is independent of m:

$$\begin{cases} b^{(m)} = b \\ k_i^{(m)} = k_i, \ i = 1, \ldots, a. \end{cases}$$

Hence, we may define a sequence of unramified unitary characters $\mu_1, \ldots, \mu_b$ of $F^\times$ by

$$\lim_m \mu_i^{(m)}(\varpi) = \mu_i(\varpi).$$

Next, since there are only finitely many strictly negative representations in

$$\cup_{0 \leq m \leq n} \mathrm{Irr}^{u,unr}(S_m),$$

we may assume that

$$\sigma_{sn}^{(m)} = \sigma_{sn}$$

is independent of m. We write σ_{sn} in the form $\sigma^{S_b}(\lambda_1, \ldots, \lambda_b)$ (see Theorem 1-2). Now, we have that $\varphi_{S_n}(\sigma_m)$ is the W–orbit of the n–tuple:

$$\begin{aligned}
(q^{-\frac{l_1-1}{2} - \alpha_1^{(m)}} \chi_1^{(m)}(\varpi), q^{-\frac{l_1-1}{2} + 1 - \alpha_1^{(m)}} \chi_1^{(m)}(\varpi), \ldots, q^{\frac{l_1-1}{2} - \alpha_1^{(m)}} \chi_1^{(m)}(\varpi),& \\
\ldots,& \\
q^{-\frac{l_a-1}{2} - \alpha_a^{(m)}} \chi_a^{(m)}(\varpi), q^{-\frac{l_a-1}{2} + 1 - \alpha_a^{(m)}} \chi_a^{(m)}(\varpi), \ldots, q^{\frac{l_a-1}{2} - \alpha_a^{(m)}} \chi_a^{(m)}(\varpi),& \\
q^{-\frac{k_1-1}{2}} \mu_1^{(m)}(\varpi), q^{-\frac{k_1-1}{2} + 1} \mu_1^{(m)}(\varpi), \ldots, q^{\frac{k_1-1}{2}} \mu_1^{(m)}(\varpi),& \\
\ldots,& \\
q^{-\frac{k_b-1}{2}} \mu_b^{(m)}(\varpi), q^{-\frac{k_b-1}{2} + 1} \mu_b^{(m)}(\varpi), \ldots, q^{\frac{k_b-1}{2}} \mu_b^{(m)}(\varpi),& \\
\lambda_1(\varpi), \ldots, \lambda_b(\varpi)).&
\end{aligned}$$

Clearly, the sequence $(\varphi_{S_n}(\sigma_m))$ converges to the W–orbit of the n–tuple:

$$\begin{aligned}(q^{-\frac{l_1-1}{2}-\alpha_1}\chi_1(\varpi), q^{-\frac{l_1-1}{2}+1-\alpha_1}\chi_1(\varpi), \ldots, q^{\frac{l_1-1}{2}-\alpha_1}\chi_1(\varpi),\\ \ldots,\\ q^{-\frac{l_a-1}{2}-\alpha_a}\chi_a(\varpi), q^{-\frac{l_a-1}{2}+1-\alpha_a}\chi_a(\varpi), \ldots, q^{\frac{l_a-1}{2}-\alpha_a}\chi_a(\varpi),\\ q^{-\frac{k_1-1}{2}}\mu_1(\varpi), q^{-\frac{k_1-1}{2}+1}\mu_1(\varpi), \ldots, q^{\frac{k_1-1}{2}}\mu_1(\varpi),\\ \ldots,\\ q^{-\frac{k_b-1}{2}}\mu_b(\varpi), q^{-\frac{k_b-1}{2}+1}\mu_b(\varpi), \ldots, q^{\frac{k_b-1}{2}}\mu_b(\varpi),\\ \lambda_1(\varpi), \ldots, \lambda_b(\varpi)).\end{aligned}$$

The corresponding representation $\sigma \in \mathrm{Irr}^{unr}(S_n)$ is unitary (since $\varphi_{S_n}(\mathrm{Irr}^{u,unr}(S_n))$ is a closed subset of D_n^W; see Theorem 3-7), and clearly the unique irreducible unramified subquotient of

$$\times_{i=1}^{a}\ \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{\alpha_i}\chi_i)}\rangle \times_{i=1}^{b}\ \langle[-\frac{k_i-1}{2}, \frac{k_i-1}{2}]^{(\mu_i)}\rangle \rtimes \sigma_{sn}. \tag{11-7}$$

We summarize the assumptions on the sequence:

$$\begin{aligned}&\sigma_m \simeq \times_{i=1}^{a}\ \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{\alpha_i^{(m)}}\chi_i^{(m)})}\rangle \rtimes \sigma_{neg}^{(m)},\\ &\mathbf{e}^{(m)} = \{(l_i, \chi_i^{(m)}, \alpha_i^{(m)}),;\ i = 1, \ldots, a\}, \quad (\mathbf{e}^{(m)},\ \ \sigma_{neg}^{(m)}) \in \mathcal{M}^{u,unr}(S_n)\\ &\sigma_{neg}^{(m)} \hookrightarrow \times_{i=1}^{b}\ \langle[-\frac{k_i-1}{2}, \frac{k_i-1}{2}]^{(\mu_i^{(m)})}\rangle \rtimes \sigma_{sn}.\\ &\lim_m \chi_i^{(m)}(\varpi) = \chi_i(\varpi), \quad \lim_m \mu_i^{(m)}(\varpi) = \mu_i(\varpi), \quad \lim_m \alpha_i^{(m)} = \alpha_i.\end{aligned} \tag{11-8}$$

Now, in order to complete the proof of Theorem 11-3, we need to prove the following lemma:

Lemma 11-9. *Assume that $\sigma \in \mathrm{Irr}^{u,unr}(S_n)$ is strongly negative such that (1) and (2) of Theorem 11-3 hold. Then for every sequence $(\sigma_m)_{m\in\mathbb{Z}_{>0}}$, satisfying (11-8), such that $\varphi_{S_n}(\sigma_m) \to \varphi_{S_n}(\sigma)$, there exists m_0 such that $\sigma_m = \sigma_{sn}$ for $m \geq m_0$.*

Proof. Assume that $(\sigma_m)_{m\in\mathbb{Z}_{>0}}$ is a sequence that satisfies (11-8). Assume that $\varphi_{S_n}(\sigma_m) \to \varphi_{S_n}(\sigma)$ but there is no m_0 such that $\sigma_m = \sigma_{sn}$ for $m \geq m_0$. Then passing to a subsequence we may assume that $a + b > 0$ for all $m > 0$. (a and b are defined in (11-8).) We show that this is not possible.

Put $G = S_n$. We begin by computing the Langlands lift $\tau := \sigma^{GL(N,F)}$ of σ to $GL(N, F)$. (See (10-1) for the definition of the lift and the first displayed formula in Section 10 for the definition of the number N.) We can compute the lift in two ways. First, since by our assumption σ is strongly negative, we have the following:

$$\tau \simeq \times_{(l,\chi)\in\mathrm{Jord}(\sigma)}\langle[-\frac{l-1}{2}, \frac{l-1}{2}]^{(\chi)}\rangle. \tag{11-10}$$

Also, since σ is limit of the sequence $(\sigma_m)_{m\in\mathbb{Z}_{>0}}$, it is an irreducible subquotient of the induced representation given by (11-7). Therefore, we obtain the following:

$$\begin{aligned}&\tau \text{ is the unique unramified irreducible subquotient of}\\ &\times_{i=1}^{a} \ \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{\alpha_i}\chi_i)}\rangle \times_{i=1}^{b} \ \langle[-\frac{k_i-1}{2}, \frac{k_i-1}{2}]^{(\mu_i)}\rangle\times\\ &\times_{(l',\chi')\in \mathrm{Jord}(\sigma_{sn})}\langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\chi')}\rangle\times\\ &\times_{i=1}^{a} \ \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{-\alpha_i}\chi_i^{-1})}\rangle \times_{j=1}^{b} \ \langle[-\frac{k_i-1}{2}, \frac{k_i-1}{2}]^{(\mu_i^{-1})}\rangle\end{aligned} \tag{11-11}$$

Now, since only $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$ appear in (11-10), we see that

$$\chi_1, \ldots, \chi_a, \mu_1, \ldots, \mu_b \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}.$$

Likewise, we have $\alpha_1, \ldots, \alpha_a \in \frac{1}{2}\mathbb{Z}$. Since $\lim_m \alpha_i^{(m)} = \alpha_i$ and $\alpha_i^{(m)} \in]0,1[$ (see Definition 5-13), we see that

$$\begin{aligned}&\text{if } a>0, \text{ then } \ \alpha_1, \ldots, \alpha_a \in \{0, \frac{1}{2}, 1\}, \ \chi_1, \ldots, \chi_a \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}\\ &\text{if } b>0, \text{ then } \ \mu_1, \ldots, \mu_b \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}.\end{aligned} \tag{11-12}$$

Using this we analyse (11-11). First, we observe that the unique irreducible unramified subquotient of

$$\langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{\alpha_i}\chi_i)}\rangle \times \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{-\alpha_i}\chi_i^{-1})}\rangle,$$

for $\alpha_i = \frac{1}{2}$, is

$$\langle[-\frac{l_i}{2}, \frac{l_i}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-2}{2}, \frac{l_i-2}{2}]^{(\chi_i)}\rangle.$$

Next, we observe that the unique irreducible unramified subquotient of

$$\langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{\alpha_i}\chi_i)}\rangle \times \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{-\alpha_i}\chi_i^{-1})}\rangle,$$

for $\alpha_i = 1$, is

$$\begin{cases}\langle[-\frac{l_i+1}{2}, \frac{l_i+1}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-3}{2}, \frac{l_i-3}{2}]^{(\chi_i)}\rangle; & l_i \geq 2\\ \nu\chi_i \times \nu^{-1}\chi_i; & l_i = 1.\end{cases}$$

Therefore, (11-11) implies that τ is an irreducible subquotient of

$$\begin{aligned}&\times_{i,\ \alpha_i=0} \ \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\chi_i)}\rangle\times\\ &\times_{i,\ \alpha_i=\frac{1}{2}} \ \langle[-\frac{l_i}{2}, \frac{l_i}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-2}{2}, \frac{l_i-2}{2}]^{(\chi_i)}\rangle\times\\ &\times_{i,\ \alpha_i=1,\ l_i\geq 2} \ \langle[-\frac{l_i+1}{2}, \frac{l_i+1}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-3}{2}, \frac{l_i-3}{2}]^{(\chi_i)}\rangle\times\\ &\times_{i=1}^{b} \ \langle[-\frac{k_i-1}{2}, \frac{k_i-1}{2}]^{(\mu_i)}\rangle \times \langle[-\frac{k_i-1}{2}, \frac{k_i-1}{2}]^{(\mu_i)}\rangle\\ &\times_{(l',\chi')\in \mathrm{Jord}(\sigma_{sn})} \ \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\chi')}\rangle\times\\ &\times_{i,\ \alpha_i=1,\ l_i=1} \ \nu\chi_i \times \nu^{-1}\chi_i.\end{aligned} \tag{11-13}$$

We show that there is no i such that $\alpha_i = 1$ and $l_i = 1$. If this is not the case, then (11-10) and (11-13) imply that $(3, \chi_i) \in \mathrm{Jord}(\sigma)$ for some i such that $\alpha_i = 1$ and $l_i = 1$. This contradicts (2) of Theorem 11-3. Now, we have that τ is isomorphic to

$$\begin{aligned}
&\times_{i,\ \alpha_i=0} \quad \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\chi_i)}\rangle\times \\
&\times_{i,\ \alpha_i=\frac{1}{2}} \quad \langle[-\frac{l_i}{2}, \frac{l_i}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-2}{2}, \frac{l_i-2}{2}]^{(\chi_i)}\rangle\times \\
&\times_{i,\ \alpha_i=1,\ l_i\geq 2} \quad \langle[-\frac{l_i+1}{2}, \frac{l_i+1}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-3}{2}, \frac{l_i-3}{2}]^{(\chi_i)}\rangle\times \\
&\times_{i=1}^{b} \quad \langle[-\frac{k_i-1}{2}, \frac{k_i-1}{2}]^{(\mu_i)}\rangle \times \langle[-\frac{k_i-1}{2}, \frac{k_i-1}{2}]^{(\mu_i)}\rangle \\
&\times_{(l',\chi')\in \mathrm{Jord}(\sigma_{sn})} \quad \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\chi')}\rangle.
\end{aligned} \tag{11-14}$$

Since $\mathrm{Jord}(\sigma)$ is a set (see Theorem 5-8), we see that (11-10) and (11-14) imply that $b = 0$ and there is no i such that $\alpha_i = 0$. Thus, we see that τ is isomorphic to

$$\begin{aligned}
&\times_{i,\ \alpha_i=\frac{1}{2}} \quad \langle[-\frac{l_i}{2}, \frac{l_i}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-2}{2}, \frac{l_i-2}{2}]^{(\chi_i)}\rangle\times \\
&\times_{i,\ \alpha_i=1,\ l_i\geq 2} \quad \langle[-\frac{l_i+1}{2}, \frac{l_i+1}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-3}{2}, \frac{l_i-3}{2}]^{(\chi_i)}\rangle\times \\
&\times_{(l',\chi')\in \mathrm{Jord}(\sigma_{sn})} \quad \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\chi')}\rangle.
\end{aligned}$$

If there is an i such that $\alpha_i = \frac{1}{2}$, then $(l_i+1, \chi_i) \in \mathrm{Jord}(\sigma)$ and $(l_i-1, \chi_i) \in \mathrm{Jord}(\sigma)$ $(l_i \geq 2)$. Now, if $l_i = 1$, then $(2, \chi_i) \in \mathrm{Jord}(\sigma)$. This contradicts (2) of Theorem 11-3. On the other hand, if $l_i \geq 2$, then $(l_i \pm 1, \chi_i) \in \mathrm{Jord}(\sigma)$. Clearly, if we put $a = l_i + 1$, then $a_- = l_i - 1$. Hence $a - a_- = 2$. This contradicts (1) of Theorem 11-3. Thus, we see that τ is isomorphic to

$$\begin{aligned}
\times_{i,\ \alpha_i=1,\ l_i\geq 2} \quad &\langle[-\frac{l_i+1}{2}, \frac{l_i+1}{2}]^{(\chi_i)}\rangle \times \langle[-\frac{l_i-3}{2}, \frac{l_i-3}{2}]^{(\chi_i)}\rangle\times \\
&\times_{(l',\chi')\in \mathrm{Jord}(\sigma_{sn})} \quad \langle[-\frac{l'-1}{2}, \frac{l'-1}{2}]^{(\chi')}\rangle.
\end{aligned} \tag{11-15}$$

To complete the proof we need to show that there is no i such that $\alpha_i = 1$ and $l_i \geq 2$. Assume that this is not the case. Let us fix some i_0 such that $\alpha_{i_0} = 1$ and $l_{i_0} \geq 2$. Then, (11-15) implies that

$$(l_{i_0} + 2, \chi_{i_0}) \in \mathrm{Jord}(\sigma) \tag{11-16}$$

and

$$\sigma_m \simeq \times_{i,\ \alpha_i=1,\ l_i\geq 2} \ \langle[-\frac{l_i-1}{2}, \frac{l_i-1}{2}]^{(\nu^{\alpha_i^{(m)}}\chi_i^{(m)})}\rangle \rtimes \sigma_{sn}, \ \text{for all } m > 0.$$

Since $\lim_m \alpha_i^{(m)} = \alpha_i = 1$, we may assume that $\alpha_i^{(m)} \in]\frac{1}{2},\ 1[$. Then since σ_m is unitary, Theorem 5-14 implies that $\mathbf{e}^{(m)}(l_{i_0}, \chi_{i_0}^{(m)})$ satisfies Definition 5-13 (3). In particular, $\chi_{i_0}^{(m)}(\varpi) = -1$. Now, $\lim_m \chi_{i_0}^{(m)}(\varpi) = \chi_{i_0}(\varpi)$ implies that we have $\chi_{i_0}^{(m)} = \chi_{i_0}$ for all $m > 0$. Next, according to Definition 5-13 (3) (a) (applied to any σ_m) we have the two cases.

Assume $(l_{i_0}, \chi_{i_0}^{(m)}) \in \mathrm{Jord}(\sigma_{sn})$. Then (11-15) implies that $(l_{i_0}, \chi_{i_0}) \in \mathrm{Jord}(\sigma)$. If we put $a = l_{i_0} + 2$ and apply (11-16), then we obtain that $a_- = l_{i_0}$. This contradicts (1) of Theorem 11-3.

Assume $(l_{i_0}, \chi_{i_0}^{(m)}) \notin \mathrm{Jord}(\sigma_{sn})$. Then, according to Definition 5-13 (3) (a) (applied to any σ_m), there must exist $i \neq i_0$ such that $l_i = l_{i_0}$, $\chi_i = \chi_{i_0} = \chi_{i_0}^{(m)} = \chi_i^{(m)}$, and $\alpha_i^{(m)} \in \mathbf{e}^{(m)}(l_{i_0}, \chi_{i_0}^{(m)})$. Hence $l_i = l_{i_0}$, $\chi_i = \chi_{i_0}$, and $\alpha_i = \alpha_{i_0} = 1$. Then (11-15) and (11-16) imply that $(l_{i_0} + 2, \chi_{i_0})$ appears twice in $\mathrm{Jord}(\sigma)$. This contradicts the fact that $\mathrm{Jord}(\sigma)$ is a set. (See Theorem 5-8.) $\square$

12. Examples

In this section we give examples of our algorithm presented in Section 10. We use the notation introduced there. In particular, when we speak about steps we mean the steps of the algorithm in Section 10. We begin by the following remark:

REMARK 12-1. *Suppose we consider a representation*

$$\sigma = \sigma^G(\chi_1, \dots, \chi_n). \tag{12-2}$$

Let χ be a unitary (unramified) character such that $\chi = \chi_i^u$ for some index i. Consider the subsequence $\varphi_1, \dots, \varphi_m$ of $\chi_1, \dots, \chi_n$ formed by χ_i for which $\chi_i^u \in \{\chi, \chi^{-1}\}$, and the representation

$$\sigma^G(\varphi_1, \dots, \varphi_m). \tag{12-3}$$

From the classification theorem (see Theorem 5-14 and Definition 5-13) is clear that if (12-2) is unitarizable, then (12-3) is unitarizable. The converse also holds: if (12-3) is unitarizable for all χ as above, then (12-2) is unitarizable.

Therefore, it is enough to understand how the algorithm works in the case that all χ_i^u belong to one $\{\chi, \chi^{-1}\}$. We consider below only such examples.

A. First consider the easy case: $\chi \neq \chi^{-1}$, i.e., $\chi \notin \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. In this group of examples we always assume that G is not a symplectic group. (If one adds the segment $\{\mathbf{1}_{F^\times}\}$ in the multiset η, then one would obtain examples for symplectic groups.)

12.1. Example. Look at $\sigma = \sigma^G(\chi, \nu\chi)$. Now, steps 1 and 2 give

$$\eta = \{[0,1]^{(\chi)}, [-1,0]^{(\chi^{-1})}\} = \{[-\frac{1}{2}, \frac{1}{2}]^{(\frac{1}{2}\chi)}, [-\frac{1}{2}, \frac{1}{2}]^{(-\frac{1}{2}\chi^{-1})}\}.$$

Step 3 is not executed here. In step 4, $\eta_{nsd,+} = \{[-\frac{1}{2}, \frac{1}{2}]^{(\frac{1}{2}\chi)}\}$. Since $1/2 \geq 1/2$, σ is not unitarizable.

12.2. Example. Look at $\sigma = \sigma^G(\chi, \nu\chi, \nu\chi^{-1})$. Now, steps 1 and 2 give

$$\eta = \{[-1,1]^{(\chi)}, [-1,1]^{(\chi^{-1})}\}.$$

In step 3 we have $\mathrm{Jord} = \{(3,\chi)\,(3,\chi^{-1})\}$, $\eta_{nsd,unit} = \{[-1,1]^{(\chi)}, [-1,1]^{(\chi^{-1})}\}$, and the new η is $\eta - \eta_{nsd,unit}$. The steps 4–8 are not executed for the new η (since it is empty). Step 9 sends us directly to step 10. Step 10 implies that σ is negative (therefore unitarizable) with $\mathrm{Jord}(\sigma) = \{(3,\chi)\,(3,\chi^{-1})\}$ (and $\mathbf{e} = \emptyset$).

12.3. Example. Look at $\sigma = \sigma^G(\nu^{-1/4}\chi, \nu^{3/4}\chi)$. Now, steps 1 and 2 give

$$\eta = \{[-1/4, 3/4]^{(\chi)}, [-3/4, 1/4]^{(\chi^{-1})}\} = \{[-\frac{1}{2}, \frac{1}{2}]^{(\frac{1}{4}\chi)}, [-\frac{1}{2}, \frac{1}{2}]^{(-\frac{1}{4}\chi^{-1})}\}.$$

Step 3 is not executed. In step 4 we have $\eta_{nsd,+} = \{[-\frac{1}{2}, \frac{1}{2}]^{(\frac{1}{4}\chi)}\}$ Now, we have a non-unitarizability since the Hermitian condition is not satisfied: $\eta_{nsd,+} \neq \bar{\eta}_{nsd,+}$. (The exponents are $< 1/2$.)

B. Now, consider the case $\chi = \chi^{-1}$. This means $\chi \in \{\mathbf{1}_{F^\times}, \mathbf{sgn}_u\}$. We shall deal with χ through the constant $\alpha_\chi \in \{0, \frac{1}{2}, 1\}$ defined in Definition 5-2. We write every χ_i (uniquely) in the form $\nu^{e(\chi_i)}\chi$ where $e(\chi_i) \in \mathbb{R}$, and instead of assigning $\sigma = \sigma^G(\chi_1, \ldots, \chi_n)$, we assign the sequence of exponents $e(\chi_1), \ldots, e(\chi_n)$. (Repeated exponents as in the sequence $1, 1, 1, 2, 2, 3$ shall be written as follows: $1^{(3)}, 2^{(2)}, 3$.) In what follows $[a, b]$ means $[a, b]^{(\chi)} = [-\frac{b-a}{2}, \frac{b-a}{2}]^{(\nu^{\frac{a+b}{2}}\chi)}$. We shall start with simple examples.

12.4. Example. We consider the exponents $1, 2$. Let $\alpha_\chi = 1/2$. Then σ is a representation of $SO(5, F)$. Now, step 2 gives $\eta = \{[-2, -1], [1, 2]\}$. Now, step 6 is relevant and it implies $[1, 2] \in \eta_{sd,\{0,1\},+}$. Since $\frac{1+2}{2} \geq 1$, σ is not unitarizable.

12.5. Example. We consider the exponents $1, 2$. Let $\alpha_\chi = 0$. Now, step 2 gives $\eta = \{[-2, -1], [1, 2], \ldots\}$. (In addition to the displayed segments, one needs to include $\{\mathbf{1}_{F^\times}\}$ if $G = Sp(4, F)$; in which case $\chi = \mathbf{sgn}_u$.) Now, step 5 is relevant and it implies $[1, 2] \in \eta_{sd,\frac{1}{2},+}$. Since $\frac{1+2}{2} \geq 1/2$, σ is not unitarizable.

12.6. Example. We consider the exponents $1, 2$. Consider $\alpha_\chi = 1$. (Then σ is a representation of $Sp(4, F)$ and $\chi = \mathbf{1}_{F^\times}$; see Definition 5-2). Step 2 gives $\eta = \{[-2, 2]\}$. Since $(5-1)/2 = 2$ and $5 - (2\alpha_\chi + 1) = 2 \in 2\mathbb{Z}$, step 7 is relevant. It implies Jord $= \{(5, \mathbf{1}_{F^\times})\}$. We remove $[-2, 2]$ from η and proceed further to step 9. Step 9 takes us to step 10. Step 10 shows that σ is (strongly) negative.

12.7. Example. We consider the exponents $1/2, 3/2$. We assume that σ is a representation of $\mathrm{O}(4, F)$. Then $\alpha_\chi = 0$. Now step 2 gives $\eta = \{[-3/2, 3/2]\}$. Since $(4-1) = 3/2$ and $4 - (2\alpha_\chi + 1) = 3 \notin 2\mathbb{Z}$, we proceed to step 8. Since $4 \geq 3$, we see that σ is not unitarizable by (c) of (ii) in step 8.

12.8. Example. We consider the exponents $1/2, 3/2$. Consider $\alpha_\chi = 1/2$. Then σ is a representation of $\mathrm{SO}(5, F)$. Again step 2 gives $\eta = \{[-3/2, 3/2]\}$. Since $(4-1) = 3/2$ and $4 - (2\alpha_\chi + 1) = 2 \in 2\mathbb{Z}$, we proceed to step 7. We remove $[-2, 2]$ from η and proceed further to step 9. Step 9 takes us to step 10. Step 10 shows that σ is (strongly) negative.

12.9. Example. We consider the exponents $1/2, 3/2$. Consider $\alpha_\chi = 1$. Then σ is a representation of $\mathrm{Sp}(4, F)$ and $\chi = \mathbf{1}_{F^\times}$. Step 2 gives $\eta = \{[-3/2, 3/2], [0, 0]\}$. Step 7 will put $(1, \mathbf{1}_{F^\times})$ in Jord, but we get non-unitarizability from (c) of (ii) in step 8 (applied to $\Delta = [-3/2, 3/2]$).

Now come some slightly more complicated examples.

12.10. Example. We consider the exponents $0^{(4)}, 1^{(5)}, 2^{(3)}, 3$. Then $n = 4 + 5 + 3 + 1 = 13$. We assume that σ is a representation of $\mathrm{O}(26, F)$ or $\mathrm{SO}(27, F)$. We perform step 2

characters of σ^{GL}	η
$-3, -2^{(3)}, -1^{(5)}, 0^{(8)}, 1^{(5)}, 2^{(3)}, 3$	$\emptyset$
$-2^{(2)}, -1^{(4)}, 0^{(7)}, 1^{(4)}, 2^{(2)}$	$[-3, 3]$
$-2, -1^{(3)}, 0^{(6)}, 1^{(3)}, 2$	$[-3, 3], [-2, 2]$
$-1^{(2)}, 0^{(5)}, 1^{(2)}$	$[-3, 3], 2 \cdot [-2, 2]$
$-1, 0^{(4)}, 1$	$[-3, 3], 2 \cdot [-2, 2], [-1, 1]$
$0^{(3)}$	$[-3, 3], 2 \cdot [-2, 2], 2 \cdot [-1, 1]$
$\emptyset$	$[-3, 3], 2 \cdot [-2, 2], 2 \cdot [-1, 1], 3 \cdot [0, 0].$

Consider first $\alpha_\chi = 0$. Then σ is a representation of $\mathrm{O}(26, F)$. Now, step 7 gives $\mathrm{Jord} = \{(7, \chi), 2 \cdot (5, \chi), 2 \cdot (3, \chi), (1, \chi)\}$ and $\mathbf{e} = \emptyset$. We proceed to step 10. The representation σ is a negative representation attached to Jord.

Now consider reducibility at $\alpha_\chi = 1/2$. Then σ is a representation of $\mathrm{SO}(27, F)$. Now, step 8 is relevant. In the first iteration of step 8 the largest possible l is $l = 2 \cdot 3 + 1 = 7$ and the corresponding Δ is $\Delta = [-3, 3]$. We apply (ii) (c), to see that σ is not unitarizable.

12.11. Example. Consider the exponents $0^{(6)}, 1^{(8)}, 2^{(3)}, 3^{(2)}, 4$. Let $n = 6 + 8 + 3 + 2 + 1 = 20$. Assume that σ is a representation of $\mathrm{SO}(41, F)$. Then $\alpha_\chi = 1/2$. As in the previous example, step 2 gives:

$$\eta = \{[-4, 4], [-3, 3], [-2, 2], 5 \cdot [-1, 1], 4 \cdot [0, 0]\}.$$

Applying step 8 (ii) (c) twice and step 8 (i) twice, we obtain

$$\mathbf{e} = \{(8, \chi, 1/2), (4, \chi, 1/2)\}, \quad \mathrm{Jord} = \{4 \cdot [-1, 1], 4 \cdot [0, 0]\}.$$

We proceed directly to step 10. We obtain unitarizability.

12.12. Example. Consider the exponents 1/4, 4/6 and 5/6. Let $n = 1 + 1 + 1 = 3$. Assume that σ is a representation of $\mathrm{Sp}(6, F)$. Let $\chi = \mathbf{1}_{F^\times}$. Then $\alpha_\chi = 1$. Step 2 gives the multiset

$$\eta = \{[-1/4, -1/4], [-4/6, -4/6], [-5/6, -5/6], [0, 0], [1/4, 1/4], [4/6, 4/6], [5/6, 5/6]\}.$$

Now, step 6 implies

$$\mathbf{e} = \{(1, \chi, 1/4), (1, \chi, 4/6), (1, \chi, 5/6)\}.$$

Next, step 7 implies $\mathrm{Jord} = \{(1, \chi)\}$. Step 10 implies unitarizability.

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