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A CONSTRUCTION OF ELEMENTS IN THE BERNSTEIN CENTER FOR QUASI-SPLIT GROUPS

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Abstract.

The Bernstein center of a reductive p-adic group is the algebra of conjugation invariant distributions on the group which are essentially compact, i.e., invariant distributions whose convolution against a locally constant compactly supported function is again locally constant compactly supported. The center acts naturally on any smooth representation, and if the representation is irreducible, each element of the center acts as a scalar. For a quasi-split group, we show certain linear combinations of orbital integrals belong to the Bernstein center. Furthermore, when these combinations are projected to a Bernstein component, they form an ideal in the Bernstein center which can be explicitly described and is often a principal ideal. The elements constructed here should have applications to various questions in harmonic analysis.

1. Introduction

1.1. An indispensable tool in the representation theory of reductive Lie groups is the center $\mathcal{Z}(\mathfrak{U}(\mathrm{Lie}(G)))$ of the universal enveloping algebra $\mathfrak{U}(\mathrm{Lie}(G))$ of the Lie algebra $\mathrm{Lie}(G)$ of a Lie group G. In the 1980's, J. Bernstein introduced into the representation theory of reductive p-adic groups an analogue of the center of the universal enveloping algebra (see [BD]). Suppose F is a non-archimedean local field of characteristic zero, i.e., a p-adic field and $G = \mathsf{G}(F)$ is the group of F-rational points of a connected reductive group G. Bernstein's center has two realizations, which are referred to as the geometrical and spectral realizations.

Let $\mathcal{C}_c^\infty(G)$ denote the space of locally constant compactly supported functions on G. In the geometrical realization, the Bernstein center is a space $\mathcal{Z}(G)$ of G-invariant distributions which can act on any smooth representation. The later requirement means the invariant distributions should satisfy a property known as essential compactness, i.e.,

$$\text{(1.1a)}\quad \mathcal{Z}(G) = \{ \text{ invariant distributions } D \mid D \star f = f \star D \in \mathcal{C}_c^\infty(G) \ \forall \ f \in \mathcal{C}_c^\infty(G) \} \ .$$

If $g \in G$, it is elementary the delta distribution δ_g belongs to $\mathcal{Z}(G)$ precisely when it is G-invariant, i.e., when g is a central element of G.

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It is natural to view $\mathcal{Z}(G)$ as a subspace of a larger space of distributions $\mathcal{U}(G)$. This is done by dropping the G-invariance condition, but keeping the essential compactness property. So

$$\text{(1.1b)} \qquad \mathcal{U}(G) \ := \ \{ \text{ distributions } D \mid \text{ both } D \star f, \ f \star D \in \mathcal{C}_c^\infty G \ \ \forall \ f \in \mathcal{C}_c^\infty G \, \} \ .$$

It is elementary that $\mathcal{U}(G)$ can be made into a convolution algebra, so that $\mathcal{Z}(G)$ is its center. Furthermore, $\mathcal{U}(G)$ acts naturally on any smooth representation of G, and the restriction of the action to $\mathcal{Z}(G)$ is precisely the original action of $\mathcal{Z}(G)$. A more detailed discussion of $\mathcal{U}(G)$ and $\mathcal{Z}(G)$ is in section 4.

Denote by $\widetilde{G}$ the smooth dual of G. It carries a natural topology [T], and one can then produce a Hausdorffization $\Omega(G)$ of $\widetilde{G}$. The natural algebraic group structure on the unramified quasi-characters of Levi subgroups of G defines an algebraic variety structure on $\Omega(G)$. The spectral realization of the Bernstein center is as the algebra

$$\text{(1.1c)} \qquad \mathfrak{Z}(G) \ = \ \{ \text{ regular functions on } \Omega(G) \, \} \ .$$

The connection between the geometrical and spectral realization of the Bernstein center is that any invariant essentially compact distribution $z \in \mathcal{Z}(G)$ naturally acts as a scalar on any irreducible smooth representation, and so determines a map $\tilde{z} : \widetilde{G} \to \mathbb{C}$. The map $z \to \tilde{z}$ factors through the Hausdorffization $\Omega(G)$ and is a regular function on $\Omega(G)$. Then, $z \to \tilde{z}$ is a isomorphism of the algebras $\mathcal{Z}(G)$ and $\mathfrak{Z}(G)$.

1.2. The space of G-invariant essentially compact distributions is vast. For example, if G is semi-simple, and π is an irreducible supercuspidal representation of G, then the character Θ_π of π belongs to $\mathcal{Z}(G)$. But supercuspidal characters are rather mysterious objects, and indeed so too is the Bernstein center. Besides the delta distribution of central elements, characters of supercuspidal representations, and recent work of the authors for $\mathsf{SL}(2)(F)$ [MT1,MT2], only one other explicit distribution can be found in the literature. We describe it. Suppose ψ is a nontrivial additive character of the p-adic field F. In the notes [Bn], Bernstein mentions the distribution on $\mathsf{SL}(n)(F)$ represented by the function

$$\text{(1.2a)} \qquad g \mapsto \psi(\mathrm{Trace}(g))$$

is essentially compact, and thus lies in the Bernstein center. In those notes, Bernstein raised the question of explicit construction of invariant essentially compact distributions.

1.3. The distributions in $\mathcal{U}(G)$, and hence in the Bernstein center are known to be tempered [MT3]. A very natural, important, and relatively simple source of G-invariant tempered distributions on reductive groups are orbital integrals. An orbital integral is essentially compact if and only if the orbit is compact. In particular, if G has no compact factors, then aside from the delta distributions on central elements, orbital integrals do not belong to $\mathcal{Z}(G)$.

1.4. Let $G = \mathsf{SL}(2)(F)$. One recent striking discovery of [MT2] is that certain linear combinations of orbital integrals, in particular certain differences, are essentially compact, and therefore in the Bernstein center.

Suppose two non-empty subsets $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of $\mathsf{G}(F)$ are each the finite union of conjugacy classes of $\mathsf{G}(F)$. It is shown in [MT2] that if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have the same asymptotic behavior at infinity (see [MT2] for the definition of same asymptotic behavior), then the difference of the normalized orbital integrals of the classes in $\mathcal{Q}_1$, and those in $\mathcal{Q}_2$ lies in the Bernstein center. For example, two hyperbolic regular elements have the same asymptotic behavior at infinity. Therefore, the difference of their normalized orbital integrals lies in the Bernstein center. The proof of the above fact in [MT2] is based on exploiting the significant amount of existing explicit knowledge of the representations and harmonic analysis of $\mathsf{SL}(2)(F)$, namely, explicit descriptions of (i) the smooth dual, (ii) the Plancherel measure [SaSh1], (iii) the characters of the irreducible representations [Sa,SaSh3], and formulae [SaSh2] for the expansion of orbital integrals. Such explicit information is currently unavailable for other groups.

1.5. After discussions with Dan Barbasch, the first author reformulated some of the results of [MT2] in a way which allows formulation of a partial extrapolation of the results for $\mathsf{SL}(2)(F)$ to quasi-split groups. Suppose G is quasi-split over F, and $A_\emptyset = \mathsf{A}_\emptyset(F)$ is the F-rational points of a maximal F-split torus $\mathsf{A}_\emptyset$ of G. Let $\mathsf{M}_\emptyset$ be the centralizer $\mathsf{A}_\emptyset$ and $M_\emptyset = \mathsf{M}_\emptyset(F)$. Let W denote the Weyl F-group of $A_\emptyset$. Suppose $\gamma_0, \gamma \in M_\emptyset$, satisfy the property that for each $w \in W$, the product $\gamma_0\, w(\gamma)$ is W-regular, i.e., if $w' \in W$, and $w'(\gamma_0\, w(\gamma)) = \gamma_0\, w(\gamma)$, then $w' = 1$. Consider the distribution

$$f \;\longrightarrow\; F^{M_\emptyset}_f(\,\gamma_0\ w\cdot\gamma\,) \qquad f \in \mathcal{C}^\infty_c(G) \tag{1.5a}$$

which is the normalized orbital integral over the conjugacy class $\mathcal{O}(\gamma_0\, w\cdot\gamma)$ of $\gamma_0\, w(\gamma)$. Then, our first key result (Theorem 7.9f) is that the (W-skew) linear combination of normalized orbital integrals

$$f \;\longrightarrow\; S^{M_\emptyset}(\gamma_0,\gamma)(f) \;:=\; \sum_{w\in W} \operatorname{sgn}(w)\, F^{M_\emptyset}_f(\,\gamma_0\ w\cdot\gamma\,) \qquad f \in \mathcal{C}^\infty_c(G) \tag{1.5b}$$

is an element of the Bernstein center. This sum over W has obvious similarities to sums in the Weyl and Harish-Chandra character formulae, but this is the first time such a sum has appeared in the context of the Bernstein center. The methodology of the proof, which is the one originally used in [MT2] for $\mathsf{SL}(2)(F)$, is to use knowledge of the Plancherel measure, and a criterion for when an invariant distribution belongs to the Bernstein center. More specifically, the Fourier transform of these distributions have support on components $\mathcal{Q}$, where $\mathcal{Q}$ is a connected component in the set $\widetilde{M_\emptyset}$ of quasi-characters of $M_\emptyset$ (see sections 5.6 and 7.3). The criterion is that, for all $\mathcal{Q}$, the restriction of the Fourier Transform to $\mathcal{Q}$ is a regular function.

As already remarked, an orbital integral is rarely in the Bernstein center. For example if $\gamma \in M_\emptyset$ is regular, then spectrally, the Fourier transform of $F^{M_\emptyset}(\gamma)$ has singularities on the unitary principal series with respect to the Plancherel density, but is regular almost everywhere. It is then natural to seek finite combinations of orbital integrals for which the Fourier transform singularities cancel. For $\mathsf{SL}(2)(F)$, this can be accomplished by taking differences of orbital integrals, but differences do not work for groups of higher rank. Skew sums, over the Weyl group, such as the distributions $S^{M_\emptyset}(\gamma_0,\gamma)$ are a natural combination to consider. The Fourier transform of $S^{M_\emptyset}(\gamma_0,\gamma)$ turns out to be the product of two skew symmetric factors. In this regard, it is very useful to introduce a generalization of the Weyl denominator (see section 7.7). The generalized Weyl denominator has a divisibility property with respect to skew functions analogous to the ordinary Weyl denominator and this is crucial. In particular, the skew symmetry of each factor means it must vanish on certain fixed points. This vanishing, via Hilbert's Nullstellensatz, establishes the Fourier transform of $S^{M_\emptyset}(\gamma_0,\gamma)$ is regular on the unitary principal series. To complete the proof that $S^{M_\emptyset}(\gamma_0,\gamma)$ lies in the Bernstein center we show the analytic continuation of the Fourier transform vanishes on tempered representations which show up in principal series induced by non-unitary characters. We get this from the Plancherel formula, since these tempered representations show up where the Plancherel density has a pole.

Once the distributions of type (1.5b) are known to be in the Bernstein center it is natural to ask what can be established about the subspace spanned by them as γ_0 and γ vary over $M_\emptyset$. Each component $\mathcal{Q}$ has a canonically defined idempotent $e_{_\mathcal{Q}}$ in the Bernstein center. It is the distribution whose Fourier Transform has constant value one on $\mathcal{Q}$ and zero elsewhere. Let $\mathcal{FT}(\mathcal{Q})$ denoted the space spanned by the convolution of the distributions of type (1.5b) with the idempotent $e_{_\mathcal{Q}}$. Our second key result is an explicit description of $\mathcal{FT}(\mathcal{Q})$. In particular, we show $\mathcal{FT}(\mathcal{Q})$ is an ideal in the Bernstein center (see Proposition 8.2h). It is often the principal ideal generated by the element which is the quotient of the generalized Weyl denominator by the Plancherel density, but not always (see Proposition 8.7a, Theorem 8.8g, and example 8.6). Here, the generalized Weyl denominator again plays a crucial role.

1.6. Now we describe the manuscript according to its sections. In section 2, we introduce some notation, and as an aide to the reader, we give some of the conventions we follow in choosing notation. In section 3, we recall notation and facts related to parabolic data. In section 4, we introduce the convolution algebra of essentially compact distributions, and recall its center is one realization of the Bernstein center. Section 5 recalls Plancherel densities, the Plancherel formula, and the spectral realization of the Bernstein center. In section 6, we establish some results on root systems needed to define our generalization of the Weyl denominator. In section 7, we take up the distributions (1.5b) and compute their Fourier Transform and establish they belong to the Bernstein center. In section 8, we determine the space $\mathcal{FT}(\mathcal{Q})$.

1.7. Of keen interest is whether a formula of type (1.5b) can be extended to other more elliptic tori. For elliptic tori of $G = \mathsf{SL}(2)(F)$ it is shown in [MT2] that certain differences of normalized elliptic orbital integrals do belong to the Bernstein center, so there is evidence that (1.5b) does have an extension.

We note that because the combination (1.5b) lies in the Bernstein center, we have the following: Suppose $J \subset G$ is any open compact subgroup of G. The convolution of the distribution (1.5b) with the characteristic function 1_J of J must belong to $\mathcal{C}_c^\infty(G)$. This means for g outside some compact set M of G, one must have

$$\sum_{w\in W} \operatorname{sgn}(w)\ \operatorname{meas}(gJ \cap \mathcal{O}(\gamma_0\, w\cdot\gamma)) \ = 0\ . \tag{1.7a}$$

Conversely, it is elementary to see that the vanishing statement (1.7a) also implies the combination (1.5b) of normalized orbital integral lies in the Bernstein center. It would be very interesting if property (1.7a) could be established in a geometrical manner. For $\mathsf{SL}(2)(F)$, there is such a proof of (1.7a) in [MT2]. In general, if (1.7a) can be established in an extended generality that encompasses arbitrary tori in G, it would lead to a very general construction of elements of the Bernstein center.

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2. Notation

2.1. Let F be a non-archimedean local field with modulus character $|\ \ |_F$, i.e., if dx is a Haar measure on F, then $d(ax) = |a|_F dx$ for any $a \in F^\times$. Denote by val_F the valuation map $\mathrm{val}_F : F \to \mathbb{Z}$.

Suppose G is a connected reductive group defined over F, and $G = \mathsf{G}(F)$ is its group of F-rational points. In all that follows, we will use the terminological convenience but logical imprecision of referring to the group of F-rational points $H := \mathsf{H}(F)$ of a subgroup $\mathsf{H} \subset \mathsf{G}$ as a F-subgroup H.

2.2. To facilitate the reader's understanding of our notation, we shall mainly adhere to the following conventions:

(i) Enclosed parenthesis will usually indicate rational points or a dependence on an F-subgroup. For example, if G is an algebraic group, then $\mathsf{G}(F)$ is its subgroup of F-rational points, and $\mathrm{Lie}(\mathsf{G})$ is its algebraic Lie algebra.

(ii) A superscript on the right will usually indicate invariants, or isotypical component of a group action.

(iii) A subscript on the right will indicate a dependence on a group, e.g., a Levi or parabolic subgroup, or a subset, e.g., a subset of the simple roots. Thus, for example, if M is a F-Levi subgroup, the maximal split subgroup in the center is denoted A_M, and the Weyl group of M is denoted W_M. If a choice of positive roots has been made, the positive roots in a unipotent radical N is denoted by $\Sigma_+(A_M)$, etc.

(iv) A superscript on the left is used only when that notation is common in the literature.

3. Parabolic data

3.1. Fix $A_\emptyset$ to be a maximal split F-torus of G. Let Σ_G denote the $A_\emptyset$-roots, i.e., non-trivial characters of $A_\emptyset$, which occur in $\mathrm{Lie}(G)$. Recall a semi-standard parabolic F-subgroup, with respect to $A_\emptyset$, is a parabolic F-subgroup containing $A_\emptyset$. Recall also that given a semi-standard parabolic subgroup Q, there is a unique opposite semi-standard parabolic subgroup $\overline{Q}$ so that $M := Q \cap \overline{Q}$ is a Levi subgroup of both Q and

$\overline{Q}$. The uniquely defined Levi subgroup M is called a semi-standard Levi F-subgroup. Denote the maximal split F-torus in the center of M as A_M, and the unipotent radicals of Q and $\overline{Q}$ by N_Q and $N_{\overline{Q}}$ respectively. The F-torus A_M is a subgroup of $A_\emptyset$, and M equals $C_G(A_M)$, the centralizer of A_M in G. In particular, note that $M_\emptyset := C_G(A_\emptyset)$ is contained in any $M = C_G(A_M)$, and $A_{M_\emptyset} = A_\emptyset$.

Let Σ_M denote the $A_\emptyset$-roots which occur in Lie(M). The complementary set $\Sigma_G \backslash \Sigma_M$ of $A_\emptyset$-roots to Σ_M is the $A_\emptyset$-roots which occur in Lie(N_Q) and Lie$(N_{\overline{Q}})$. These roots are precisely the $A_\emptyset$-roots whose restriction to A_M is non-trivial. Let $\Sigma(A_M)$ denote the characters of A_M obtained from these roots. Note that $\Sigma_G = \Sigma(A_\emptyset)$. These A_M-characters are called the A_M-roots. Recall that

(3.1a) the A_M-roots form a root system of characters of A_M .

Following a convention in [He], we set

(3.1b) $$\Sigma(Q) \ := \ \{ \, \alpha \in \Sigma(A_M) \, \big| \, \alpha \text{ occurs as an } A_M\text{-root in Lie}(N_Q) \ \} \, .$$

Designate the reduced roots in $\Sigma(A_M)$, and $\Sigma(Q)$ as $\Sigma_{\mathrm{red}}(A_M)$, and $\Sigma_{\mathrm{red}}(Q)$ respectively. Clearly, $\Sigma(A_M)$ is the disjoint union of $\Sigma(Q)$ and $-\Sigma(Q)$.

Let M be a semi-standard Levi subgroup. Denote by $W_G(A_M)$ the quotient group of $N_G(A_M)$, the normalizer of A_M, by the centralizer $C_G(A_M)$. Recall that $C_G(A_M)$ equals M. Set $W_\emptyset := W_G(A_\emptyset)$. Each coset of the quotient $W_G(A_M)$ can be represented by an element of $N_G(A_\emptyset)$ which is unique modulo $M_{A_\emptyset} := C_G(A_\emptyset)$.

3.2. Suppose $P_\emptyset \supset A_\emptyset$ is a minimal parabolic F-subgroup of G. In this situation, let $M_\emptyset$ denote the unique semi-standard Levi F-subgroup, with respect to $A_\emptyset$, contained in $P_\emptyset$, and let $P_\emptyset = M_\emptyset N_\emptyset$ be the Levi decomposition. The selection of $P_\emptyset$ determines a splitting of Σ_G into positive $\Sigma_{G,+}$ and negative roots $-\Sigma_{G,+}$; in particular, it determines a set $\Delta_G \subset \Sigma_{G,+}$ of simple positive roots.

For notational convenience, we abbreviate Σ_G, $\Sigma_{G,+}$, and Δ_G to Σ, Σ_+, and Δ respectively.

A parabolic F-subgroup Q is called standard, with respect to $P_\emptyset$, if $Q \supset P_\emptyset$. Recall the standard parabolic subgroups are in one-one correspondence with the subsets of Δ. For a subset $J \subset \Delta$, let P_J be the smallest F-group containing $P_\emptyset$ and the root groups $U_{\pm\alpha}$, and $U_{\pm 2\alpha}$ for all $\alpha \in J$. The group $M = M_J$ generated by $A_\emptyset$, and the root groups $U_{\pm\alpha}$, and $U_{\pm 2\alpha}$ for all $\alpha \in J$ is the unique semi-standard Levi subgroup of P_J. As in the above paragraph (3.1), let Σ_M denote the roots of $A_\emptyset$ occurring in Lie(M). Then, the subset $\Delta_M := \Delta \cap \Sigma_M$ of simple roots, in Σ_M, with respect to $P_\emptyset$, is J.

We denote by both A_{M_J}, and A_J – the latter for notational convenience – the maximal split F-torus in the center of M_J, i.e., $A_J = \{ \, a \in A_\emptyset \, | \, \alpha(a) = 1 \ \forall \, \alpha \in J \, \}$.

As in paragraph (3.1), we let $\Sigma(A_J)$ denote the root system of A_J obtained as the non-trivial restrictions of roots of $A_\emptyset$. The choice of $P_\emptyset$ determines a subset $\Sigma_+(A_J) \subset \Sigma(A_J)$ of positive roots, as well as a subset $\Delta_+(A_J)$ set of simple roots. The characters $\Sigma_+(A_J)$ of A_J are precisely those which occur in the unipotent radical N_{P_J} of P_J. For notational convenience we shorten N_{P_J} to N_J.

Denote the reduced roots in $\Sigma(A_J)$, and $\Sigma_+(A_J)$ by $\Sigma_{\mathrm{red}}(A_J)$, and $\Sigma_{\mathrm{red},+}(A_J)$ respectively.

3.3. Suppose L is a connected reductive F-group. Let $X_L = \mathrm{Hom}_F(L, \mathrm{GL}(1))$ denote the additive group of one-dimensional F-characters. It is a free finite rank $\mathbb{Z}$-module, i.e., a lattice. The natural map

$$L \times X_L \longrightarrow \mathrm{GL}(1) \xrightarrow{\mathrm{val}_F} \mathbb{Z} \tag{3.3a}$$

defines a perfect pairing of lattices

$$\Lambda(L) \times X_L \longrightarrow \mathbb{Z} \ , \tag{3.3b}$$

where

$$\begin{aligned} \Lambda(L) \ &:= \ L/{}^{\circ}L \ , \quad \text{with} \\ {}^{\circ}L \ &:= \ \bigcap_{\chi \in X_L} \mathrm{Ker}\,(\,\mathrm{val}_F \circ \chi\,) \quad (\text{equivalently Ker } |\chi|_F\,) \ . \end{aligned} \tag{3.3c}$$

The pairing for $\ell \cdot {}^{\circ}L \in L/{}^{\circ}L$, and $\chi \in X_L$ is

$$\langle\, \ell \cdot {}^{\circ}L \,,\, \chi \,\rangle \ = \ \mathrm{val}_F(\chi(\ell)) \ . \tag{3.3d}$$

Recall our notational convention has the maximal split F-torus in the center of L denoted as A_L. Both X_L and X_{A_L} are of rank the dimension of A_L, and the restriction of a F-character of L to A_L is an embedding $X_L \to X_{A_L}$ with finite cokernel. In fact, we have

$$\begin{array}{ccccc} \Lambda(L) & \times & X_L & \longrightarrow & \mathbb{Z} \\ \cup & & \cap & & \\ \Lambda(A_L) & \times & X_{A_L} & \longrightarrow & \mathbb{Z} \end{array} \ , \tag{3.3e}$$

and the embedding $X_L \to X_{A_L}$ is dual to an embedding $\Lambda(A_L) \to \Lambda(L)$.

Under the same hypothesis that L is a reductive F-subgroup, we set

$$\begin{aligned} \mathfrak{a}_L \ &:= \ \Lambda(L) \otimes_{\mathbb{Z}} \mathbb{R} \ \supset \ \Lambda(L) \\ \mathfrak{a}_L^* \ &:= \ X_L \otimes_{\mathbb{Z}} \mathbb{R} \ \supset \ X_L \\ \left(\mathfrak{a}_L^*\right)_{\mathbb{C}} \ &:= \ X_L \otimes_{\mathbb{Z}} \mathbb{C} \ \supset \mathfrak{a}_L^* \\ \Psi(L) \ &:= \ \mathrm{Hom}(\Lambda(L)\,,\, \mathbb{C}^{\times}\,) \ . \end{aligned} \tag{3.3f}$$

The first two objects $\mathfrak{a}_L$ and $\mathfrak{a}_L^*$ are the real Lie algebra and real dual attached to A_L. The pairing (3.3d) extends uniquely to pairings $\mathfrak{a}_L \times \mathfrak{a}_L^* \to \mathbb{R}$ and $\mathfrak{a}_L \times (\mathfrak{a}_L^*)_{\mathbb{C}} \to \mathbb{C}$. We keep the same notation $\langle\, ,\, \rangle$ for these extended pairings.

The set $\Psi(L)$, whose notation follows J. Bernstein's notes [Bn], is a complex torus of dimension $\dim(A_L)$, and it is canonically the set of unramified characters of L.

Define $H_L : L \to \mathfrak{a}_L$ to be the composition of the obvious maps $L \to \Lambda(L) \hookrightarrow \mathfrak{a}_L$. Recall the complex vector space $(\mathfrak{a}^*_L)_{\mathbb{C}}$ can be used to realize the unramified characters of L as follows. For $\nu \in (\mathfrak{a}^*_L)_{\mathbb{C}}$, set

$$\chi_\nu \ := \ \ell \to q^{\,-\langle H_L(\ell)\,,\,\nu\,\rangle} \quad , \quad \ell \in L \ . \tag{3.3g}$$

In particular, if $\nu = \phi \in X_L \subset \mathfrak{a}^*_L$, then $\chi_\phi := \ell \to |\phi(\ell)|$. Obviously,

$$\begin{aligned} &\text{(i)} \ \ \text{the character } \chi_\nu \text{ is unitary if and only if } \nu \in \sqrt{-1}\,\mathfrak{a}^*_L \,, \\ &\text{(ii)} \ \ \chi_\nu \equiv 1 \text{ if and only if } \nu(\ell \cdot {}^{\circ}L) \in \frac{2\pi\sqrt{-1}}{\ln(q)}\mathbb{Z}, \text{ for all } \ell \in L \ . \end{aligned} \tag{3.3h}$$

Since the pairing (3.3d) is perfect, condition (ii) holds precisely when ν belongs to the lattice

$$\mathcal{L}_L \ := \ \frac{2\pi\sqrt{-1}}{\ln(q)} X_L \quad \subset \quad \sqrt{-1}\,\mathfrak{a}^*_L \ , \tag{3.3i}$$

and the map $\nu \to \chi_\nu$ gives a natural isomorphism of $\big(\mathfrak{a}^*_L\big)_{\mathbb{C}} / \mathcal{L}_L$ to $\Psi(L)$.

3.4. With $A_\emptyset$ as in paragraph (3.1), let $M_\emptyset := C_G(A_\emptyset)$ denote the minimal semi-standard Levi F-subgroup. Suppose $M_2 \supset M_1 \supset M_\emptyset$ are two additional semi-standard Levi F-subgroups. Restriction of a character $\chi \in X_{M_2}$ to M_1 is an injection of X_{M_2} into X_{M_1}. These restrictions induce embeddings in (3.4b) below. Dual to this restriction map is a map $\pi : M_1/{}^{\circ}M_1 \to M_2/{}^{\circ}M_2$, i.e., $\Lambda(M_2)$ given by

$$\ell \cdot {}^{\circ}M_1 \xrightarrow{\ \pi\ } \ell \cdot {}^{\circ}M_2 \ . \tag{3.4a}$$

The kernel of π is the set of elements $\ell \cdot {}^{\circ}M_1$, with $\ell \in {}^{\circ}M_2 \cap M_1$. The Iwasawa decomposition for M_2 with respect to M_1 asserts $M_2 = M_1NK$, where M_1N is a parabolic subgroup of M_2 with unipotent radical N, and K is an appropriate compact subgroup of M_2. Since the subgroups N and K are subgroups of ${}^{\circ}M_2$, we see the map π is surjective, so $M_2/{}^{\circ}M_2$ is canonically a quotient of $M_1/{}^{\circ}M_1$. In particular, we have canonical injection and quotient maps

$$\mathfrak{a}_{M_2} \hookrightarrow \mathfrak{a}_{M_1} \hookrightarrow \mathfrak{a}_{M_\emptyset} = \mathfrak{a}_{A_\emptyset} \quad \text{and} \quad \mathfrak{a}^*_{M_2} \leftarrow \mathfrak{a}^*_{M_1} \leftarrow \mathfrak{a}^*_{M_\emptyset} = \mathfrak{a}^*_{A_\emptyset} \ . \tag{3.4b}$$

If we fix a $W_G(A_\emptyset)$ invariant form $(\ ,\ )$ on $\mathfrak{a}_{A_\emptyset}$, then because every coset element of $W_G(A_{M_i})$ has a representative in $N_G(A_\emptyset)$, it follows the restriction of $(\ ,\ )$ to $\mathfrak{a}_{A_{M_i}}$ is $W_G(A_{M_i})$ invariant.

3.5. We now recall some additional facts about semi-standard Levi subgroups following [He]. With $A_\emptyset$ as in paragraph (3.1), let M be a semi-standard Levi F-subgroup. For $\alpha \in \Sigma_{\text{red}}(A_M)$, let M_α denote the smallest semi-standard Levi F-subgroup containing M and the root groups U_β, $\beta \in \Sigma(A_\emptyset)$ such that the restriction of β to A_M is α.

The semi-simple rank of the Levi M_α is one more than that of M. The map π of (3.4a) yields an exact sequence

$$1 \ \to \ \big({}^{\circ}M_\alpha \cap M\big) \,/\, {}^{\circ}M \ \to \ M \,/\, {}^{\circ}M \xrightarrow{\ \pi\ } M_\alpha \,/\, {}^{\circ}M_\alpha \ \to \ 1 \ . \tag{3.5a}$$

There is a unique generator $h_\alpha \cdot {}^{\circ}M \in M/\,{}^{\circ}M$ for the free rank one $\mathbb{Z}$-module $\ker(\pi)$ satisfying

$$\langle\, H_M(h_\alpha)\,,\, \alpha\,\rangle \ \ \text{is a positive integer}\,. \tag{3.5b}$$

The pairing is the $\mathfrak{a}_M \times \mathfrak{a}^*_M$ pairing of (3.3d). Observe, by the Cartan decomposition, the element h_α can be taken to be in $M_\emptyset$. If we replace the root α by its negative $-\alpha$, note that

$$h_{-\alpha} \cdot {}^{\circ}M \ = \ h_\alpha^{-1} \cdot {}^{\circ}M \ . \tag{3.5c}$$

If we apply $\mathrm{Hom}(-, \mathbb{C}^\times)$ to (3.5a), we obtain the exact sequence

$$1 \ \to \ \Psi(M_\alpha) \xrightarrow{\ \pi^*\ } \Psi(M) \ \longrightarrow \ \mathrm{Hom}\,\Big(\big({}^{\circ}M_\alpha \cap M\big) \,/\, {}^{\circ}M \,,\, \mathbb{C}^\times\Big) \ \to \ 1 \ . \tag{3.5d}$$

The character group $\Psi(M_\alpha)$ is precisely the set of unramified characters of M which are trivial on ${}^{\circ}M_\alpha \cap M$, equivalently trivial on $h_\alpha \cdot {}^{\circ}M$.

In the situation when $P = MN$ is a maximal semi-standard parabolic F-subgroup, then $\Sigma(A_M)$ consists of two elements $\pm\alpha$. The Weyl group $W_G(A_M)$, in this situation, is either trivial or of order two. For the latter, let $w_\alpha \in N_G(A_\emptyset)$ be a representative for the non-trivial element. Then, conjugation by w_α induces an involution on $\mathfrak{a}_M$ which is the identity on $\mathfrak{a}_G \subset \mathfrak{a}_M$. We conclude

$$(w_\alpha h_\alpha w_\alpha^{-1}) \cdot {}^{\circ}M \ = \ h_\alpha^{-1} \cdot {}^{\circ}M \ . \tag{3.5f}$$

Remark 3.5g. As an example, consider $G = \mathsf{GL}(2)(F)$. Let $d(a,b)$, denote the diagonal matrix with diagonal entries a and b, and set $A_\emptyset \ := \ \{\, d(a,b) \mid a,\, b \in F^\times \,\}$, the group of diagonal matrices. Denote the two roots of $A_\emptyset$ in G as $\Sigma(A_\emptyset) = \{\pm\alpha\}$. Then $\{\, h_\alpha, h_{-\alpha} \,\} = \{\, d(\varpi, \varpi^{-1})^{\pm 1} \,\}$.

4. The algebra of essentially compact distributions and its center

4.1. Suppose $G = \mathsf{G}(F)$ is an F-group. Recall the open compact subgroups of G form a fundamental system of open neighborhoods of the identity element $1_G \in G$. We assume G is unimodular, e.g., this is true when G is reductive. Let $\mathcal{C}^\infty_c(G)$ denote the vector space of complex-valued functions on G which are locally constant and have compact support. A distribution is, by definition, a linear functional $D : \mathcal{C}^\infty_c(G) \longrightarrow \mathbb{C}$. Let $\mathcal{C}^\infty_c(G)'$ denote the complex vector space of distributions. For $g \in G$, let λ_g, and ρ_g denote respectively the left and right translation action g on $\mathcal{C}^\infty_c(G)$. Similarly for $\mathcal{C}^\infty_c(G)'$.

If D is a distribution, and $g \in G$, define g not to be in the support of D if there exists an open neighborhood $\mathcal{V}$ of g such that $D(f) = 0$ for any $f \in \mathcal{C}_c^\infty(G)$ which vanishes outside $\mathcal{V}$. The set of elements $g \in G$ which do not belong to the support of D is obviously an open subset of G. Its complement, which we denote as support(D), or supp(D) is a closed subset of G. Define a distribution D to be compact if its support is a compact subset of G.

4.2. Suppose $\theta, f \in \mathcal{C}_c^\infty(G)$. The convolution product $\theta \star f \in \mathcal{C}_c^\infty(G)$ is defined as:

$$\theta \star f \ := \ x \longrightarrow \int_G \theta(g)\, f(g^{-1}x)\, dg \ . \tag{4.2a}$$

The distribution

$$D_\theta(f) \ := \ \int_G \theta(g)\, f(g)\, dg \tag{4.2b}$$

satisfies

$$\begin{aligned} D_\theta(f) \ &= \ \int_G \theta(g)\, f(g)\, dg \ = \ \int_G \theta(g)\, \check{f}(g^{-1})\, dg \ , \quad \text{where } \check{f}(g) \ := \ f(g^{-1}) \\ &= \ (\theta \star \check{f})\,(1) \ . \end{aligned} \tag{4.2c}$$

We deduce

$$(\theta \star f) \ = \ x \longrightarrow D_\theta(\lambda_x(\check{f})) \ . \tag{4.2d}$$

With (4.2d) as a model, for an arbitrary distribution D, and $f \in \mathcal{C}_c^\infty(G)$, we define the convolution of $D \star f$ to be the function $G \to \mathbb{C}$ given by

$$D \star f \ := \ x \longrightarrow D(\lambda_x(\check{f})) \ . \tag{4.2e}$$

Similarly, we define

$$f \star D \ := \ x \longrightarrow D(\rho_{x^{-1}}(\check{f})) \ . \tag{4.2f}$$

If D is G-invariant, i.e., $D(f) = D(\lambda_g \rho_g f)$ for all $g \in G$, then $D \star f = f \star D$. Both $D \star f$, and $f \star D$ are locally constant functions on G, but a-priori there is no reason they should be in $\mathcal{C}_c^\infty(G)$. A nice example of this is orbital integrals. Suppose $y \in G$. Let $\mathcal{O} := \mathcal{O}(y)$ denote the conjugacy class of y. Then, $\mathcal{O}$ is a manifold isomorphic to the homogeneous space $G/C_G(y)$, where $C_G(y)$ is the centralizer of y in G. Recall there is a G-invariant measure $d\mu_{\mathcal{O}}$ on $\mathcal{O}$, which is unique up to scalar. Then,

$$\mu_{\mathcal{O}}(f) \ := \ \int_{\mathcal{O}} f(g)\, d\mu_{\mathcal{O}}(g) \tag{4.2g}$$

is a G-invariant distribution. If 1_J is the characteristic function of an open compact subgroup J, then $\lambda_g \check{1_J}$ is the characteristic function of gJ, and

$$\int_{\mathcal{O}} 1_{gJ}\, d\mu_{\mathcal{O}} \ = \ \mu_{\mathcal{O}}(\, gJ \cap \mathcal{O}\,) \ . \tag{4.2h}$$

In particular, the function $\mu_{\mathcal{O}} \star 1_J$ is compactly supported if and only if $\mathcal{O}$ is a compact orbit. An elementary argument then says for arbitrary $f \in \mathcal{C}_c^\infty(G)$, the convolution $\mu_{\mathcal{O}} \star f$ is compactly supported if and only if $\mathcal{O}$ is a compact orbit. The conjugacy class of a central element $z \in G$, for which the associated G-invariant distribution is the delta function δ_z, is an example of such compact orbits.

4.3. If D is a compactly supported distribution and $f \in \mathcal{C}_c^\infty(G)$, it is elementary both $D \star f$ and $f \star D$ are in $\mathcal{C}_c^\infty(G)$.

Definition 4.3a. *A distribution D is essentially compact if for any $f \in \mathcal{C}_c^\infty(G)$, both $D \star f$ and $f \star D$ belong to $\mathcal{C}_c^\infty(G)$.*

Let $\mathcal{U}(G)$ denote the vector space of essentially compact distributions. If $D_1, D_2 \in \mathcal{U}(G)$ their convolution product $D_1 \star D_2$ is defined as follows: For any $f \in \mathcal{C}_c^\infty(G)$, the convolution product $D_2 \star (D_1 \star \check{f})$, by definition, lies in $\mathcal{C}_c^\infty(G)$. We set

$$(D_2 \star D_1)(f) \ := \ \big(D_2 \star (D_1 \star \check{f}) \big)\,(1)\ . \tag{4.3b}$$

Clearly, the convolution $(D_2 \star D_1)$ is an essentially compact distribution, and the convolution product makes $\mathcal{U}(G)$ a (convolution) algebra whose identity element is the delta function δ_{1_G} at the identity 1_G. Note that $\forall g \in G$, the delta distribution δ_g at g belongs to $\mathcal{U}(G)$. Furthermore, a choice of Haar measure on G determines an embedding of the vector space $\mathcal{C}_c^\infty(G)$ into $\mathcal{U}(G)$.

Since for any $g \in G$, the delta function δ_g at g belongs to $\mathcal{U}(G)$, we deduce that any $\mathcal{U}(G)$-module V is a representation of the group G. Smooth representations of G are precisely the $\mathcal{U}(G)$-modules V which are non-degenerate, i.e., for any $v \in V$ there exists an open compact subgroup J_v so that $\delta_k\, v = v$ for all $k \in J_v$.

The center of $\mathcal{U}(G)$ is the subspace:

$$\mathcal{Z}(G) \ := \ G\text{-invariant essentially compact distributions on } G\,. \tag{4.3c}$$

5. Plancherel densities and the Bernstein center

5.1. Suppose L is a connected reductive F-group. Let $\widetilde{L}$ denote the smooth dual of L. We adopt the logical imprecision, but short hand convenience of identifying an irreducible representation of L with its class. We recall that twisting a smooth irreducible representation σ of L by an unramified character $\psi \in \Psi(L)$ defines an action of $\Psi(L)$ on $\widetilde{L}$, and the $\Psi(L)$-orbits in $\widetilde{L}$. Recall a smooth representation of a reductive group is called essentially square integrable if it becomes square integrable representation modulo the center after a twist by a quasi-character. If a $\Psi(L)$-orbit of representations consists of cuspidal (resp. essentially square integrable) representations, we call the orbit a connected component of irreducible cuspidal (resp. essentially square integrable) representations of L. When G is a connected reductive F-group, connected components of irreducible cuspidal representations of Levi F-subgroups $L \subset G$, as well as connected components of essentially square integrable representations are key ingredients in the formulation of the Bernstein center and Plancherel formula respectively. If σ is smooth irreducible representation of L, let

$$\Omega(\sigma) \subset \widetilde{L} \quad \text{denote the } \Psi(L)\text{-orbit of } \sigma\ . \tag{5.1a}$$

We denote a typical $\Psi(L)$-orbit in $\widetilde{L}$ as Ω. We write $\widetilde{L} = \coprod \Omega$ for the partition of $\widetilde{L}$ into $\Psi(L)$-orbits.

In general, a connected component Ω may not contain any unitary representations. However, in the situations which matter for the formulation of the Bernstein center, and the Plancherel formula, the components do have unitary representation, and therefore, in all that follows, we assume Ω has unitary representations. To denote the subset of unitary representations in Ω, we adopt the notation Ω_u. Set

$$\Omega(L) \ := \ \coprod_{\Omega_u \neq \emptyset} \Omega \ \subset \ \widetilde{L} \ . \tag{5.1b}$$

By this same convention, we denote the subset of unitary characters in $\Psi(L)$ as $\Psi_u(L)$. In particular, $\Psi_u(L)$ acts transitively on Ω_u. It is elementary the stabilizer in $\Psi(L)$ of an representation $\sigma \in \Omega$ is both independent of σ, and is finite, and whence contained in $\Psi_u(L)$. Because of the independence, we use the notation

$$\mathrm{Stab}_\Psi(\Omega) \ := \ \{ \, \chi \in \Psi(L) \mid \chi\sigma \simeq \sigma \ \ \forall \, \sigma \in \Omega \, \} \tag{5.1c}$$

to denote the stabilizer. Then,

$$\mathcal{L}_L(\Omega) \ := \ \{ \, \nu \in (\mathfrak{a}_L^*)_{\mathbb{C}} \mid \chi_\nu \sigma \cong \sigma \, \} \tag{5.1d}$$

is a lattice which contains the lattice $\mathcal{L}_L$ (3.3i), and the quotient $\mathcal{L}_L(\Omega)/\mathcal{L}_L$ is naturally isomorphic to $\mathrm{Stab}_\Psi(\Omega)$. Obviously, if we fix $\sigma \in \Omega_u$, then the maps

$$\nu \ \to \ \chi_\nu \quad \text{and} \quad \chi \ \to \ \chi\sigma \tag{5.1e}$$

produces bijections among $(\mathfrak{a}_L^*)_{\mathbb{C}}/\mathcal{L}_L(\Omega)$, and $\Psi(L)/\mathrm{Stab}_\Psi(\Omega)$, and Ω, as well as between $\sqrt{-1}(\mathfrak{a}_L^*)/\mathcal{L}_L(\Omega)$, and $\Psi_u(L)/\mathrm{Stab}_\Psi(\Omega)$, and Ω_u. Additionally, and most importantly for the formulation of the Bernstein center, and the Plancherel formula, the complex variety, and the topological group structures on $\Psi(L)$ and $\Psi_u(L)$ respectively, determine corresponding structures on Ω and Ω_u. Thus, a function $f : \Omega \to \mathbb{C}$ is, by definition, (complex) regular, if the composed map of f and the quotient map from $\Psi(L)$ to Ω is (complex) regular as a function on $\Psi(L)$.

Suppose now L is a Levi F-subgroup of a reductive connected F-group G. The finiteness of $\mathrm{Stab}_\Psi(\Omega) \subset \Psi(L)$ means the set of values $\mathrm{Im}(\chi)$ of any character $\chi \in \mathrm{Stab}_\Psi(\Omega)$ is a finite, hence cyclic, subgroup of $\mathbb{C}^\times$. Therefore, with α and h_α as in (3.5b), the set

$$\{ \, \chi(h_\alpha) \, | \, \chi \in \mathrm{Stab}_\Psi(\Omega) \, \} \tag{5.1f}$$

is a finite, hence cyclic, subgroup of $\mathbb{C}^\times$, and so it is generated by some root of unity $e^{2\pi\sqrt{-1}/r}$, where

$$r = r(\Omega, \alpha) \tag{5.1g}$$

depends on Ω and α.

Lemma 5.1h. [He] Lemma 3.2.

$$\chi(h^r_\alpha) \;=\; 1 \iff \chi \in \text{subgroup of } \Psi(L) \text{ generated by } \mathrm{Stab}_\Psi(\Omega),\ \text{and } \{\psi|_L \mid \psi \in \Psi(L_\alpha)\}\ .$$

We conclude this section by introducing some notation for later use. Suppose L is a Levi F-subgroup of a connected reductive F-group G. For $w \in W_G(A_L)$, let $\widetilde{w} \in N_G(A_L)$ (normalizer in G of A_L) be a representative of w. If τ is a smooth irreducible representation of L, let $w \cdot \tau$ denote the usual action of w on σ, i.e., $m \to \tau(\widetilde{w}^{-1} m \widetilde{w})$. For Ω a $\Psi(L)$-orbit in $\widetilde{L}$, set

$$W_G(A_L, \Omega) \;:=\; \{\ w \in W_G(A_L) \mid w \cdot \sigma \in \Omega\ ,\ \forall\ \sigma \in \Omega\ \}\ . \tag{5.1i}$$

Lemma 5.1j. *The lattice $\mathcal{L}_L(\Omega)$ is $W_G(A_L,\Omega)$-invariant.*

Proof. Suppose $\nu \in \mathcal{L}_L(\Omega)$. For any $\sigma \in \Omega$, we have $\chi_\nu \sigma \simeq \sigma$. If $w \in W_G(A_L,\Omega)$, then $w \cdot (\chi_\nu \sigma) \simeq w \cdot \sigma$; hence, $w(\chi_\nu)\ (w \cdot \sigma) \simeq w \cdot \sigma$. We then easily deduce $w(\chi_\nu) \in \mathrm{Stab}_\Psi(\Omega)$, i.e., $\nu \in \mathcal{L}_L(\Omega)$. $\square$

5.2. Plancherel densities. Let $\mathcal{B}(G)$ denote the extended Bruhat-Tits building of G. Suppose, a choice of a maximal split F-torus $A_\emptyset$ has been made, i.e., we are in the setting of paragraph (3.1). Let $\mathcal{B}(A_\emptyset) \subset \mathcal{B}(G)$ denote the apartment associated to $A_\emptyset$. Let K be a parahoric subgroup associated to any special point x of $\mathcal{B}(G)$ which lies in $\mathcal{B}(A_\emptyset)$. Then, for any semi-standard parabolic F-subgroup, P, we have the decomposition $G = PK$.

Write $P = MN$ for the semi-standard decomposition of P. To define Plancherel densities, and to formulate, in the next section, the Plancherel formula, we respectively recall Harish-Chandra's γ and c-factors associated to the Levi F-subgroup M. If S is a closed subgroup of G and ds a Haar measure on S (left or right), then we shall always assume that it is normalized so the measure of $S \cap K$ is one. Let $d\bar{n}$ denote the Haar measure on $\bar{N}$. Use the decomposition $G = PK$ to extend the modular function δ_P of P to a function δ'_P on all G, i.e., $\delta'_P(pk) = \delta_P(p)$ for $p \in P$ and $k \in K$. Then, Harish-Chandra's γ-factor $\gamma(G|M)$ is defined as:

$$\gamma(G|M) = \int_{\bar{N}} \delta'_P(\bar{n})\, d\bar{n}\,. \tag{5.2a}$$

To define Harish-Chandra's c-factor $c(G|M)$, suppose $\alpha \in \Sigma_{\mathrm{red}}(P)$. Let M_α be as in paragraph (3.5), and define $\gamma(M_\alpha|M)$ as above, i.e., (5.2a). Then,

$$c(G|M) \;:=\; \gamma(G|M)^{-1} \prod_{\alpha \in \Sigma_{\mathrm{red}}(P)} \gamma(M_\alpha|M)\ . \tag{5.2b}$$

Define

$$\widetilde{\mathcal{E}}^2(M) \;:=\; \text{disjoint union of } \Psi(M)\text{-orbits } \Omega \subset \widetilde{M} \text{ of irreducible essentially square integrable representations.} \tag{5.2c}$$

Suppose Ω is a $\Psi(M)$-orbit of $\widetilde{\mathcal{E}}^2(M)$. For $\tau \in \Omega$, let $\mathrm{Ind}_P^G(\tau)$ denote the right translations representation parabolically induced from τ. For $w \in W_G(A_M)$, choose a representative $\widetilde{w} \in N_G(A_M)$ for w, and define $N_w := N \cap \widetilde{w}\overline{N}\widetilde{w}^{-1}$ to be the subgroup of N generated by the root groups U_α, $\alpha \in \Sigma_G$ satisfying $\widetilde{w}^{-1}U_\alpha\widetilde{w} \subset \overline{N}$. Recall, there is a non-empty open subset $\mathcal{V} \subset \Omega$ so that for all $\tau \in \mathcal{V}$, $g \in G$, and $f \in \mathrm{Ind}_P^G(\tau)$ the integral

$$A_w(\tau)(f)\,(g) \;:=\; \int_{N_w} f(\widetilde{w}^{-1}ng)\,dn \tag{5.2d}$$

converges, and continues regularly on a Zariski open subset of Ω. Then, the operator $A_w(\tau)$ intertwines the induced representation $\mathrm{Ind}_P^G(\tau)$ with $\mathrm{Ind}_P^G(w\cdot\tau)$, i.e.,

$$A_w(\tau) : \mathrm{Ind}_P^G(\tau) \longrightarrow \mathrm{Ind}_P^G(w\cdot\tau)\ . \tag{5.2e}$$

The parabolic subgroup P determines a length function on the Weyl group $W_G(A_M)$. Designate the longest element in $W_G(A_M)$, i.e., the one which conjugates N to $\overline{N}$ as w_P. Then, since $\mathrm{Ind}_P^G(w_P\cdot\tau)$ is irreducible for all τ in a Zariski open subset $\mathcal{X}(\Omega) \subset \Omega$, the composition operator

$$A_{w_P}(\tau) \circ A_{w_P^{-1}}(w_P\cdot\tau) : \mathrm{Ind}_P^G(w_P\cdot\tau) \longrightarrow \mathrm{Ind}_P^G(w_P\cdot\tau) \tag{5.2f}$$

is a scalar operator. The Plancherel density $\mu_{G|M,\Omega}$, on $\mathcal{X}(\Omega)$, is the function defined on the Zariski open subset $\mathcal{X}(\Omega)$ by the requirement

$$A_{w_P}(\tau) \circ A_{w_P^{-1}}(w_P\cdot\tau) \;=\; \mu_{G|M,\Omega}(\tau)^{-1}\ \gamma(G|M)^2\ . \tag{5.2g}$$

The disjoint union

$$\mathcal{X} \;:=\; \coprod_{\Omega\subset\widetilde{\mathcal{E}}^2(M)} \mathcal{X}(\Omega) \tag{5.2h}$$

is a Zariski open subset of $\widetilde{\mathcal{E}}^2(M)$. We patch together the densities $\mu_{G|M,\Omega}$ on the pieces $\mathcal{X}(\Omega)$ to obtain a Plancherel density $\mu_{G|M}$, on $\mathcal{X}$.

Theorem 5.2i. [W]. *The Plancherel density $\mu_{G|M}$ defined by (5.2g) satisfies the following properties:*

(1) *On each Ω, $\mu_{G|M,\Omega}$ extends from the non-empty Zariski open set $\mathcal{X}(\Omega)$ to a rational function on Ω.*

(2) *$\mu_{G|M,\Omega}$ is regular, i.e., has no poles, and is non-negative on Ω_u.*

(3) *As a rational function on $\widetilde{\mathcal{E}}^2(M)$: $\mu_{G|M}(w\cdot\tau) = \mu_{G|M}(\tau)$ for all $w \in W_G(A_M)$.*

(4) *$\mu_{G|M}(\tau^\vee) = \mu_{G|M}(\tau)$, where $\tau^\vee$ is the contragredient representation to τ.*

(5)

$$\mu_{G|M}(\tau) \;=\; \prod_{\alpha\in\Sigma_{\mathrm{red}}(P)} \mu_{M_\alpha|M}(\tau)\ . \tag{5.2j}$$

In particular, statement (5) reduces the calculation of arbitrary Plancherel densities to the situation of maximal parabolic subgroups.

We emphasize that $\mu_{G|M}$, and the $\mu_{M_\alpha|M}$'s are (canonically defined) functions on $\widetilde{\mathcal{E}}^2(M)$, which are rational functions on each connected component Ω. At times, by selection of a base point in Ω, we can view it as a quotient of $\Psi(M)$. Then $\mu_{G|M,\Omega}$, and $\mu_{M_\alpha|M,\Omega}$ become functions on the complex torus $\Psi(M)$.

Remark 5.2k. Suppose $w \in W_G(A_M)$, and $\tilde{w} \in G$ is a representative for w. The map $\mathrm{Ad}(\tilde{w})(x) := \tilde{w}^{-1}x\tilde{w} : M \longrightarrow M$, is independent of the representative $\tilde{w}$ and yields bijective maps

$$\Sigma_{\mathrm{red}}(A_M) \longrightarrow \Sigma_{\mathrm{red}}(A_M) \quad \text{and} \quad \Psi(M) \longrightarrow \Psi(M)\ , \tag{5.2l}$$

which, for brevity of notation, we also denote as w. We have $\tilde{w}M_\alpha\tilde{w}^{-1} = M_{w(\alpha)}$, and for any $\chi \in \Psi(M)$, and $\tau \in \Omega$, a $\Psi(M)$-orbit of $\widetilde{\mathcal{E}}^2(M)$,

$$\mu_{M_\alpha|M}(\chi\tau) = \mu_{M_{w(\alpha)}|M}(\, w(\chi)\ (\tilde{w} \cdot \tau)\,)\ . \tag{5.2m}$$

In particular, if $w \in W_G(A_M, \Omega)$, then the function $\chi \to \mu_{M_\alpha|M,\Omega}(\chi\tau)$ is non-constant if and only if the function $\chi \to \mu_{M_{w(\alpha)}|M,\Omega}(\chi\tau)$ is non-constant. $\mu_{M_\alpha|M,\Omega}$

5.3. Maximal parabolic subgroups.

Theorem 5.3a. [He] Proposition 4.1. *Suppose $P = MN$ is a maximal semi–standard parabolic F-subgroup of G, and $\Sigma_{\mathrm{red}}(P) = \{\alpha\}$, so G is M_α. Suppose further, that σ is an irreducible cuspidal unitary representation of M, and $\Omega = \Omega(\sigma)$, and Ω_u are respectively its connected component and connected unitary component.*

(i) *If $\mu_{G|M}$ is non-constant on Ω_u, then it has a zero in Ω_u.*

(ii) *When $\mu_{G|M}$ is non-constant on Ω_u, replace σ by a unitary twist so that $\mu_{G|M}(\sigma) = 0$. Let r be defined as in Lemma 5.1h. Define a finite order character $\eta_{\{M,\alpha\}}$ by [He:p9]*

$$\eta_{\{M,\alpha\}}(m) := e^{\pi\sqrt{-1}\,\langle H_M(m),\, \widetilde{\alpha}\rangle}\ , \tag{5.3b}$$

where $\widetilde{\alpha}$ is the fractional multiple of α so that $\langle H_M(h^r_\alpha),\, \widetilde{\alpha}\rangle = 1$. For $\chi \in \Psi(M)$, set

$$z_\alpha := \chi(h^r_\alpha)\ . \tag{5.3c}$$

Then:

(1) *If* $\mu_{G|M}(\eta_{\{M,\alpha\}} \cdot \sigma) \neq 0$, *then there exists* $c > 0$ *and* $k > 0$ *so that* $\mu_{G|M} = c\,\mu^{+}_{G|M}$, *where*

$$\mu^{+}_{G|M}(\chi\,\sigma) \;:=\; \frac{1 - z_\alpha}{1 - q^k z_\alpha}\;\frac{1 - z_\alpha^{-1}}{1 - q^k z_\alpha^{-1}}\;. \tag{5.3d}$$

(2) *If* $\mu_{G|M}(\eta_{\{M,\alpha\}} \cdot \sigma) = 0$, *then there exists* $c > 0$ *and* $k, \ell > 0$ *so that* $\mu_{G|M} = c\,\mu^{+}_{G|M}\,\mu^{-}_{G|M}$, *where* $\mu^{+}_{G|M}$ *is as in (5.3d), and*

$$\mu^{-}_{G|M}(\chi\,\sigma) \;:=\; \frac{1 + z_\alpha}{1 + q^\ell z_\alpha}\;\frac{1 + z_\alpha^{-1}}{1 + q^\ell z_\alpha^{-1}}\;. \tag{5.3e}$$

Theorem 5.3f. [He] Proposition 4.1, [Si] p. 576. *Suppose* $P = MN$ *is an arbitrary semi–standard parabolic* F*-subgroup of* G*, and* σ *is an irreducible cuspidal unitary representation of* M*. If the Plancherel density* $\mu_{G|M} : \Omega(\sigma) \to \mathbb{C}$ *is non constant, replace* σ *by a unitary twist so that* $\mu_{G|M}$ *vanishes at* σ*. Then:*

(1) *The set*

$$\{\, \alpha \in \Sigma_{\mathrm{red}}(A_M) \;\big|\; \mu_{M_\alpha|M}(\sigma) = 0 \,\} \tag{5.3g}$$

forms a root system.

(2) *For any* $\tau \in \Omega$*, if the induced representation* $\mathrm{Ind}_P^G(\tau)$ *contains an essentially square integrable subquotient representation, then* $\mu_{G|M}$ *has a pole at* τ*.*

(3) *In the situation when* $P = MN$ *is a maximal parabolic subgroup of* G*, then* $W_G(A_M, \Omega(\sigma)) = W_G(A_M)$*, has order two, and* σ *is a fixed point of the action of* $W_G(A_M)$ *on* $\Omega(\sigma)$*.*

Remarks 5.3h. Suppose $P = MN$ is a maximal semi-standard parabolic F-subgroup of G.

(1) Suppose that $\mu_{G|M}(\sigma) = 0$ for some $\sigma \in \widetilde{\Omega}_u$. By Theorem 5.3f, $W_G(A_M)$ has order two, say $W_G(A_M) = \{\, 1\,, w_\alpha \,\}$, and $w_\alpha(\sigma) = \sigma$. Consider the function $\chi \to \mu_{G|M}(\chi\sigma)$. Define $n_\alpha \in \{1, 2\}$ as follows:

$$n_\alpha = \begin{cases} 1 & \text{if } \; \mu_{G|M} = c\,\mu^{+}_{G|M} \\ 2 & \text{if } \; \mu_{G|M} = c\,\mu^{+}_{G|M}\mu^{-}_{G|M}\;. \end{cases} \tag{5.3i}$$

Then, Theorem 5.3a asserts that for $\chi \in \Psi(M)$,

$$\mu_{G|M}(\chi\sigma) = 0 \quad \text{if and only if} \quad \chi(h_\alpha^r)^{n_\alpha} = 1\;. \tag{5.3j}$$

(2) Suppose $\mu_{G|M}$, and σ are as in (1), and suppose $\mu_{G|M}(\chi\sigma) = 0$ for some $\chi \in \Psi(M)$. Conceivably, $\chi\sigma$ may not be unitary, in which case Theorem 5.3f cannot be used

directly to assert $w_\alpha(\chi\sigma) = \chi\sigma$. We shall show here that $w_\alpha(\chi\sigma) = \chi\sigma$. From remark (1), we know $\chi(h^{r_\alpha})^{n_\alpha} = 1$, thus $\chi(h^{r_\alpha}) = \pm 1$. Consider first the case when $\chi(h^{r_\alpha}) = 1$. Lemma 5.1h asserts $\chi = (\psi|_M)\phi$ for some $\phi \in \Psi(M)$ satisfying $\phi\sigma \cong \sigma$, and some $\psi \in \Psi(G)$. Obviously, w_α fixes ψ since ψ is trivial on the derived subgroup. Now $w_\alpha(\phi\sigma) \cong w_\alpha(\sigma) \cong \sigma$, and $\sigma \cong w_\alpha(\phi\sigma) \cong w_\alpha(\phi)\sigma$. Thus, $w_\alpha(\chi\sigma) = w_\alpha((\psi|_M)\phi)\sigma = (\psi|_M)\, w_\alpha(\phi)\sigma = (\psi|_M)\,(w_\alpha(\phi)\sigma) \cong (\psi|_M)\,\sigma \cong (\psi|_M)\,(\phi\sigma) = ((\psi|_M)\phi)\,\sigma = \chi\sigma$.

The remaining case to consider is when $n_\alpha = 2$, and $\chi(h^r_\alpha) = -1$. Here, the character $\eta_{\{M,\alpha\}}$ satisfies $\eta_{\{M,\alpha\}}(h^r_\alpha) = -1$ since $\langle H_M(h^r_\alpha)\,,\,\tilde{\alpha}\rangle = 1$. Now, by our assumption, $\mu_{G|M}(\eta_{\{M,\alpha\}}\sigma) = 0$. Furthermore: (i) $\eta_{\{M,\alpha\}}\sigma$ is unitary, so Theorem 5.3f applies and asserts $\eta_{\{M,\alpha\}}\sigma$ is fixed by w_α, and (ii) $\mu_{G|M}(\,(\chi\eta^{-1}_{\{M,\alpha\}})\,(\eta_{\{M,\alpha\}}\sigma)) = 0$. Since $\chi\,\eta^{-1}_{\{M,\alpha\}}\,(h^r_\alpha) = 1$, the first case implies again $w_\alpha(\chi\sigma) \cong \chi\sigma$.

(3) The subgroup $\mathcal{K} := \{\, \chi \in \Psi(M) \mid z_\alpha = \chi(h^r_\alpha) = 1 \,\}$ is a (complex) codimension one subgroup of $\Psi(M)$, and we have $\mu_{G|M}(\chi\sigma) = \mu_{G|M}(\sigma)$ for any $\chi \in \mathcal{K}$. Consider the function $\chi \to \mu_{|G|M}(\chi\sigma)$. If $\mu_{G|M}(\chi\sigma) = c\,\mu^+_{G|M}(\chi\sigma)$, then the zero set equals $\mathcal{K}$. If $\mu_{G|M}(\chi\sigma) = c\,\mu^+_{G|M}(\chi\sigma)\;\mu^-_{G|M}(\chi\sigma)$, then the zero set is the two $\mathcal{K}$-cosets $\{\chi \,|\, z_\alpha = 1\}$, and $\{\chi \,|\, z_\alpha = -1\}$.

(4) If the orbit Ω consists of one-dimensional representations, i.e., quasi-characters, then $r = 1$. In particular, if $\mu_{G|M}(\chi\sigma) = 0$ for some $\chi \in \Psi(M)$, and $\mu_{G|M}(\sigma) = 0$ as in (1), then $w_\alpha(\chi) = \chi$. Furthermore, the subgroup $\mathcal{K}$ of the previous remark is a connected subgroup of $\Psi(M)$, and the zero set of the Plancherel density $\chi \to \mu_{G|M}(\chi\sigma)$ has one or two connected components depending on whether $\mu_{G|M} = c\,\mu^+_{G|M}$ or $\mu_{G|M} = c\,\mu^+_{G|M}\;\mu^-_{G|M}$ respectively.

(5) In sections 6 and 7 we will focus on the situation when the cuspidal representation σ is one-dimensional. In this setting, we have $\mathrm{Stab}_\Psi(\Omega) = \{1\}$, and $r = r(\Omega,\sigma) = 1$, and $\eta_{\{M,\alpha\}}$ is a character which is trivial on ${}^\circ\!M$ and -1 at h_α. In the case when $\mathrm{GL}(2) = \mathsf{GL}(\mathbf{2})(F)$, and Ω is the unramified component, then $\tilde{\alpha} = \frac{1}{2}\alpha$, and $\eta_{\{M,\alpha\}}$ has order 4. The Weyl group acts on the unramified component, and the fixed point set is a connected one-dimensional variety. If instead, we consider $\mathrm{PGL}(2) = \mathsf{PGL}(\mathbf{2})(F) = \mathsf{SO}(\mathbf{2,1})(F) = \mathrm{SO}(2,1)$, then $\tilde{\alpha} = \alpha$ instead of $\frac{1}{2}\alpha$. In particular, the character $\eta_{\{M,\alpha\}}$ for PGL(2) is of order 2. The fixed point set of the Weyl action on the unramified component has 2 fixed points. Finally, if we consider $\mathrm{SL}(2) = \mathsf{SL}(\mathbf{2})(F)$, and again the unramified component, the character $\eta_{\{M,\alpha\}}$ for SL(2) is of order 2. It is the restriction of the character $\eta_{\{M,\alpha\}}$ for GL(2) to SL(2). The fixed point set of the Weyl group action on the unramified component has 2 fixed points.

5.4. Example for SU(2,1).

For the group $\mathbf{SU(2,1)}(F)$, we describe the Plancherel densities found in [JKM]. We reiterate our assumption that F is a p-adic field, i.e., $\mathrm{char}(F) = 0$. Let E be a quadratic extension of F, and for $x \in E$, let $\overline{x}$ denote the Galois conjugate of x. Define G to be the algebraic F-group $\mathbf{SU(2,1)}$ which preserves the form in three variables

$$x\,\overline{z} \;+\; y\,\overline{y} \;+\; z\,\overline{x} \quad . \tag{5.4a}$$

Let $G = \mathsf{G}(F)$ denote the group of F-rational points. For a, b, $c \in E^\times$, denote by $d(a,b,c)$, the diagonal matrix in $\mathbf{GL(3)}(E) \supset G$ with diagonal entries a, b, and c. Then,

$$A_\emptyset \ := \ \{ \, d(a,1,a^{-1}) \mid a \in F^\times \, \} \tag{5.4b}$$

is a maximal split F-torus in G, and its centralizer is the maximal F-torus

$$M_\emptyset := C_G(A_\emptyset) \ = \ \{ \, d(a,\overline{a}/a,1/\overline{a}) \mid a \in E^\times \, \} \ . \tag{5.4c}$$

An irreducible cuspidal unitary representation of $M_\emptyset$ is a unitary character; so, let λ denote a unitary character of $M_\emptyset$, and $\Omega(\lambda) = \Psi(M_\emptyset) \cdot \lambda$. Let ϖ_E denote a prime of E. An unramified character $\psi \in \Psi(M_\emptyset)$ is completely determined by it value at $d(\varpi_E, \overline{\varpi_E}/\varpi_E, 1/\overline{\varpi_E})$. Set

$$z \ = \ z(\psi) \ := \ \psi(\ d(\varpi_E, \overline{\varpi_E}/\varpi_E, 1/\overline{\varpi_E}) \ ) \ = \ \frac{1}{q^s} \tag{5.4d}$$

We describe the Plancherel density on $\Omega(\lambda)$. We consider two situations: E/F unramified and ramified, and the behavior of λ restricted to ${}^\circ M_\emptyset$. The later is naturally isomorphic to the units in the ring of integers $\mathcal{R}_E$ of E.

E/F unramified:
$\lambda\big|_{{}^\circ M_\emptyset} \ \equiv \ 1$:
The Plancherel density has two zeros on $\Omega(\lambda)$. If we select λ so that

$$\lambda(\, d(\varpi_E, \overline{\varpi_E}/\varpi_E, 1/\overline{\varpi_E}) \,) = 1 \ , \tag{5.4e}$$

then the two zeros occur at $\lambda\psi$ when $z(\psi) = \pm 1$. The density is:

$$\frac{(1-z)(1-z^{-1})}{(q^2-z)(q^2-z^{-1})} \cdot \frac{(1+z)(1+z^{-1})}{(q+z)(q+z^{-1})} \ , \tag{5.4f}$$

The double zeros at $z = 1$ and $z = -1$ means the principal series there are irreducible, and complementary series axes extend out from the unitary 'axis' at $z = 1$ as well as $z = -1$. Discrete series occur at the poles $z \in \{q^2, q^{-2}\}$, and $z \in \{-q, -q^{-1}\}$.

$\lambda\big|_{{}^\circ M_\emptyset} \ \not\equiv \ 1$, but $\lambda\big|_{{}^\circ A_\emptyset} \ \equiv \ 1$:
The Plancherel density has a single zero on $\Omega(\lambda)$. If we select λ so that

$$\lambda(\, d(\varpi_E, \overline{\varpi_E}/\varpi_E, 1/\overline{\varpi_E}) \,) = -1 \ , \tag{5.4g}$$

then the zero occurs at $\lambda\psi$ when $z(\psi) = 1$. The density is:

$$\frac{(1-z)(1-z^{-1})}{(q-z)(q-z^{-1})} \ , \tag{5.4h}$$

The double zero at $z = 1$ means the principal series there is irreducible and so gives rise to a complementary series. A discrete series representation occurs at the pole $z \in \{q, q^{-1}\}$.

$\lambda\big|_{{}^\circ A_\emptyset} \not\equiv 1$:
The Plancherel density is constant.

E/F ramified:
Let $\mathrm{sgn}_{E/F}$ denote the quadratic character of $F^\times$ corresponding, by class field theory, to E. We transfer this character to $A_\emptyset$ and denote it also by $\mathrm{sgn}_{E/F}$.

$\lambda\big|_{{}^\circ A_\emptyset} \equiv \mathrm{sgn}_{E/F}$:
The Plancherel density has two zeros on $\Omega(\lambda)$. If we select λ so that

$$\lambda(\, d(\varpi_E, \overline{\varpi_E}/\varpi_E, 1/\overline{\varpi_E})\,) = 1 \ , \tag{5.4i}$$

then the two zeros occur at $\lambda\psi$ when $z(\psi) = \pm 1$. The density is:

$$\frac{(1-z)(1-z^{-1})}{(q^{\frac{1}{2}}-z)(q^{\frac{1}{2}}-z^{-1})} \cdot \frac{(1+z)(1+z^{-1})}{(q^{\frac{1}{2}}+z)(q^{\frac{1}{2}}+z^{-1})} \ , \tag{5.4j}$$

The double zeros at $z = 1$ and $z = -1$ means the principal series there are irreducible, and complementary series axes extend out from the unitary 'axis' at $z = 1$ as well as $z = -1$. Discrete series occur at the poles $z \in \{q^{\frac{1}{2}}, q^{-\frac{1}{2}}\}$, and $z \in \{-q^{\frac{1}{2}}, -q^{-\frac{1}{2}}\}$.

$\lambda\big|_{{}^\circ M_\emptyset} \equiv 1$:
The Plancherel density has a single zero on $\Omega(\lambda)$. If we select λ so that

$$\lambda(\, d(\varpi_E, \overline{\varpi_E}/\varpi_E, 1/\overline{\varpi_E})\,) = 1 \ , \tag{5.4k}$$

then the zero occurs at $\lambda\psi$ when $z(\psi) = 1$. The density is:

$$\frac{(1-z)(1-z^{-1})}{(q-z)(q-z^{-1})}. \tag{5.4l}$$

The double zero at $z = 1$ gives rise to a complementary series, with a discrete series representation occurring at the pole $z \in \{q, q^{-1}\}$.

$\lambda\big|_{{}^\circ M_\emptyset} \not\equiv 1$, but $\lambda\big|_{{}^\circ A_\emptyset} \equiv 1$:
The Plancherel density is constant, and there are two points of reducibility in $\Omega_u(\lambda)$.

$\lambda\big|_{{}^\circ A_\emptyset} \not\equiv$ either 1, or $\mathrm{sgn}_{E/F}$:
The Plancherel density is constant, and there are no reducibility points in $\Omega_u(\lambda)$.

5.5. Plancherel formula.

Suppose M is a semi-standard Levi F-subgroup of G. Let $\mathcal{E}^2(M) \subset \widetilde{\mathcal{E}}^2(M)$ denote the set of classes of irreducible unitary essentially square integrable representations of M. As we previously remarked, for convenience, we permit the imprecision of identifying a representation with its class. If $\sigma \in \mathcal{E}^2(M)$, let $\Omega = \Omega(\sigma)$, and $\Omega_u = \Omega_u(\sigma)$ denote respectively, the $\Psi(M)$-orbit and $\Psi_u(M)$-orbit of σ. As already mentioned in section (5.1), Ω, and Ω_u are naturally isomorphic to $\Psi(M)/\mathrm{Stab}_\Psi(\Omega)$, and $\Psi_u(M)/\mathrm{Stab}_\Psi(\Omega)$ respectively. In particular, we can transfer the Haar measure on $\Psi_u(M)/\mathrm{Stab}_\Psi(\Omega)$ to Ω_u. Denote this measure at $\omega \in \Omega_u \subset \mathcal{E}^2(M)$ as $d\omega$.

Let $P = MN$ be a semi-standard parabolic F-subgroup containing M. For $\alpha \in \Sigma_{\mathrm{red}}(P)$, let $M_\alpha \supset M$ be as in section (3.5), and let $\gamma(G|M)$, and $c(G|M)$ be as in (5.2a) and (5.2b). Let

$$a(G|M) \ := \ \frac{1}{c(G|M)^2 \ \gamma(G|M) \ \#|W_M(A_\emptyset)|} \ . \tag{5.5a}$$

If $f \in C_c^\infty(G)$, and π is a smooth representation of finite length, set

$$\widehat{f}\,(\pi) \ := \ \mathrm{trace}\,(\pi(f)) \ . \tag{5.5b}$$

Then, Harish-Chandra's Plancherel formula [W] states for $f \in C_c^\infty(G)$, and more generally f in Harish-Chandra's Schwartz space, that

$$f(1) \ = \ \sum_M \ a(G|M) \int_{\mathcal{E}^2(M)} \widehat{f}\,(\mathrm{Ind}_P^G(\omega)) \ d_\omega \ \mu_{G|M}(\omega) \ d\omega \quad , \tag{5.5c}$$

where the sum runs over all semi-standard Levi subgroups up to conjugacy. We note that the induced representation $\mathrm{Ind}_P^G(\omega)$ is independent of the semi-standard parabolic $P = MN$ containing M.

Let $\widehat{G}$ denote the unitary dual of G. The Plancherel formula is usually stated as an integration over the unitary dual $\widehat{G}$ of the scalar Fourier transform $\widehat{f}(\pi)$ with respect to a measure μ_{PL} on $\widehat{G}$. The support of μ_{PL} is precisely the classes of irreducible tempered representations of G.

We shall now rewrite (5.5c). First, we write the topological space $\mathcal{E}^2(M)$ as a disjoint union of connected components, i.e., $\Psi_u(M)$-orbits,

$$\mathcal{E}^2(M) \ = \ \coprod \mathcal{O} \ . \tag{5.5d}$$

Each $\mathcal{O}$ is a $\Psi_u(M)$-orbit in $\mathcal{E}^2(M)$. The integral over $\mathcal{E}^2(M)$ equals, trivially, a sum of integrals over the $\Psi_u(M)$-orbits $\mathcal{O}$, i.e.,

$$f(1) \ = \ \sum_M \ \sum_{\mathcal{O} \subset \mathcal{E}^2(M)} a(G|M) \int_{\mathcal{O}} \ \widehat{f}\,(\mathrm{Ind}_P^G(\omega)) \ d_\omega \ \mu_{G|M}(\omega) \ d\omega \quad , \tag{5.5e}$$

Define

$$\mathrm{Orb}(M) \ := \quad \text{the set of } W_G(A_M)\text{-orbits of } \Psi_u(M)\text{-orbits in } \mathcal{E}^2(M). \tag{5.5f}$$

Suppose $\mathcal{O} \subset \mathcal{E}^2(M)$ is a connected component, and $w \in W_G(A_M)$. Since the Weyl action commutes with parabolic induction Ind_{MN}^G, we have

$$
\begin{aligned}
\int_{\mathcal{O}} \widehat{f}\,(\mathrm{Ind}_P^G(\omega))\ \mu_{G|M}(\omega)\ d\omega\ &=\ \int_{\mathcal{O}} \widehat{f}\,(\mathrm{Ind}_P^G(w\cdot\omega))\ \mu_{G|M}(\omega)\ d\omega \\
&=\ \int_{w\cdot\mathcal{O}} \widehat{f}\,(\mathrm{Ind}_P^G(\omega))\ \mu_{G|M}(\omega)\ d\omega\ .
\end{aligned} \tag{5.5g}
$$

Define $\mathcal{Q} := W_G(A_M)\cdot\mathcal{O} \in \mathrm{Orb}(M)$, and let $\#|W_G(A_M)\cdot\mathcal{O}|$ denote the number of connected components in $\mathcal{Q}$. Then,

$$
\int_{\mathcal{Q}} \widehat{f}\,(\mathrm{Ind}_P^G(\omega))\,\mu_{G|M}(\omega)\,d\omega\ =\ \#|W_G(A_M)\cdot\mathcal{O}| \int_{\mathcal{O}} \widehat{f}\,(\mathrm{Ind}_P^G(\omega))\,\mu_{G|M}(\omega)\,d\omega\ , \tag{5.5h}
$$

and so

$$
f(1)\ =\ \sum_{M}\ {\sum_{\mathcal{O}}}'\ a(\mathcal{O}) \int_{\mathcal{O}} \widehat{f}\,(\mathrm{Ind}_P^G(\omega))\ \mu_{G|M}(\omega)\ d\omega\ \ , \tag{5.5i}
$$

where the sum $\sum{}'$ is over representatives $\mathcal{O}$ of the $W_G(A_M)$-orbits of connect components of $\mathcal{E}^2(M)$, and $a(\mathcal{O}) = \#|W_G(A_M)\cdot\mathcal{O}|\ a(G|M)\ d_\omega > 0$.

5.6. The spectral realization of the Bernstein center.

Suppose M is a Levi F-subgroup and σ is a cuspidal representation of M. We refer to the pair (M,σ) as a cuspidal pair. Recall our notation has $\Omega(\sigma)$ denoting the connected component of twists of σ by the unramified characters $\Psi(M)$ of M. Since σ is cuspidal, the connected component $\Omega(\sigma)$ will contain an unramified twist of σ which is unitary. For convenience we can therefore assume σ is unitary. We shall do so in what follows. Then $\Omega_u(\sigma)$, the set of unitary representations of $\Omega(\sigma)$, equals the twists of σ by the unramified unitary characters $\Psi_u(M)$. Define two cuspidal pairs (M_1,σ_1), (M_2,σ_2) to be G-equivalent if they are in the same G-adjoint orbit. Let $[M,\sigma]_G$ denote the G-adjoint orbit of (M,σ), and following common notational convention, set $\Omega(\sigma) = \Omega(M,\sigma)$ as follows:

$$
\Omega(\sigma)\ :=\ \left\{\, [M,\psi\sigma]_G \,\middle|\, \psi\in\Psi(M)\,\right\} \tag{5.6a}
$$

The collection of G-adjoint orbits $\Omega(G)$ is a disjoint union of the $\Omega(\sigma)$'s. Since we are only interested in the G-adjoint orbits, we can assume M is a semi-standard Levi F-subgroup of G. Then $\Omega(\sigma)$ is a quotient $\Omega(\sigma) = \Psi(M)/\mathrm{Stab}_\Psi(\sigma)$ of $\Psi(M)$ by a subgroup $\mathrm{Stab}_\Psi(\sigma) \subset W_G(A_M)$. In particular, the quotient map $\Psi(M) \to \Omega(\sigma)$ allows one to define a complex algebraic variety structure on $\Omega(\sigma)$, and hence on $\Omega(G)$. The sets $\Omega(\sigma)$ are open and closed connected components in $\Omega(G)$.

The cuspidal support of a smooth irreducible representation $\pi \in \widetilde{G}$, i.e.,

$$\begin{aligned} \pi \ \rightarrow\ & G\text{-adjoint orbit of a cuspidal pair } (M,\sigma) \text{ such that} \\ & \pi \text{ is isomorphic to a subquotient of } \mathrm{Ind}_{MN}^{G}(\sigma)\ , \end{aligned} \tag{5.6b}$$

is a surjection $\Pi : \widetilde{G} \longrightarrow \Omega(G)$.

Let $\mathfrak{Z}(G)$ denote the algebra of regular functions on $\Omega(G)$. Recall that the complex vector space $\mathcal{U}(G)$ of essentially compact distributions on G acts naturally on any smooth representation of G, and the action of the center

$$\mathcal{Z}(G) \ :=\ \text{subalgebra of } G\text{-invariant essentially compact distributions on } G \tag{5.6c}$$

commutes with the representation action. If τ is a smooth irreducible representation of a Levi F-subgroup M, and $D \in \mathcal{Z}(G)$, then D acts as a scalar on $\mathrm{Ind}_{MN}^{G}(\tau)$. We denote this scalar by $\hat{D}(M,\tau)$. If (M_1,τ_1) and (M_2,τ_2) belong to the same G-adjoint orbit then $\hat{D}(M_1,\tau_1) = \hat{D}(M_2,\tau_2)$. Although defined for any smooth irreducible representation τ, the situation when τ is an essentially square integrable representation is most relevant for the Bernstein center. In particular, $(M,\tau) \rightarrow \hat{D}(M,\tau)$ canonically becomes a regular function on $\Omega(G)$. The fundamental result of Bernstein related to $\mathcal{Z}(G)$ and $\mathfrak{Z}(G)$ is that $D \rightarrow \hat{D}$ is an isomorphism of $\mathcal{Z}(G)$ onto $\mathfrak{Z}(G)$. From [BD], we have the following inversion formula: In the notation of (5.5i),

$$D(f) \ =\ \sum_{M} \ {\sum_{\mathcal{O}}}' \ a(\mathcal{O}) \int_{\mathcal{O}} \ \widehat{f}\,(\mathrm{Ind}_{P}^{G}(\omega))\ \hat{D}(M,\omega)\ \mu_{G|M}(\omega)\ d\omega\ . \tag{5.6d}$$

6. Some results on root systems of quasi-split groups

6.1. In this section we establish some results about root systems when $G = \mathsf{G}(F)$ is the group of F-rational points of a reductive quasi-split F-group G. We retain the notation convention of section 3, so $A_\emptyset$ is the F-rational points of maximal F-split torus of G, $\Sigma(A_\emptyset)$ is the roots of $A_\emptyset$, $\Sigma_{\mathrm{red}}(A_\emptyset)$ is the reduced roots, $M_\emptyset$ is the centralizer of $A_\emptyset$, etc.

For $\alpha \in \Sigma_{\mathrm{red}}(A_\emptyset)$, let w_α be the associated reflection in the Weyl group $W_G(A_\emptyset)$, and define groups and subgroups

$$\begin{aligned} \Lambda \ &:=\ \Lambda(M_\emptyset) \\ \Lambda^{w_\alpha,+} \ &:=\ \{\, v \in \Lambda \mid s_\alpha v = v \,\} \\ \Lambda^{w_\alpha,-} \ &:=\ \{\, v \in \Lambda \mid s_\alpha v = -v \,\}\ . \end{aligned} \tag{6.1a}$$

We shall write these groups additively. The intersection of $\Lambda^{w_\alpha,+}$ and $\Lambda^{w_\alpha,-}$ is the zero subgroup. So, we can consider their direct sum

$$\Lambda^{\alpha} := \Lambda^{w_\alpha,+} + \Lambda^{w_\alpha,-} \tag{6.1b}$$

inside Λ. For any $v \in \Lambda$, we have $w_\alpha v + v \in \Lambda^{w_\alpha,+}$, $-w_\alpha v + v \in \Lambda^{w_\alpha,-}$, and $2v = (w_\alpha v + v) + (-w_\alpha v + v)$. It follows $2v \in \Lambda^\alpha$. In particular, Λ^α is finite index in Λ. Furthermore, every element of the quotient Λ/Λ^α has order 2.

The $\mathbb{Z}$-rank of $\Lambda^{w_\alpha,-}$ is 1. Up to additive inverse, it has an unique generator d_α. We can use the standard pairing (3.3d) to select d_α so that

$$\langle\, d_\alpha\,,\,\alpha\,\rangle \;:=\; \alpha(d_\alpha) \;>\; 0\;. \tag{6.1c}$$

Lemma 6.1d. $d_\alpha \;=\; h_\alpha{}^\circ M_\emptyset$.

Proof. As in the notation of section 3.5, let M_α denote smallest parabolic subgroup containing $M_\emptyset$ and the root groups U_α. The element $h_\alpha{}^\circ M_\emptyset$ is a generator of the kernel of the map $\pi : M_\emptyset/{}^\circ M_\emptyset \to M_\alpha/{}^\circ M_\alpha$. But the kernel is precisely the elements in $\Lambda^{w_\alpha,-}$. So, $d_\alpha \;=\; \pm h_\alpha{}^\circ M_\emptyset$. That d_α equals $h_\alpha{}^\circ M_\emptyset$ is because both have positive pairing against the root α. $\square$

Proposition 6.1e.

(i) The index $[\Lambda : \Lambda^\alpha]$ is either 1 or 2.

(ii) The linear map $\Lambda \to \Lambda$ given by $t \to w_\alpha(t) - t$ has d_α in its image if and only if $[\Lambda : \Lambda^\alpha] = 2$.

Proof.

Part (i). Suppose $[\Lambda : \Lambda^\alpha] > 2$. Select distinct cosets $v_1 + \Lambda^\alpha$, $v_2 + \Lambda^\alpha$, $v_3 + \Lambda^\alpha$ of Λ/Λ^α. Set

$$\frac{1}{2}\Lambda^{w_\alpha,-} \;:=\; \{\,\frac{v}{2}\mid v \in \Lambda^{w_\alpha,-}\,\}\;, \tag{6.1f}$$

so $P(v) := \frac{v - w_\alpha(v)}{2}$ defines a projection map $\Lambda \longrightarrow \frac{1}{2}\Lambda^{w_\alpha,-}$. Note that P takes any $v \in \Lambda^{w_\alpha,-}$ to itself. A-priori, the map P may not be onto. Since $[\frac{1}{2}\Lambda^{w_\alpha,-} : \Lambda^{w_\alpha,-}] = 2$, at least two of $P(v_1)$, $P(v_2)$, $P(v_3)$ differ by an element of $\Lambda^{w_\alpha,-}$. We can assume $P(v_1 - v_2) \in \Lambda^{w_\alpha,-}$. Set $Q(v) := \frac{v + w_\alpha(v)}{2}$, so that

$$Q(v_1 - v_2) \;=\; (v_1 - v_2) \;-\; P(v_1 - v_2)\;. \tag{6.1g}$$

Therefore, $Q(v_1 - v_2) \in \Lambda$, and clearly it is w_α symmetric; so, $Q(v_1 - v_2) \in \Lambda^{w_\alpha,+}$, and whence $(v_1 - v_2) \;=\; Q(v_1 - v_2) \,+\, P(v_1 - v_2) \;\in \Lambda^\alpha$, i.e., $v_1 + \Lambda^\alpha = v_2 + \Lambda^\alpha$. This is a contradiction. Thus, $[\Lambda : \Lambda^\alpha]$ is 1 or 2.

To prove part (ii), we consider the two cases:

Case $\Lambda = \Lambda^\alpha$. Here, we have a direct sum $\Lambda = \Lambda^{w_\alpha,+} \oplus \Lambda^{w_\alpha,-}$, so any $t \in \Lambda$ has a unique decomposition $t = x + y$, with $x \in \Lambda^{w_\alpha,+}$, and $y \in \Lambda^{w_\alpha,-}$. Then, $w_\alpha(t) = x - y$, and so $w_\alpha(t) - t = 2y$. Obviously, $2y$ can not be the generator d_α.

Case $[\Lambda : \Lambda^\alpha] = 2$. The crucial fact is the lattice $\Lambda^{w_\alpha,-}$ is rank one. Choose $x \in \Lambda$ so that the element $w_\alpha(x) - x \in \Lambda^{w_\alpha,-}$ generates a sublattice of Λ of minimal index. Since the generator d_α satisfies $w_\alpha(d_\alpha) - d_\alpha = -d_\alpha - d_\alpha = -2d_\alpha$, the minimal index must be either one or two. If the index is two, for all $v \in \Lambda$, we have $w_\alpha(v) - v \in \mathbb{Z}(-2d_\alpha)$. Then,

$$w_\alpha(v) - v = k(-2d_\alpha) = w_\alpha(kd_\alpha) - (kd_\alpha)\ , \tag{6.1h}$$

so $w_\alpha(v - kd_\alpha) = v - kd_\alpha$, i.e., $v - kd_\alpha \in \Lambda^{w_\alpha,+}$. Thus, $v = (v - kd_\alpha) + kd_\alpha \in \Lambda^{w_\alpha,+} + \Lambda^{w_\alpha,-}$ for all $v \in \Lambda$, which contradicts $\Lambda \neq \Lambda^\alpha$. □

Set

$$e_\alpha \ := \begin{cases} 2 & \text{when } \ [\Lambda : \Lambda^\alpha] = 1\ , \\ 1 & \text{when } \ [\Lambda : \Lambda^\alpha] = 2\ . \end{cases} \tag{6.1i}$$

The integer e_α is very useful in characterizing the unramified quasi-characters $\psi \in \Psi(M_\emptyset)$ fixed by w_α. Recall that such characters are trivial on ${}^oM$, and so we view them as quasi-characters of Λ.

Corollary 6.1j. *An unramified quasi-character $\psi \in \Psi(M_\emptyset)$ is fixed by w_α precisely when $\psi(d_\alpha)^{e_\alpha} = 1$.*

Proof. The condition that ψ is fixed by w_α is evidently the following: $\forall\ t \in \Lambda$, we have $\psi(w_\alpha(t)) = \psi(t)$, i.e., $\psi(w_\alpha(t) - t) = 1$. By Proposition 6.1e, the image of the map $t \to w_\alpha(t) - t$ is $\mathbb{Z}\, e_\alpha d_\alpha$, and this immediately implies the assertion. □

6.2. Suppose $\mathcal{O}$ is a $\Psi(M_\emptyset)$-orbit, i.e., connected component, in the set of quasi-characters $\widetilde{M_\emptyset}$ of $M_\emptyset$. Recall that $w \in W_\emptyset$ stabilizes $\mathcal{O}$, if $w(\mathcal{O}) = \mathcal{O}$, i.e., by definition, if for any $\chi \in \mathcal{O}$, there exists $\psi \in \Psi(M_\emptyset)$ so that the function $t \to (w\cdot\chi)(t) := \chi(w^{-1}\cdot t)$ on $M_\emptyset$ equals the function $t \to \psi(t)\chi(t)$.

Proposition 6.2a. *Suppose $G = \mathsf{G}(F)$ is group of F-rational points of a connected quasi-split group G defined over F. Let $A_\emptyset$ be a maximal split F-torus of G, and let σ be a cuspidal representation of $M_\emptyset = C_G(A_\emptyset)$. In particular, σ is one-dimensional. Suppose $\alpha \in \Sigma_{\rm red}(A_\emptyset)$, and M_α is the smallest Levi F-subgroup containing $M_\emptyset$ and the root group U_α. Suppose Plancherel density $\psi \to \mu_{M_\alpha|M_\emptyset}(\psi\sigma)$ vanishes at $\psi \equiv 1$. If the density has the form $\mu_{M_\alpha|M_\emptyset} = c\,\mu^+_{M_\alpha|M_\emptyset}\mu^-_{M_\alpha|M_\emptyset}$ of Theorem 5.3a Part (ii.2), then $[\Lambda : \Lambda^\alpha] = 1$, i.e., $e_\alpha = 2$.*

Proof. The assumption that the density vanishes at $\psi \equiv 1$ means, by Theorem 5.3f, that σ is w_α-invariant. By Theorem 5.3a Part (ii.2), if $\mu_{M_\alpha|M_\emptyset} = c\,\mu^+_{M_\alpha|M_\emptyset}\mu^-_{M_\alpha|M_\emptyset}$ then $\psi = \eta_{M_\emptyset,\alpha}$ is also a zero of the density. Again, by Theorem 5.3f, $\eta_{M_\emptyset,\alpha}\,\sigma$ must be w_α-invariant, and therefore $\eta_{M_\emptyset,\alpha}$ must be w_α-invariant, i.e., $\forall h \in M_\emptyset$, we have $\eta_{M_\emptyset,\alpha}(\,w_\alpha(h)\,h^{-1}\,) = 1$. From the definition of $\eta_{M_\emptyset,\alpha}$ in (5.3b) we see the condition becomes the character $h \to e^{\pi\sqrt{-1}\,\langle\, H_{M_\emptyset}(w_\alpha(h)h^{-1})\,,\,\tilde{\alpha}\,\rangle}$ is trivial. This is true exactly when $H_{M_\emptyset}(w_\alpha(h)h^{-1})$ is always an even multiple of d_α. By Proposition 6.1e, this happens precisely when $[\Lambda : \Lambda^\alpha] = 1$. □

7. Orbital integrals

In the remaining two sections 7 and 8 we assume G is a connected reductive quasi-split F-group. Fix a maximal split F-torus $A_\emptyset$ as in section (3.1), and let $M_\emptyset$ denote the centralizer of $A_\emptyset$. Designate the group of unitary characters of $M_\emptyset$ as $\widehat{M_\emptyset}$. The group $\widetilde{M_\emptyset}$ of quasi-characters of $M_\emptyset$ equals $\Psi(M_\emptyset) \cdot \widehat{M_\emptyset}$.

7.1. Suppose $P = M_\emptyset N_\emptyset$ is a semi-standard parabolic with the semi-standard Levi subgroup $M_\emptyset$. We use the abbreviation $W_\emptyset$ for $W_G(A_\emptyset)$. For $\chi \in \widehat{M_\emptyset}$, recall the well known elementary formula for the character of the induced principal series representation $\pi_\chi := \mathrm{Ind}_P^G(\chi)$. Set

$$\Theta_{\pi_\chi}(g) \ := \begin{cases} \dfrac{\sum\limits_{w \in W_\emptyset} (w \cdot \chi)(t)}{D(t)^{1/2}} & \text{if } g \text{ is conjugate to } t \in M_\emptyset \text{ , and} \\[2ex] 0 & \text{otherwise .} \end{cases} \tag{7.1a}$$

The denominator term D is the Weyl denominator. For $f \in C_c^\infty(G)$, set

$$\pi_\chi(f) \ := \ \int_G f(g)\, \pi_\chi(g)\, dg \ . \tag{7.1b}$$

Then, the distributional trace $f \ \rightarrow \mathrm{trace}(\pi_\chi(f))$ has the expression

$$\mathrm{trace}(\pi_\chi(f)) \ = \ \int_G f(g)\, \Theta_{\pi_\chi}(g)\, dg \ . \tag{7.1c}$$

For $t \in M_\emptyset$, recall the normalized orbital integral of the conjugacy class $\mathrm{Ad}(G)(t)$ is defined as

$$F_f^{M_\emptyset}(t) \ = \ D(t)^{1/2} \int_{G/M_\emptyset} f(hth^{-1}) \ dh \ . \tag{7.1d}$$

Then, by the Weyl integration formula, we have the following formula for $\mathrm{tr}(\pi(f))$:

$$\begin{aligned} \mathrm{tr}(\pi(f)) \ &= \int_G f(g)\, \Theta_\pi(g)\, dg \\ &= \int_{M_\emptyset} \frac{1}{\#|W_\emptyset|} \ D(t) \, \Big(\int_{G/M_\emptyset} f(hth^{-1})\, \Theta_\pi(hth^{-1}) \ dh \Big) \ dt \\ &= \int_{M_\emptyset} \frac{1}{\#|W_\emptyset|} \ D(t)^{1/2} \, \Big(\ D(t)^{1/2} \int_{G/M_\emptyset} f(hth^{-1}) \ dh \ \Big) \ \Theta_\pi(t) \ dt \\ &= \int_{M_\emptyset} \frac{1}{\#|W_\emptyset|} \ D(t)^{1/2} \ F_f^{M_\emptyset}(t) \ \Theta_\pi(t) \ dt \ . \end{aligned} \tag{7.1e}$$

We shall invert this to get a spectral formula for $F_f^{M_\emptyset}(t)$. Observe that

$$\begin{aligned} \text{tr}(\pi(f)) &= \int_{M_\emptyset} \frac{1}{\#|W_\emptyset|}\ D(t)^{1/2}\ F_f^{M_\emptyset}(t)\ \Big(D(t)^{-1/2} \sum_{w\in W} (w\cdot\chi)(t)\Big)\ dt \\ &= \int_{M_\emptyset} \frac{1}{\#|W_\emptyset|}\ F_f^{M_\emptyset}(t)\ \sum_{w\in W} (w\cdot\chi)(t)\ dt\ . \end{aligned} \tag{7.1f}$$

Set

$$B(w,f,\chi) \ = \ \int_{M_\emptyset} F_f^{M_\emptyset}(t)\ \ (w\cdot\chi)(t)\ dt\ . \tag{7.1g}$$

Since $F_f^{M_\emptyset}(t) = F_f^{M_\emptyset}(\,w\cdot t\,)$, it follows that $B(w,f,\chi)$ is independent of w. Thus,

$$\Theta_{\pi_\chi}(f) \ = \ \int_{M_\emptyset} F_f^{M_\emptyset}(t)\ \chi(t)\ dt\ . \tag{7.1h}$$

We invert the latter by abelian Fourier inversion to get

$$F_f^{M_\emptyset}(t) \ = \ \int_{\widehat{M_\emptyset}} \Theta_{\pi_\chi}(f)\ \chi^{-1}(t)\ \ d\chi \tag{7.1i}$$

7.2. Definition of the distribution $S^{M_\emptyset}(\gamma_0,\gamma)$.

Suppose γ_0, $\gamma\in M_\emptyset$ is such that $\gamma_0\,(w\cdot\gamma)$ is regular for every $w\in W_G(A_\emptyset)$, i.e., if $w'\in W$, and $w'(\gamma_0\,w(\gamma)) = \gamma_0\,w(\gamma)$, then $w'=1$. Define a distribution $S^{M_\emptyset}(\gamma_0,\gamma)$ as follows:

$$\forall f\in C_c^\infty(G)\ ,\quad f\mapsto S^{M_\emptyset}(\gamma_0,\gamma)(f)\ :=\ \sum_{w\in W_G(A_\emptyset)} \text{sgn}(w)\ \ F_f^{M_\emptyset}(\,\gamma_0\ w\cdot\gamma\,)\ \ . \tag{7.2a}$$

Our goal is to show this G-invariant distribution belongs to the Bernstein center. We formulate later results on the subspace of the Bernstein center spanned by these distributions. It is clear we can use (7.1i) to obtain a spectral formula for $S^{M_\emptyset}(\gamma_0,\gamma)$. For γ, $\gamma_0\in M_\emptyset$, and $\chi\in\widetilde{M_\emptyset}$, set

$$\text{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\ :=\ \sum_{w\in W_G(A_\emptyset)} \text{sgn}(w)\ \chi^{-1}(\,\gamma_0\ w\cdot\gamma\,)\ . \tag{7.2b}$$

Then

$$S^{M_\emptyset}(\gamma_0,\gamma)(f)\ :=\ \int_{\widehat{M_\emptyset}} \Theta_{\pi_\chi}(f)\ \ \text{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\ d\chi\ . \tag{7.2c}$$

7.3. Criteria for $S^{M_\emptyset}(\gamma_0,\gamma)$ to lie in the Bernstein center.

For any $f \in C_c^\infty(G)$, we decompose the integral (7.2c) into a discrete sum of integrals over the connected components of $\widehat{M_\emptyset}$ and then collect together the components which are in the same $W_G(A_{M_\emptyset})$-Weyl orbit. For brevity, set $W_\emptyset := W_G(A_{M_\emptyset})$. Now,

$$\begin{aligned} S^{M_\emptyset}(\gamma_0,\gamma)(f) \ &= \ \int_{\widehat{M_\emptyset}} \Theta_{\pi_\chi}(f) \ \ \mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi \\ &= \ \sum_{\mathcal{O}_u\ \subset\ \widehat{M_\emptyset}} \ \int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f) \ \ \mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi \qquad (7.3a) \\ &= \ \sum_{\mathcal{O}_u}{}' \int_{W_\emptyset\cdot\mathcal{O}_u} \Theta_{\pi_\chi}(f) \ \ \mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi\ . \end{aligned}$$

The sum $\sum'$ is as in (5.5i). For any $w \in W_\emptyset$, we have

(7.3b)
$$\begin{aligned} \int_{w\cdot\mathcal{O}_u} \Theta_{\pi_\chi}(f)\ \mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\,\chi \ &= \ \int_{\mathcal{O}_u} \Theta_{\pi_{w\cdot\chi}}(f) \ \ \mathrm{skew}(w\cdot\chi\,,\,\gamma_0\,,\,\gamma)\; d\,(w\cdot\chi) \\ &= \ \int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f) \ \ \mathrm{skew}(w\cdot\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi\ . \end{aligned}$$

For each connected component $\mathcal{O}_u$, and in (7.3b), let $\mathcal{O}$ denote the connected component of $\widetilde{M_\emptyset}$ which contains $\mathcal{O}_u$ as its unitary points. Clearly, $\mathrm{Stab}(\mathcal{O}_u) := W_G(A_\emptyset, \mathcal{O}_u)$, equals $\mathrm{Stab}(\mathcal{O}) := W_G(A_\emptyset, \mathcal{O})$. We deduce

(7.3c)
$$\int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f) \ \ \mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi \ = \frac{1}{\#|\mathrm{Stab}(\mathcal{O})|} \int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f) \sum_{v\in\mathrm{Stab}(\mathcal{O})} \mathrm{skew}(v\cdot\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi$$

For $\chi \in \mathcal{O}_u$, define

$$\mathrm{sym}_{\,\mathcal{O}_u}\,\mathrm{skew}\,(\chi\,,\,\gamma_0\,,\,\gamma) \ := \ \sum_{v\in\mathrm{Stab}(\mathcal{O})} \mathrm{skew}(v\cdot\chi\,,\,\gamma_0\,,\,\gamma)\ . \qquad (7.3d)$$

We conclude from (7.3a) and (7.3b)

(7.3e)

$$S^{M_\emptyset}(\gamma_0,\gamma)(f) \;=\; \sum_{\mathcal{O}_u \,\subset\, \widehat{M_\emptyset}} \int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f)\;\; \mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi$$

$$= \sum_{\mathcal{O}_u}{}' \sum_{\mathcal{Q}_u\subset W_\emptyset\cdot\mathcal{O}_u} \int_{\mathcal{Q}_u} \Theta_{\pi_\chi}(f)\;\; \mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi$$

$$= \sum_{\mathcal{O}_u}{}' \sum_{\mathcal{Q}_u\subset W_\emptyset\cdot\mathcal{O}_u} \frac{1}{\#|\mathrm{Stab}(\mathcal{O})|} \int_{\mathcal{Q}_u} \Theta_{\pi_\chi}(f)\;\; \mathrm{sym}_{\,\mathcal{Q}_u}\,\mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi$$

$$= \sum_{\mathcal{O}_u}{}' \sum_{w\in W_\emptyset/\mathrm{Stab}(\mathcal{O})} \frac{1}{\#|\mathrm{Stab}(\mathcal{O})|} \int_{w\cdot\mathcal{O}_u} \Theta_{\pi_\chi}(f)\;\; \mathrm{sym}_{\,w\cdot\mathcal{O}_u}\,\mathrm{skew}(\chi\,,\,\gamma_0\,,\,\gamma)\; d\chi$$

$$= \sum_{\mathcal{O}_u}{}' \sum_{w\in W_\emptyset/\mathrm{Stab}(\mathcal{O})} \frac{1}{\#|\mathrm{Stab}(\mathcal{O})|} \int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f)\; \mathrm{sym}_{\,w\cdot\mathcal{O}_u}\,\mathrm{skew}(w\cdot\chi\,,\,\gamma_0\,,\,\gamma)\; d(w\cdot\chi)$$

$$= \sum_{\mathcal{O}_u}{}' \frac{1}{\#|\mathrm{Stab}(\mathcal{O})|} \int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f)\,\cdot \\ \Bigg(\sum_{w\in W_\emptyset/\mathrm{Stab}(\mathcal{O})} \frac{\mathrm{sym}_{\,w\cdot\mathcal{O}_u}\,\mathrm{skew}(w\cdot\chi\,,\,\gamma_0\,,\,\gamma)}{\mu_{G|M_\emptyset}(w\cdot\chi)} \Bigg)\;\cdot\;\mu_{G|M_\emptyset}(w\cdot\chi)\; d(w\cdot\chi)$$

$$= \sum_{\mathcal{O}_u}{}' \frac{1}{\#|\mathrm{Stab}(\mathcal{O})|} \int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f)\,\cdot \\ \Bigg(\sum_{w\in W_\emptyset/\mathrm{Stab}(\mathcal{O})} \frac{\mathrm{sym}_{\,w\cdot\mathcal{O}_u}\,\mathrm{skew}(w\cdot\chi\,,\,\gamma_0\,,\,\gamma)}{\mu_{G|M_\emptyset}(w\cdot\chi)} \Bigg)\;\cdot\;\mu_{G|M_\emptyset}(\chi)\;\; d\chi$$

Further manipulation gives

(7.3f)

$$S^{M_\emptyset}(\gamma_0,\gamma)(f) \;=\; \sum_{\mathcal{O}_u}{}' \frac{1}{\#|\mathrm{Stab}(\mathcal{O})|} \int_{\mathcal{O}_u} \Theta_{\pi_\chi}(f)\,\cdot \\ \Bigg(\sum_{w\in W_\emptyset/\mathrm{Stab}(\mathcal{O})} \mathrm{sgn}(w)\; \frac{\mathrm{sym}_{\,\mathcal{O}_u}\,\mathrm{skew}(\chi\,,\,w\cdot\gamma_0\,,\,\gamma)}{\mu_{G|M_\emptyset}(\chi)} \Bigg)\;\cdot\;\mu_{G|M_\emptyset}(\chi)\;\; d\chi$$

For $\mathcal{Q}_u = w\cdot\mathcal{O}_u$, write $\mathcal{Q} \supset \mathcal{Q}_u$ for the $\Psi(M_\emptyset)$-orbit containing $\mathcal{Q}_u$ as its unitary subset. Define a function $\mathcal{Q} \to \mathbb{C}$

(7.3g)
$$\chi \mapsto \mathrm{sym}_{\,\mathcal{Q}}\,\mathrm{skew}(\chi, \gamma_0, \gamma)$$

as in (7.3d). The function $\chi \mapsto \mathrm{sym}_{\mathcal{Q}}\, \mathrm{skew}(\chi, \gamma_0, \gamma)$ is the regular extension to $\mathcal{Q}$ of the function $\chi \mapsto \mathrm{sym}_{\mathcal{Q}_u}\, \mathrm{skew}(\chi, \gamma_0, \gamma)$ on $\mathcal{Q}_u$.

We remark that we have defined the distribution $S^{M_\emptyset}(\gamma_0, \gamma)(-)$ under the hypothesis $\gamma_0\, w(\gamma)$ is regular for all $w \in W_\emptyset$. The right hand side of (7.3f) provides a canonical way to define distributions $S^{M_\emptyset}(\gamma_0, \gamma)(-)$ for all $\gamma_0, \gamma \in M_\emptyset$. We assume this in all that follows.

The formula (7.3e) for the distribution $S^{M_\emptyset}(\gamma_0, \gamma)(-)$ is easily compared with the formula (5.6d) for the Fourier transform of a distribution. We draw two conclusions. Firstly, we see that the spectral support of the Fourier transform of $S^{M_\emptyset}(\gamma_0, \gamma)(-)$ is contained in the tempered representations which are constituents of unitary principal series. Secondly, by §5.6 and (7.3f), we have the following.

Criteria 7.3h. *The distribution $S^{M_\emptyset}(\gamma_0, \gamma)(-)$ belongs to the Bernstein center if and only if the following two conditions hold:*

(i) Regularity. For each component $\mathcal{Q}_u \subseteq \widehat{M_\emptyset}$, the rational function on $\mathcal{Q}$ given by

$$\chi \longrightarrow \frac{\mathrm{sym}_{\mathcal{Q}}\, \mathrm{skew}(\chi\, ,\, \gamma_0\, ,\, \gamma)}{\mu_{G|M_\emptyset}(\chi)}\ , \tag{7.3i}$$

is a regular function.

(ii) Vanishing. Suppose π is an irreducible tempered representation of G which occurs in a non-unitary principal series $\mathrm{Ind}_{P_\emptyset}^G(\chi_0)$, $\chi_0 \in \mathcal{Q} \subset \widetilde{M_\emptyset}$. Then, the regular function (7.3i) vanishes at χ_0.

7.3j. Remark. Recall that if an irreducible tempered representation π of G occurs in a non-unitary principal series, then π is a subrepresentation of a parabolically induced representation $\mathrm{Ind}_{MU}^G(\tau)$, where τ is an irreducible square integrable representation of a Levi subgroup M which strictly contains $M_\emptyset$.

7.4. For the remainder of this manuscript, we shall be primarily concerned with the Plancherel densities associated to full principal series. For facility of notation, it is convenient for us to abbreviate some notation. For $\alpha \in \Sigma_{\mathrm{red}}(P_\emptyset)$, set

$$M_\alpha = (M_\emptyset)_\alpha\ , \quad \mu = \mu_{G|M_\emptyset}\ , \quad \text{and} \quad \mu_\alpha = \mu_{M_\alpha|M_\emptyset}\ . \tag{7.4a}$$

We have

$$\mu \ = \prod_{\alpha \in \Sigma_{\mathrm{red}}(P_\emptyset)} \mu_\alpha. \tag{7.4b}$$

Viewed as a function on $\widetilde{M_\emptyset}$, μ satisfies the transformation rule

$$\mu(\chi) = \mu(w \cdot \chi)\ , \quad \text{for all } \ w \in W_\emptyset := W_G(A_\emptyset)\ . \tag{7.4c}$$

The factor μ_α, satisfies $\mu_\alpha(\chi) = \mu_\alpha(w_\alpha \cdot \chi)$, where w_α is the non-trivial element of the Weyl group $W_{M_\alpha}(A_\emptyset)$. Furthermore,

$$\mu_\alpha(\chi) = \mu_\alpha(\chi^{-1})\,, \quad \text{and thus} \quad \mu(\chi) = \mu(\chi^{-1}) \quad \text{as well.} \tag{7.4d}$$

Express μ_α as

$$\mu_\alpha = \frac{p_\alpha}{q_\alpha}\ , \tag{7.4e}$$

where p_α, and q_α are regular functions on $\widetilde{M_\emptyset}$.

Each connected component $\mathcal{Q}$ of $\widetilde{M_\emptyset}$ is, as a complex variety, isomorphic to a complex torus, i.e., $(\mathbb{C}^\times)^{\dim(\mathcal{Q})}$. The ring of regular functions $\mathcal{R}[\mathcal{Q}]$ on $\mathcal{Q}$ is thus, the localization of a polynomial ring in $\dim(\mathcal{Q})$ variables, and so is also a unique factorization domain (UFD). We can, of course, restrict the functions in (7.4e) to the connected component $\mathcal{Q}$ as well as its Weyl orbit $W_\emptyset \cdot \mathcal{Q}$. By the transformation formula (7.4c), the functions on $W_\emptyset \cdot \mathcal{Q}$ are completely determined from their restrictions to the any of the connected components of the Weyl orbit, e.g., $\mathcal{Q}$. When we wish to emphasize the restriction, we shall write $\mu_{\alpha,\mathcal{Q}}$, $p_{\alpha,\mathcal{Q}}$, and $q_{\alpha,\mathcal{Q}}$, but for brevity of notation, we shall usually suppressed this dependence. We shall now fix an arbitrary connected component $\mathcal{Q}$, and establish the criteria of (7.3h).

We observe

Lemma 7.4f.
(i) For any $\beta \in \Sigma_{\mathrm{red}}(P_\emptyset)\backslash\{\alpha\}$, the functions $p_{\beta,\mathcal{Q}}$, and $q_{\beta,\mathcal{Q}}$ are relatively prime, in $\mathcal{R}[\mathcal{Q}]$, to both $p_{\alpha,\mathcal{Q}}$, and $q_{\alpha,\mathcal{Q}}$.
(ii) The factorization of $q_{\alpha,\mathcal{Q}}$ is square free.
(iii) If $p_{\alpha,\mathcal{Q}}$ is divisible by an irreducible element r, then, in fact, $p_\alpha, \mathcal{Q}$ is divisible by the square r^2, but not by the cube r^3.

Proof. In the setting of Theorem 5.3a, for $\chi \in \Psi(M_\emptyset)$, set $z_\alpha := \chi(h_\alpha)$, and $z_\beta := \chi(h_\beta)$ (note r_α and r_β are both 1), and consider the (prime) linear polynomials $1 - z_\alpha$, and $1 - z_\beta$. The real vectors $H_{M_\emptyset}(h_\alpha)$, $H_{M_\emptyset}(h_\beta) \in \mathfrak{a}_{M_\emptyset}$ are linearly independent, and therefore the zero sets $V(1-z_\alpha) := \{\, \chi \in \Psi(M_\emptyset) \,|\, 1-z_\alpha = 0\,\}$, and $V(1-z_\beta) := \{\, \chi \in \Psi(M_\emptyset) \,|\, 1 - z_\beta = 0\,\}$ are incomparable, i.e., neither is contained in the other, and so $1 - z_\alpha$, and $1 - z_\beta$ are relatively prime. Similarly $1 - z_\alpha$ is relatively prime to $1 + z_\beta$, as well as $1 - q^k z_\beta$, and $1 + q^\ell z_\beta$, for $k, \ell > 0$. We thus deduce (i). Parts (ii), and (iii) are obvious from Theorem 5.3a. □

Now, for all $\alpha \in \Sigma_{\mathrm{red}}(P_\emptyset)$, both $p_{\alpha,\mathcal{Q}}$, and $q_{\alpha,\mathcal{Q}}$ are regular functions on $\widetilde{M_\emptyset}$, and

$$\frac{1}{\mu_{\mathcal{Q}}(\chi)} \;=\; \prod_{\alpha\in\Sigma_{\mathrm{red}}(P_\emptyset)} \frac{q_{\alpha,\mathcal{Q}}(\chi)}{p_{\alpha,\mathcal{Q}}(\chi)}\ . \tag{7.4g}$$

The truth of the two criteria of (7.3h) are consequences of the following:

Proposition 7.4h.

(i) *Regularity. For any $\alpha \in \Sigma_{\mathrm{red}}(P_\emptyset)$, the function $\chi \mapsto \mathrm{sym}_{\mathcal{Q}}\,\mathrm{skew}(\chi,\gamma_0,\gamma)$ of (7.3i) is divisible by $\chi \mapsto p_{\alpha,\mathcal{Q}}(\chi)$. In particular,*

$$\frac{\mathrm{sym}_{\mathcal{Q}}\,\mathrm{skew}(\chi,\gamma_0,\gamma)}{\mu(\chi)} \;=\; \Big(\frac{\mathrm{sym}_{\mathcal{Q}}\,\mathrm{skew}(\chi,\gamma_0,\gamma)}{\prod\limits_{\alpha\in\Sigma_{\mathrm{red}}(P_\emptyset)} p_{\alpha,\mathcal{Q}}(\chi)}\Big)\Big(\prod_{\alpha\in\Sigma_{\mathrm{red}}(P_\emptyset)} q_{\alpha,\mathcal{Q}}(\chi)\Big)\ , \tag{7.4i}$$

and the first factor on the right hand side is regular.

(ii) *Vanishing. Suppose π is an irreducible tempered representation of G which occurs in a non-unitary principal series $\mathrm{Ind}_{P_\emptyset}^G(\chi_0)$, $\chi_0 \in \mathcal{Q} \subset \widetilde{M_\emptyset}$. Then, there exists $\alpha \in \Sigma_{\mathrm{red}}(P_\emptyset)$ such that*

$$q_{\alpha,\mathcal{Q}}(\chi_0) \;=\; 0\ . \tag{7.4j}$$

7.5. Proof of the vanishing statement of Proposition 7.4h.

Proof. Suppose $\chi \in \mathcal{Q} \subset \widetilde{M_\emptyset}$, is a non-unitary quasi-character, such that $\mathrm{Ind}_{P_\emptyset}^G(\chi)$ contains a tempered subquotient π. Then, there exists a standard parabolic subgroup $P = LN \supsetneq P_\emptyset$, and an irreducible square integrable representation τ of L such that π is isomorphic to a subrepresentation of $\mathrm{Ind}_P^G(\tau)$. Since the cuspidal support of π is the minimal parabolic subgroup $P_\emptyset$, the same is true for τ; thus, there exists a quasi-character $\chi' \in \widetilde{M_\emptyset}$ so that τ appears as a subquotient of $\mathrm{Ind}_{P_\emptyset\cap L}^L(\chi')$. That π is a subquotient of both $\mathrm{Ind}_{P_\emptyset}^G(\chi)$, and $\mathrm{Ind}_{P_\emptyset}^G(\chi')$ means $\chi = w\cdot\chi'$ for some $w \in W_\emptyset$. We can replace χ' by a Weyl conjugate $w\cdot\chi'$ which lies in the same connected component $\mathcal{Q}$ as χ, and then replace L by $w\cdot L$. This replacement allows us to assume χ and χ' are both in $\mathcal{Q}$.

The hypothesis that τ is a square integrable representation of L, and a subquotient of $\mathrm{Ind}_{P_\emptyset\cap L}^L(\chi')$ means, by Theorem 5.3f, that the Plancherel density $\mu_{L|M_\emptyset}(\lambda)$ has a pole at $\lambda = \chi'$. Then, by the product formula

$$\mu_{L|M_\emptyset}(\lambda) \;=\; \prod_{\alpha\in\Sigma_{\mathrm{red}}(P_\emptyset\cap L)} \mu_{M_\alpha|M_\emptyset}(\lambda)\ , \tag{7.5a}$$

of Theorem 5.2g, and the expressions in Theorem 5.3a for the factors $\mu_{M_\alpha|M_\emptyset}(\lambda) = \frac{p_\alpha(\lambda)}{q_\alpha(\lambda)}$, it follows $q_\beta(\chi') = 0$ for some $\beta \in \Sigma_{\mathrm{red}}(P_\emptyset\cap L)$. Furthermore, for $\alpha \in \Sigma_{\mathrm{red}}(P_\emptyset\cap L)$, the factors $\mu_\alpha = \mu_{M_\alpha|M_\emptyset}$ is the same in G as in L; thus $\mu_{L|M_\emptyset}$ is a subproduct of

$$\mu_{G|M_\emptyset}(\lambda) \;=\; \prod_{\alpha\in\Sigma_{\mathrm{red}}P_\emptyset} \mu_{M_\alpha|M_\emptyset}(\lambda)\ . \tag{7.5b}$$

The vanishing condition of is then a consequence of the regular functions $\mathcal{R}[\mathcal{Q}]$ on $\mathcal{Q}$ being a unique factorization domain, and part (i) of Lemma 7.4f. $\square$

7.6. Factorization of $\operatorname{sym}_{\mathcal{Q}} \operatorname{skew}(\chi, \gamma_0, \gamma)$**.**

Here, we obtain a factorization of $\operatorname{sym}_{\mathcal{Q}} \operatorname{skew}(\chi, \gamma_0, \gamma)$. We have

$$
\begin{aligned}
\operatorname{sym}_{\mathcal{Q}} \operatorname{skew}(\chi\,,\,\gamma_0\,,\,\gamma) \;&:=\; \sum_{v \in \operatorname{Stab}(\mathcal{Q})} \operatorname{skew}(v\cdot\chi\,,\,\gamma_0\,,\,\gamma) \\
&= \sum_{v \in \operatorname{Stab}(\mathcal{Q})} \sum_{w \in W_\emptyset} \operatorname{sgn}(w)\; (v\cdot\chi)^{-1}\,(\gamma_0\, w(\gamma)) \\
&= \sum_{v \in \operatorname{Stab}(\mathcal{Q})} \sum_{w \in W_\emptyset} \operatorname{sgn}(w)\; \chi^{-1}\,(v^{-1}(\gamma_0\, w(\gamma))) \\
&= \sum_{v \in \operatorname{Stab}(\mathcal{Q})} \sum_{w \in W_\emptyset} \operatorname{sgn}(w)\; \chi^{-1}\,(\,v^{-1}(\gamma_0)\; v^{-1}(w(\gamma))\,) \\
&= \sum_{v \in \operatorname{Stab}(\mathcal{Q})} \sum_{w \in W_\emptyset} \operatorname{sgn}(v\,w)\; \chi^{-1}\,(\,v^{-1}(\gamma_0)\; w(\gamma)\,) \\
&= \Big(\sum_{v \in \operatorname{Stab}(\mathcal{Q})} \operatorname{sgn}(v)\; \chi^{-1}\,(\,v(\gamma_0)\,)\Big)\,\Big(\sum_{w \in W_\emptyset} \operatorname{sgn}(w)\; \chi^{-1}\,(\,w(\gamma)\,)\Big)\;.
\end{aligned}
\tag{7.6a}
$$

Define

$$
\begin{aligned}
\operatorname{skew}_{\mathcal{Q}}(\chi, \gamma_0) \;&:=\; \sum_{v \in \operatorname{Stab}(\mathcal{Q})} \operatorname{sgn}(v)\; \chi^{-1}\,(\,v(\gamma_0)\,) \\
\operatorname{skew}(\chi, \gamma) \;&:=\; \sum_{w \in W_\emptyset} \operatorname{sgn}(w)\; \chi^{-1}\,(\,w(\gamma)\,)\;,
\end{aligned}
\tag{7.6b}
$$

so $\operatorname{sym}_{\mathcal{Q}} \operatorname{skew}(\chi\,,\,\gamma_0\,,\,\gamma) = \operatorname{skew}_{\mathcal{Q}}(\chi, \gamma_0) \cdot \operatorname{skew}(\chi, \gamma)$. It is clear the function $(\chi, \gamma_0) \to \operatorname{skew}_{\mathcal{Q}}(\chi, \gamma_0)$ is $\operatorname{Stab}(\mathcal{Q})$-skew in each of the variables, and the function $(\chi, \gamma) \to \operatorname{skew}(\chi, \gamma)$ is $W_\emptyset$-skew.

7.7. Preliminaries to the regularity statement of Proposition 7.4h.

To establish the regularity condition of Proposition 7.4h, we shall show the existence of a function $\mathcal{D}_{\mathcal{Q}}(-) : \mathcal{Q} \to \mathbb{C}^\times$ analogous to the square of the Weyl denominator. To define $\mathcal{D}_{\mathcal{Q}}$ we review a decomposition of the group $\operatorname{Stab}(\mathcal{Q})$ as a semi-direct product $\operatorname{Stab}(\mathcal{Q}) = R(\mathcal{Q}) \ltimes W(\mathcal{Q})$ of an R-group $R(\mathcal{Q})$, and a reflection group $W(\mathcal{Q})$. Set

$$
\Sigma(\mathcal{Q}) \;:=\; \{\; \alpha \in \Sigma_{\mathrm{red}}(A_\emptyset) \;\big|\; \mu_{\alpha,\mathcal{Q}} \text{ is non-constant } \;\}\;. \tag{7.7a}
$$

For $w \in W_\emptyset$, we have $\mu_{\alpha,\mathcal{Q}}(\chi) = \mu_{w(\alpha),w\cdot\mathcal{Q}}(w\cdot\chi)$ for all $\chi \in \mathcal{Q}$; so, $\Sigma(\mathcal{Q})$ is a $\operatorname{Stab}(\mathcal{Q})$-invariant set of roots. Furthermore, by Theorem 5.3a, we have $w_\alpha \in \operatorname{Stab}(\mathcal{Q})$ for all $\alpha \in \Sigma(\mathcal{Q})$. It follows $\Sigma(\mathcal{Q})$ is a root system. Set

(7.7b)

$$
\begin{aligned}
\Sigma_+(\mathcal{Q}) \;&:=\; \Sigma(\mathcal{Q}) \,\cap\, \Sigma_+\;, \\
W(\mathcal{Q}) \;&:=\; \text{reflection subgroup of } \operatorname{Stab}(\mathcal{Q}) \text{ generated by all } w_\alpha,\ \alpha \in \Sigma(\mathcal{Q})\;, \\
R(\mathcal{Q}) \;&:=\; \big\{\, w \in \operatorname{Stab}(\mathcal{Q}) \mid w(\Sigma_+(\mathcal{Q})) \,=\, \Sigma_+(\mathcal{Q}) \,\big\}\;, \\
C(\Sigma_+(\mathcal{Q})) \;&:=\; \{\, v \in \operatorname{span}(\Sigma_+(\mathcal{Q})) \mid \langle v, \alpha\rangle > 0 \;\;\forall\; \alpha \in \Sigma_+(\mathcal{Q}) \,\}\;, \\
\overline{C}(\mathcal{Q}) \;&:=\; \{\, v \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle v, \alpha\rangle \geq 0 \;\;\forall\; \alpha \in \Sigma_+(\mathcal{Q}) \,\}\;.
\end{aligned}
$$

We note the set $\overline{C}(\mathcal{Q})$ is the sum of the closure of $C(\Sigma_+(\mathcal{Q}))$ and the real span of the elements in Λ perpendicular to $\Sigma_+(\mathcal{Q})$.

Lemma 7.7c. $\mathrm{Stab}(\mathcal{Q}) = R(\mathcal{Q}) \ltimes W(\mathcal{Q})$.

Proof. Proposition 16 in section 1 of chapter VI (page 168) of [Bour]. $\square$

Note that up to a non-zero constant, $\prod\limits_{\alpha\in\Sigma_{\mathrm{red}}(P_\emptyset)} p_{\alpha,\mathcal{Q}}(\chi) \;=\; \prod\limits_{\alpha\in\Sigma_+(\mathcal{Q})} p_{\alpha,\mathcal{Q}}(\chi)$

The function $\mathcal{D}_{\mathcal{Q}}$ we define will satisfy the following three conditions:

$$\text{(7.7d)}\qquad \begin{array}{rl} \text{(i)} & \mathcal{D}_{\mathcal{Q}}(w\cdot\chi) = \mathcal{D}_{\mathcal{Q}}(\chi) \quad \forall\, w \in W(\mathcal{Q}). \\ \text{(ii)} & \text{In } \mathcal{R}[\mathcal{Q}], \text{ the function } \mathcal{D}_{\mathcal{Q}}(-) \text{ divides the function } \mathrm{sym}_{\,\mathcal{Q}}\,\mathrm{skew}(-). \\ \text{(iii)} & \text{In } \mathcal{R}[\mathcal{Q}], \text{ the function } \chi \to \prod\limits_{\alpha\in\Sigma_{\mathrm{red}}(P_\emptyset)} p_{\alpha,\mathcal{Q}}(\chi) \text{ divides the function } \mathcal{D}_{\mathcal{Q}}(-). \end{array}$$

Properties (ii) and (iii) clearly imply the regularity statement of Proposition 7.4h.

Recall the notation of section 6. For $v \in \Lambda := \Lambda(M_\emptyset)$, set

$$\text{(7.7e)}\qquad \begin{aligned} f_v : \Psi(M_\emptyset) &\to \mathbb{C} \\ \psi &\to \psi(v) \end{aligned}$$

It is elementary, a unit of the ring $\mathcal{R}[\Psi(M_\emptyset)]$ is the non-zero multiple of a function f_v, $v \in \Lambda$.

The following results gives divisibility properties of $\mathrm{Stab}(\mathcal{Q})$-skew and $W_\emptyset$-skew functions on $\Psi(M_\emptyset)$. Observe that for any $\alpha \in \Sigma_{\mathrm{red}}(P_\emptyset)$, the function $\psi \to f_{d_\alpha}(\psi) - f_{-d_\alpha}(\psi)$ is w_α-skew.

Lemma 7.7f. *Define both $d_\alpha \in \Lambda$, and $e_\alpha \in \{1,2\}$ as in section 6.1.*

(i) If $e_\alpha = 2$, then any w_α-skew function on $\Psi(M_\emptyset)$ is divisible by $f_{d_\alpha} - f_{-d_\alpha} = -f_{-d_\alpha}(1 - f^2_{d_\alpha})$.

(ii) If $e_\alpha = 1$, then there exists $v \in \Lambda$ so that $f_v(1 - f_{d_\alpha})$ is is w_α-skew on $\Psi(M_\emptyset)$, and any w_α-skew function on $\Psi(M_\emptyset)$ is divisible by $f_v(1 - f_{d_\alpha})$.

In particular, in all cases, any w_α-skew function in $\mathcal{R}[\Psi(M_\emptyset)]$ is divisible by $(1 - f^{e_\alpha}_{d_\alpha})$.

Proof. We begin with some general observations before turning to the proofs of (i) and (ii). Suppose $f \in \mathcal{R}[\Psi(M_\emptyset)]$ is w_α-skew. This means $f(w_\alpha(\psi)) = -f(\psi)$ for all $\psi \in \Psi(M_\emptyset)$. In particular, if $\psi \in \Psi(M_\emptyset)$ is fixed by w_α, then obviously $f(\psi) = 0$. The condition $w_\alpha(\psi) = \psi$ means for all $v \in \Lambda$ that $\psi(w_\alpha(v)) = \psi(v)$, i.e.,

$$\text{(7.7g)}\qquad \psi(v - w_\alpha(v)) \;=\; 1 \quad \forall\, v \in \Lambda\ .$$

To prove (i), recall that when $e_\alpha = 2$, then $\Lambda = \Lambda^{w_\alpha,+} \oplus \Lambda^{w_\alpha,-}$. Write an arbitrary $v \in \Lambda$ as $v = x + y$ with $x \in \Lambda^{w_\alpha,+}$, and $y \in \Lambda^{w_\alpha,-}$. Then, $w_\alpha(v) = x - y$, and so condition (7.7g) becomes $\psi(y^2) \;=\; 1 \quad \forall\, y \in \Lambda^{w_\alpha,-}$. Taking $y = d_\alpha$, we see any w_α-fixed ψ must satisfy $\psi(d^2_\alpha) = 1$. Conversely if $\psi(d^2_\alpha) = 1$, then $\psi(y^2) = 1$ for all

$y \in \Lambda^{w_\alpha,-}$, and ψ is w_α-fixed. Thus, any w_α-skew regular function $f \in \mathcal{R}[\Psi(M_\emptyset)]$ must vanish on the set of quasi-characters $\{\, \psi \in \Psi(M_\emptyset) \,|\, \psi(d_\alpha) = \pm 1 \,\}$. This is the subvariety

$$\{\, \psi \in \Psi(M_\emptyset) \mid (1 - f_{d_\alpha}(\psi))\,(1 + f_{d_\alpha}(\psi)) \;=\; 0 \,\} \;. \tag{7.7h}$$

The two functions $(1 - f_{d_\alpha})$, $(1 + f_{d_\alpha}) \in \mathcal{R}[\Psi(M_\emptyset)]$ are non-associate prime elements. It follows the principal ideal of $(1-f_{d_\alpha})(1+f_{d_\alpha})\mathcal{R}[\Psi(M_\emptyset)]$ is its own radical, and therefore by Hilbert's Nullstensatz $(1 - f_{d_\alpha})(1 + f_{d_\alpha})$ divides any w_α-skew function in $\mathcal{R}[\Psi(M_\emptyset)]$.

To prove (ii), we first determine what conditions on $\ell \in \Lambda$ insure the function $(1 - f_{d_\alpha})\, f_\ell$ is w_α-skew. We have

$$\begin{aligned} (1 - f_{d_\alpha})\, f_\ell \xrightarrow{w_\alpha} (1 - f_{d_\alpha}^{-1})\, f_{w_\alpha(\ell)} \;&=\; -f_{d_\alpha}^{-1}(-f_{d_\alpha} + 1)\, f_{w_\alpha(\ell)} \\ &=\; -(1 - f_{d_\alpha})\, f_\ell\, f_{w_\alpha(\ell) - \ell - d_\alpha} \end{aligned} \tag{7.7i}$$

We see $(1 - f_{d_\alpha})\, f_\ell$ is w_α-skew precisely when ℓ satisfies $w_\alpha(\ell) - \ell - d_\alpha = 0$, i.e., $d_\alpha = w_\alpha(\ell) - \ell$. By part (ii) of Proposition 6.1e, such an ℓ exists precisely $[\Lambda : \Lambda^\alpha] = 2$, i.e., when $e_\alpha = 1$. This proves the first statement of (ii). To prove the second statement, suppose $f \in \mathcal{R}[\Psi(M_\emptyset)]$ is w_α-skew. By analogous reasoning as when $e_\alpha = 2$, we get $f(\psi) = 0$ for any $\psi \in \Psi(M_\emptyset)$ fixed by w_α. The condition ψ is w_α-fixed is again (7.7g). By part (ii) of Proposition 6.1, we get $\psi(d_\alpha) = 1$. Therefore, any w_α-skew regular function f must vanish on the subvariety

$$\{\, \psi \in \Psi(M_\emptyset) \mid (1 - f_{d_\alpha})(\psi)) \;=\; 0 \,\} \;, \tag{7.7j}$$

and then, as in case (i), we conclude f is divisible by $(1 - f_{d_\alpha})$. □

Corollary 7.7k. *Any $f \in \mathcal{R}[\Psi(M_\emptyset)]$ which is $W(\mathcal{Q})$-skew is divisible by the function*

$$\prod_{\alpha \in \Sigma_+(\mathcal{Q})} (1 - f_{d_\alpha}^{e_\alpha}) \;. \tag{7.7l}$$

Proof. Suppose $\alpha \in \Sigma_+(\mathcal{Q})$. If $f \in \mathcal{R}[\Psi(M_\emptyset)]$ is $W(\mathcal{Q})$-skew, then it is w_α-skew and so by the Proposition divisible by $(1 - f_{d_\alpha}^{e_\alpha})$. Since all the $(1 - f_{d_\alpha})$'s as well as the $(1 + f_{d_\alpha})$'s when $e_\alpha = 2$ are distinct non-associate primes in $\mathcal{R}[\Psi(M_\emptyset)]$, the Corollary now follows. □

Proposition 7.7m. *Set*

$$2\rho_{_\mathcal{Q}} := \sum_{\alpha \in \Sigma_+(\mathcal{Q})} e_\alpha d_\alpha \in \Lambda \tag{7.7n}$$

and

$$D_\mathcal{Q} \;:=\; f_{-2\rho_{_\mathcal{Q}}} \prod_{\alpha \in \Sigma_+(\mathcal{Q})} (1 - f_{d_\alpha}^{e_\alpha})^2 \;. \tag{7.7o}$$

Then:

(i) $(2\rho_{\mathcal{Q}}) - w_\beta(2\rho_{\mathcal{Q}}) = 2e_\beta d_\beta$ *for all simple roots* $\beta \in \Sigma_+(\mathcal{Q})$.

(ii) As a function on $\Psi(M_\emptyset)$, $D_{\mathcal{Q}}$ *is* $W(\mathcal{Q})$*-symmetric, and if* f *and* g *are* $W(\mathcal{Q})$*-skew functions on* $\Psi(M_\emptyset)$*, then* $D_{\mathcal{Q}}$ *divides the product* fg.

(iii) The element $2\rho_{\mathcal{Q}}$ *is fixed by* $R(\mathcal{Q})$*, and as a function on* $\Psi(M_\emptyset)$, $D_{\mathcal{Q}}$ *is* $R(\mathcal{Q})$*-symmetric.*

Proof. To prove (i), we note that if β is a simple root of $\Sigma_+(\mathcal{Q})$, then w_β permutes the elements of $\Sigma_+(\mathcal{Q})\backslash\{\beta\}$. Since $w_\beta(d_\beta) = -d_\beta$, we have

$$\begin{aligned} 2\rho_{\mathcal{Q}} - w_\beta(2\rho_{\mathcal{Q}}) \ &= \ \Big(\sum_{d_\alpha \in \Sigma_+(\mathcal{Q})\backslash\{\beta\}} e_\alpha d_\alpha \ + \ e_\beta d_\beta \ \Big) \\ &\qquad - \ \Big(\sum_{d_\alpha \in \Sigma_+(\mathcal{Q})\backslash\{\beta\}} e_\alpha d_\alpha \ - \ e_\beta d_\beta \ \Big) \\ &= \ 2e_\beta d_\beta \ . \end{aligned} \tag{7.7p}$$

To prove the first assertion of (ii), we see from (i) that if β is a simple root of $\Sigma_+(\mathcal{Q})$, then

$$\begin{aligned} w_\beta(D_{\mathcal{Q}}) \ &= \ w_\beta(f_{-2\rho_{\mathcal{Q}}}) \prod_{\alpha\in\Sigma_+(\mathcal{Q})} w_\beta\big((1-f^{e_\alpha}_{d_\alpha})^2\big) \\ &= \ (f_{-w_\beta(2\rho_{\mathcal{Q}})}) \ \ w_\beta\big((1-f^{e_\beta}_{d_\beta})^2\big) \prod_{\alpha\in\Sigma_+(\mathcal{Q})\backslash\{\beta\}} w_\beta\big((1-f^{e_\alpha}_{d_\alpha})^2\big) \\ &= \ (f_{(-2\rho_{\mathcal{Q}}+2e_\beta d_\beta)}) \ \ \big((1-f^{e_\beta}_{-d_\beta})^2\big) \prod_{\alpha\in\Sigma_+(\mathcal{Q})\backslash\{\beta\}} w_\beta\big((1-f^{e_\alpha}_{d_\alpha})^2\big) \\ &= \ (f_{(-2\rho_{\mathcal{Q}})})\ (f_{(2e_\beta d_\beta)})\ (f_{(-2e_\beta d_\beta)})\ \big((1-f^{e_\beta}_{d_\beta})^2\big) \prod_{\alpha\in\Sigma_+(\mathcal{Q})\backslash\{\beta\}} w_\beta\big((1-f^{e_\alpha}_{d_\alpha})^2\big) \\ &= \ D_{\mathcal{Q}} \ . \end{aligned} \tag{7.7q}$$

The second statement of (ii) follows from the fact that if f and g are $W(\mathcal{Q})$-skew, then by Corollary 7.7k, they are divisible by the function (7.7l).

Finally, we observe assertion (iii) is an easy consequence of the definition of the R-group $R(\mathcal{Q})$. $\square$

Remark 7.7r. In (7.7n), the element

$$\rho_{\mathcal{Q}} \ := \ \frac{1}{2} \sum_{\alpha\in\Sigma_+(\mathcal{Q})} e_\alpha d_\alpha \ \in \ \frac{1}{2}\Lambda \ \subset \ \mathbb{R}\otimes_{\mathbb{Z}} \Lambda \tag{7.7s}$$

may or may not belong to the lattice Λ. If $\rho_{\mathcal{Q}}$ does belong to Λ, then the function on $\Psi(M_\emptyset)$ defined as

$$\Delta_{\mathcal{Q}} \ := \ f_{-\rho_{\mathcal{Q}}} \prod_{\alpha\in\Sigma_+(\mathcal{Q})} (1 - f^{e_\alpha}_{d_\alpha}) \ . \tag{7.7t}$$

is $W(\mathcal{Q})$-skew and it divides any $W(\mathcal{Q})$-skew function. Its square is $D_{\mathcal{Q}}$. When $\rho_{_\mathcal{Q}} \notin \Lambda$, we can define a lattice $\tilde{\Lambda} := (\Lambda) + (\rho_{_\mathcal{Q}} + \Lambda) \subset (\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$. This lattice is the character group for a double cover $\widetilde{\Psi}$ of $\Psi(M_\emptyset)$. It is then possible to define an analogous function $\widetilde{\Delta}_{\mathcal{Q}}$ on $\widetilde{\Psi}$ which divides any $W(\mathcal{Q})$-skew function on $\widetilde{\Psi}$. We discuss more about this in section 8 when we determine the span of distributions $S^{M_\emptyset}(\gamma_0, \gamma)$. Both $D_{\mathcal{Q}}$ and $\Delta_{\mathcal{Q}}$ are obviously generalizations of the Weyl denominators.

7.8. A fixed point result.

Proposition 7.8a. *Suppose $\mathcal{Q}$ is a $\Psi(M_\emptyset)$-orbit in $\widetilde{M_\emptyset}$. There exists $\sigma \in \mathcal{Q}$ which is fixed by $W(\mathcal{Q})$.*

Proof. Let $\beta_0, \beta_1, \ldots, \beta_s \in \Sigma_+(\mathcal{Q})$ be a set of simple roots. We view $M_\emptyset$ as a maximal parabolic F-subgroup of M_{β_i}. Recall, from (5.3c), the function $\psi \to z_{\beta_i}(\psi) := \psi(h_{\beta_i})$ on $\psi(M_\emptyset)$. By Theorems 5.3a, and 5.3f, the non-constant function on $\mathcal{Q}$ given by $\chi \to \mu_{_{M_{\beta_i}|M_\emptyset}}(\chi)$ has a zero $\chi_i \in \mathcal{Q}_u$, which is fixed by w_{β_i}, and the function on $\Psi(M_\emptyset)$ given by $\psi \to \mu_{_{M_{\beta_i}|M_\emptyset}}(\psi\chi_i)$ has zero set containing the variety $1 - z_{\beta_i} = 0$. Furthermore, by Lemma 6.1d and Corollary 6.1j, if $\psi \in \Psi(M_\emptyset)$ lies on the variety $1 - z_{\beta_i} = 0$, then ψ, hence $\psi\chi_i$ is w_{β_i}-invariant. Note that $\chi_i\chi_j^{-1} \in \Psi(M_\emptyset)$. Suppose we can find $\psi \in \Psi(M_\emptyset)$ so that for each $0 \le j \le s$ the quasi-character $\psi\chi_0\chi_j^{-1}$ lies in the variety $1 - z_{\beta_j} = 0$. Consider $\psi\chi_0$. For each j, since $\psi\chi_0 = (\psi\chi_0\chi_j^{-1})\chi_j$, and both $\psi\chi_0\chi_j^{-1}$ and χ_j are w_{β_j}-invariant and a zero of $\mu_{_{M_{\beta_j}|M_\emptyset}}$, we conclude the quasi-character $\sigma := \psi\chi_0 \in \mathcal{Q}$ is w_{β_j}-invariant, and a zero of $\mu_{_{M_{\beta_j}|M_\emptyset}}$. Since $W(\mathcal{Q})$ is generated by $w_{\beta_0}, \ldots, w_{\beta_s}$, the former means σ is fixed by $W(\mathcal{Q})$. To see the existence of a ψ so that $\psi\chi_0\chi_j^{-1}$ lies in the variety $1 - z_{\beta_j} = 0$, we note that as linear functions on $\Psi(M_\emptyset)$ the gradients are linearly independent, e.g., consider $\mathfrak{a}^*_{\mathbb{C}}$. A common zero for the linear functions is then obvious. $\square$

With σ as in the proposition, the map

$$(7.8b) \qquad \begin{array}{ccc} \Psi(M_\emptyset) & \longrightarrow & \mathcal{Q} \\ \psi & \longrightarrow & \psi\,\sigma \end{array}$$

is a $W(\mathcal{Q})$-equivariant bijection. It allows a $W(\mathcal{Q})$-equivariant map identification of $\mathcal{R}[\Psi(M_\emptyset)]$ and $\mathcal{R}[\mathcal{Q}]$. We assume this identification in what follows.

7.9. Proof of the regularity statement of Proposition 7.4h.

Proposition 7.9a. *Suppose $\sigma \in \mathcal{Q}$ is a $W(\mathcal{Q})$-fixed point. Then the function on $\Psi(M_\emptyset)$ given by*

$$(7.9b) \qquad \psi \longrightarrow \frac{D_{\mathcal{Q}}(\psi)}{\prod\limits_{\alpha \in \Sigma_+(\mathcal{Q})} p_{\alpha,\mathcal{Q}}(\psi\,\sigma)}$$

is regular.

Proof. Both the numerator and denominator of the right side of (7.9b) is a product over the elements of $\Sigma_+(\mathcal{Q})$. Therefore, it suffices to show the following. For all $\alpha \in \Sigma_+(\mathcal{Q})$, the function

$$\psi \longrightarrow \frac{\big(1 - f_{d_\alpha}^{e_\alpha}(\psi)\big)}{p_{\alpha,\mathcal{Q}}(\,\psi\,\sigma\,)} \tag{7.9c}$$

is regular. The regularity of this function is a consequence of Proposition 6.2a, and Theorem 5.3a. □

Proof of the regularity statement of Proposition 7.4h. Take $\sigma \in \mathcal{Q}$ to be fixed by $W(\mathcal{Q})$, and define $\mathcal{D}_{\mathcal{Q}} \in \mathcal{R}[\mathcal{Q}]$ as

$$\mathcal{D}_{\mathcal{Q}}(\psi\sigma) := D_{\mathcal{Q}}(\psi)\ . \tag{7.9d}$$

The function $\mathcal{D}_{\mathcal{Q}}$, in fact, depends on the choice of the fixed point σ, but we have suppressed notation to indicate the dependence. We have

$$\frac{\operatorname{sym}_{\mathcal{Q}}\operatorname{skew}(\chi)}{\prod\limits_{\alpha\in\Sigma_{\mathrm{red}}(P_\emptyset)} p_{\alpha,\mathcal{Q}}(\chi)} = \frac{\operatorname{sym}_{\mathcal{Q}}\operatorname{skew}(\chi)}{\mathcal{D}_{\mathcal{Q}}(\chi)}\ \frac{\mathcal{D}_{\mathcal{Q}}(\chi)}{\prod\limits_{\alpha\in\Sigma_{\mathrm{red}}(P_\emptyset)} p_{\alpha,\mathcal{Q}}(\chi)} \tag{7.9e}$$

On the right hand side of (7.9e), the first factor is regular by the factorization of $\operatorname{sym}_{\mathcal{Q}}\operatorname{skew}(\chi)$ given in section 7.6 combined with Proposition 7.7m. The second factor on the right hand side of (7.9e) is regular by Proposition 7.9a. The regularity statement obviously follows. □

An immediate consequence of Proposition 7.4h, is the following:

Theorem 7.9f. *Suppose $\gamma_0, \gamma \in M_\emptyset$ is such that $\gamma_0\,(w\cdot\gamma)$ is regular for every $w \in W_G(A_\emptyset)$, i.e., if $w' \in W$, and $w'(\gamma_0\,w(\gamma)) = \gamma_0\,w(\gamma)$, then $w' = 1$. Then, the distribution*

$$f \mapsto \sum_{w\in W_G(A_\emptyset)} \operatorname{sgn}(w)\ \ F_f^{M_\emptyset}(\,\gamma_0\ w\cdot\gamma\,) \qquad \forall f \in C_c^\infty(G) \tag{7.9g}$$

belongs to the Bernstein center.

7.10. Some subspaces of $\mathcal{R}[\Psi(M_\emptyset)]$.

We introduce some group theoretic subspaces $\mathcal{R}[\Psi(M_\emptyset)]$ which will be used in section 8. For $v \in \Lambda$, and Γ any subgroup of $\operatorname{Stab}(\mathcal{Q})$, set

$$\begin{aligned} \operatorname{sym}_\Gamma(f_v) &:= \sum_{w\in\Gamma} f_{w(v)}\ , \\ \operatorname{skew}_\Gamma(f_v) &:= \sum_{w\in\Gamma} \operatorname{sgn}(w)\, f_{w(v)}\ . \end{aligned} \tag{7.10a}$$

Let $\mathcal{R}[\Psi(M_\emptyset)]^\Gamma$ and $\mathcal{R}[\Psi(M_\emptyset)]^{\Gamma,\mathrm{sgn}}$ denote the subspace of $\mathcal{R}[\Psi(M_\emptyset)]$ of Γ-symmetric and Γ-skew functions respectively. The following is elementary.

Lemma 7.10b. *The subspaces $\mathcal{R}[\Psi(M_\emptyset)]^\Gamma$ and $\mathcal{R}[\Psi(M_\emptyset)]^{\Gamma,\mathrm{sgn}}$ are spanned by the set of functions $\{\operatorname{sym}_\Gamma(f_v)\,|\,v\in\Lambda\,\}$ and $\{\operatorname{skew}_\Gamma(f_v)\,|\,v\in\Lambda\,\}$ respectively.*

8. Component span of the distributions $S^{M_\emptyset}(\gamma_0,\gamma)$.

8.1. In this section we establish results on the linear span of the distributions $S^{M_\emptyset}(\gamma_0,\gamma)$ of section 7. Recall that we follow the usual convention and identify the algebra $\mathcal{Z}(G)$ of G-invariant essentially compact distributions on G with the algebra $\mathfrak{Z}(G)$ of regular functions on $\Omega(G)$, and call them both the Bernstein center. Suppose $\Omega(\sigma)$ is a Bernstein component of $\Omega(G)$ as in section 5.6. Associated to $\Omega(\sigma)$ is an idempotent e_σ in the Bernstein center which, as a function on $\Omega(G)$, is the characteristic function of $\Omega(\sigma)$, i.e., constant value one on $\Omega(\sigma)$ and constant value zero off $\Omega(\sigma)$. In section 7, we showed the combination of orbital integrals $S^{M_\emptyset}(\gamma_0,\gamma)$ lies in the Bernstein center, and furthermore as a function on $\Omega(G)$, it is supported on Bernstein components $\Omega(\sigma)$ with σ a character of $M_\emptyset$. The Bernstein component $\Omega(\sigma)$ is precisely the $\Psi(M_\emptyset)$-orbit of σ. We use the notation $\mathcal{Q}$ to denote this orbit so as to be compatible with section 7. The convolution

$$e_{\mathcal{Q}} \star S^{M_\emptyset}(\gamma_0,\gamma)\ , \tag{8.1a}$$

is obviously in the Bernstein center, and when it is viewed as a regular function on $\Omega(G)$, it has support on $\mathcal{Q}$. By (7.3f), we see it is

$$\begin{aligned}
\chi \ \longrightarrow\ F(\gamma_0,\gamma)(\chi)\ &:= \sum_{w\in W_\emptyset/\mathrm{Stab}(\mathcal{Q})} \mathrm{sgn}(w)\ \frac{\mathrm{sym}_{\,\mathcal{Q}}\,\mathrm{skew}(\chi\,,\,w\cdot\gamma_0\,,\,\gamma)}{\mu_{G|M_\emptyset,\mathcal{Q}}(\chi)} \\
&= \sum_{w\in W_\emptyset/\mathrm{Stab}(\mathcal{Q})} \mathrm{sgn}(w)\ \frac{\mathrm{skew}_{\mathcal{Q}}(\chi\,,\,w\cdot\gamma_0\,)\,\mathrm{skew}(\chi\,,\,\gamma)}{\mu_{G|M_\emptyset,\mathcal{Q}}(\chi)} \\
&= \frac{\mathrm{skew}(\chi\,,\,\gamma_0\,)\,\mathrm{skew}(\chi\,,\,\gamma)}{\mu_{G|M_\emptyset,\mathcal{Q}}(\chi)}\ .
\end{aligned} \tag{8.1b}$$

We remark that in the original definition of the distribution $S^{M_\emptyset}(\gamma_0,\gamma)$ in §7.2, the elements $\gamma_0,\gamma\in M_\emptyset$ are required to satisfy the property that $\gamma_0 w(\gamma)$ is regular for all $w\in W_\emptyset$. Formula (8.1b) allows one to define spectrally, i.e., by their Fourier transforms, distributions in the Bernstein center for arbitrary $\gamma_0,\gamma\in M_\emptyset$ which extrapolate the distributions $e_{\mathcal{Q}}\star S^{M_\emptyset}(\gamma_0,\gamma)$ and by consequence the distributions $S^{M_\emptyset}(\gamma_0,\gamma)$. We also remark that if we fix a finite set of components $\mathcal{Q}_1,\ldots,\mathcal{Q}_h$, then if γ_0' and γ' are sufficiently close to γ_0 and γ respectively, then $e_{\mathcal{Q}_i}\star S^{M_\emptyset}(\gamma_0',\gamma')$ equals $e_{\mathcal{Q}_i}\star S^{M_\emptyset}(\gamma_0,\gamma)$ for all $1\le i\le h$. In particular, in this situation, all these distributions occur as convolution of a distribution $S^{M_\emptyset}(\gamma_0,\gamma)$ of §7.2 with a $e_{\mathcal{Q}_i}$.

Our goal in this section is to characterize the linear span of the functions (8.1b). Equivalently we want to determine the linear span of the distributions of type (8.1a). Set

$$\mathcal{FT}(\mathcal{Q})\ :=\ \text{span of the functions (8.1b) as } \gamma_0 \text{ and } \gamma \text{ vary over } M_\emptyset \tag{8.1c}$$

We will show $\mathcal{FT}(\mathcal{Q})$ is an ideal in $\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q})}$. In particular, it follows, that the convolution of a distribution of type (8.1a) with any distribution in the Bernstein center is in the span of distributions of type (8.1a). The ideal $\mathcal{FT}(\mathcal{Q})$ is often a principal ideal, in fact generated by the element which is the quotient of the generalized Weyl denominator $\mathcal{D}_{\mathcal{Q}}$ by the Plancherel density $\mu_{G|M_\emptyset,\mathcal{Q}}$, but not always.

8.2. We make an elementary reduction.

Proposition 8.2a. *Suppose $\mathcal{Q} \subset \widetilde{M_\emptyset}$ is a $\Psi(M_\emptyset)$-orbit, and* Stab$(\mathcal{Q})$ *is the Weyl stabilizer of $\mathcal{Q}$. Then, on $\mathcal{Q}$, as γ_0 and γ range over $M_\emptyset$, the linear span of the functions of (8.1b) is the same as the span of the functions*

$$\chi \longrightarrow \frac{\mathrm{skew}_{\mathcal{Q}}(\chi\,,\,\gamma_0\,)\,\mathrm{skew}_{\mathcal{Q}}(\chi\,,\,\gamma)}{\mu_{G|M_\emptyset,\mathcal{Q}}(\chi)}\ . \tag{8.2b}$$

Proof. Trivially, the two terms skew$(\chi\,,\,\gamma_0\,)$ and skew$(\chi\,,\,\gamma)$ are symmetrical in γ_0 and γ. It suffices to show the span of the functions $\chi \to \mathrm{skew}(\chi\,,\,\gamma_0\,)$, as γ_0 varies over $M_\emptyset$, is the same as the span of the functions $\chi \to \mathrm{skew}_{\mathcal{Q}}(\chi\,,\,\gamma_0\,)$. For $y \in W_\emptyset$, we have

$$\mathrm{skew}_{\mathcal{Q}}(\chi, y(\gamma_0)) \ = \sum_{w\in \mathrm{Stab}(\mathcal{Q})} \mathrm{sgn}(w)\,\chi^{-1}(\,w(y(\gamma_0))\,)\ . \tag{8.2c}$$

So,

$$\begin{aligned} \mathrm{skew}(\chi, \gamma_0) \ &= \sum_{y\in \mathrm{Stab}(\mathcal{Q})\backslash W_\emptyset}\ \sum_{w\in \mathrm{Stab}(\mathcal{Q})} \mathrm{sgn}(w)\,\mathrm{sgn}(y)\,\chi^{-1}(\,w(y(\gamma_0))\,) \\ &= \sum_{y\in \mathrm{Stab}(\mathcal{Q})\backslash W_\emptyset} \mathrm{sgn}(y)\ \mathrm{skew}_{\mathcal{Q}}(\chi, y(\gamma_0))\ . \end{aligned} \tag{8.2d}$$

Observe that for $\gamma_c \in {}^{\mathrm{o}}M_\emptyset$, then $v(\gamma_c) \in {}^{\mathrm{o}}M_\emptyset$ for all $v \in W_\emptyset$. In particular, since the restriction of χ to ${}^{\mathrm{o}}M_\emptyset$ is Stab$(\mathcal{Q})$-invariant, it follows $\chi(\,w(y(\gamma_c))\,) = \chi(\,y(\gamma_c)\,)$, for all $w \in \mathrm{Stab}(\mathcal{Q})$ and $y \in W_\emptyset$. Consequently,

$$\begin{aligned} \mathrm{skew}_{\mathcal{Q}}(\chi, y(\gamma_0\,\gamma_c)) \ &= \sum_{w\in \mathrm{Stab}(\mathcal{Q})} \mathrm{sgn}(w)\,\chi^{-1}(\,w(y(\gamma_0))\,)\ \chi^{-1}(\,w(y(\gamma_c))\,) \\ &= \sum_{w\in \mathrm{Stab}(\mathcal{Q})} \mathrm{sgn}(w)\,\chi^{-1}(\,w(y(\gamma_0))\,)\ \chi^{-1}(\,y(\gamma_c)\,) \\ &= \ \chi^{-1}(\,y(\gamma_c)\,)\ \ \mathrm{skew}_{\mathcal{Q}}(\chi, y(\gamma_0))\ . \end{aligned} \tag{8.2e}$$

Suppose we fix γ_0. By (8.2d), (8.2e), and the fact that distinct characters are linearly independent, the span of the functions skew$(-, \gamma_0\gamma_c)$, as γ_c varies over ${}^{\mathrm{o}}M_\emptyset$ contains the functions skew$_{\mathcal{Q}}(-, y(\gamma_0))$. Whence the span of the functions skew$(-,\,\gamma_0\,)$ as γ_0 varies over $M_\emptyset$ contains the functions skew$_{\mathcal{Q}}(-,\,\gamma_0\,)$. The opposite inclusion is clear, so the two spans are equal.

To complete the proof, we consider the product function skew$(-,\,\gamma_0\,)\,$skew$(-,\,\gamma)$. If we fix γ and vary γ_0, the space spanned by the product functions equals the space spanned by the functions $\mathrm{skew}_{\mathcal{Q}}(-,\gamma_0)\,\mathrm{skew}(-,\gamma)$, where γ_0 varies over $M_\emptyset$. If we then allow γ to vary and apply the same argument, we obtain the assertion of the proposition. □

To determine the span of the functions (8.1b), it suffices to determine the span of the functions $\chi \to \mathrm{skew}_{\mathcal{Q}}(\chi\,,\,\beta\,)$ as β varies over $M_\emptyset$. Denote this vector space by

$$\mathrm{span}\left\{\,\mathrm{skew}_{\mathcal{Q}}(\,-\,,\,\beta\,)\mid \beta\in M_\emptyset\,\right\}\;. \tag{8.2f}$$

By Lemma 7.10b, we have

$$\mathrm{span}\left\{\,\mathrm{skew}_{\mathcal{Q}}(\,-\,,\,\beta\,)\mid \beta\in M_\emptyset\,\right\} \;=\; \mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q}),\mathrm{sgn}}\;. \tag{8.2g}$$

The following is then elementary.

Proposition 8.2h.

(i) The subspace of $\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q})}$ spanned, as γ_0 and γ vary over $M_\emptyset$, by the product functions $\chi \to \mathrm{skew}_{\mathcal{Q}}(\chi\,,\,\gamma_0\,)\,\mathrm{skew}_{\mathcal{Q}}(\chi\,,\,\gamma)$, is the ideal of $\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q})}$ given as follows:

$$\mathcal{I} := \mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q}),\mathrm{sgn}}\cdot\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q}),\mathrm{sgn}}\;. \tag{8.2i}$$

(ii) The space $\mathcal{FT}(\mathcal{Q})$ of (8.1c) equals

$$\frac{\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q}),\mathrm{sgn}}\cdot\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q}),\mathrm{sgn}}}{\mu_{G|M_\emptyset,\mathcal{Q}}}\;. \tag{8.2i}$$

In particular, it is an ideal of $\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q})}$.

In the remaining sections, we make more precise statements about the ideal $\mathcal{FT}(\mathcal{Q})$.

8.3. The skew sum over $W(\mathcal{Q})$.

Recall $W(\mathcal{Q})$ is a normal Coxeter subgroup of $\mathrm{Stab}(\mathcal{Q})$. For $\zeta\in M_\emptyset$, and $\chi\in\mathcal{Q}$, set

$$\mathrm{skew}_{W(\mathcal{Q})}(\chi,\zeta) \;:=\; \sum_{w\in W(\mathcal{Q})}\mathrm{sgn}(w)\chi^{-1}(\,w(\zeta)\,)\;. \tag{8.3a}$$

It is evident

$$\mathrm{skew}_{\mathcal{Q}}(\chi,\zeta) = \sum_{r\in R(\mathcal{Q})}\mathrm{sgn}(r)\;\mathrm{skew}_{W(\mathcal{Q})}(\chi, r(\zeta))\;. \tag{8.3b}$$

We determine

$$\mathrm{span}\left\{\,\mathrm{skew}_{W(\mathcal{Q})}(\,-\,,\,\beta\,)\mid \beta\in M_\emptyset\,\right\}\;. \tag{8.3c}$$

By Proposition 7.8a we can choose $\sigma\in\mathcal{Q}$ which is fixed by $W(\mathcal{Q})$ and then the bijection (7.8b) $\psi\to\psi\sigma$ is $W(\mathcal{Q})$-equivariant, and it yields a $W(\mathcal{Q})$-equivariant identification of $\mathcal{R}[\Psi(M_\emptyset)]$ with $\mathcal{R}[\mathcal{Q}]$. We consider two cases depending on whether $\rho_{\mathcal{Q}}\in\Lambda$ or $\rho_{\mathcal{Q}}\notin\Lambda$.

8.4. Case $\rho_{\mathcal{Q}}\in\Lambda$: As mentioned in remark 7.7r, the function $\Delta_{\mathcal{Q}}$ defined in (7.7t) is $W(\mathcal{Q})$-skew and it divides any $W(\mathcal{Q})$-skew function $h\in\mathcal{R}[\Psi(M_\emptyset)]$. Clearly, the quotient $h/\Delta_{\mathcal{Q}}$ is $W(\mathcal{Q})$-invariant.

Proposition 8.4a.

(i) $\mathrm{skew}_{W(\mathcal{Q})}(f_{\rho_{\mathcal{Q}}}) \;=\; (-1)^{\mathrm{card}(\Sigma_+(\mathcal{Q}))}\Delta_{\mathcal{Q}}$

(ii) Under the identification of $\mathcal{R}[\Psi(M_\emptyset)]$ *and* $\mathcal{R}[\mathcal{Q}]$,

$$\mathrm{span}\left\{ \mathrm{skew}_{W(\mathcal{Q})}(\,-\,,\,\beta\,) \mid \beta \in M_\emptyset \right\} \;=\; \Delta_{\mathcal{Q}}\ \mathcal{R}[\Psi(M_\emptyset)]^{W(\mathcal{Q})} \tag{8.4b}$$

(iii) The span of the product functions $\mathrm{skew}_{W(\mathcal{Q})}(-,\gamma_0)\ \mathrm{skew}_{W(\mathcal{Q})}(-,\gamma)$, *as* γ_0 *and* γ *vary over* $M_\emptyset$ *is*

$$\Delta_{\mathcal{Q}}^2\ \mathcal{R}[\Psi(M_\emptyset)]^{W(\mathcal{Q})}\ . \tag{8.4c}$$

In particular, it is a principal ideal of $\mathcal{R}[\Psi(M_\emptyset)]^{W(\mathcal{Q})}$

Proof. The proof of (i) is an adaptation of the proof of the analogous formula for 'rho' in the Weyl character formula. As such we only sketch it. Let w_0 denote the long Weyl element in $W(\mathcal{Q})$, so $w_0(\Sigma_+(\mathcal{Q})) = -\Sigma_+(\mathcal{Q})$. Then $w_0(-\rho_{\mathcal{Q}}) = \rho_{\mathcal{Q}}$. In combination with its product formula definition (7.7t), we see

$$\begin{aligned}(-1)^{\mathrm{card}(\Sigma_+(\mathcal{Q}))}\Delta_{\mathcal{Q}} \;&=\; w_0\,(\Delta_{\mathcal{Q}}) \;=\; w_0\,\Big(\, f_{-\rho_{\mathcal{Q}}} \prod_{\alpha\in\Sigma_+(\mathcal{Q})} (1 - f_{d_\alpha}^{\,e_\alpha})\,\Big) \\ &=\; f_{\rho_{\mathcal{Q}}} \prod_{\alpha\in\Sigma_+(\mathcal{Q})} (1 - f_{-d_\alpha}^{\,e_\alpha})\ .\end{aligned} \tag{8.4d}$$

The functions $\mathrm{skew}(f_v)$, with $v \in \Lambda \cap C(\Sigma_+(\mathcal{Q}))$ are a basis for the space of $W(\mathcal{Q})$-skew functions. To express $\Delta_{\mathcal{Q}}$ in this fashion, we can take v of the form

$$\rho_{\mathcal{Q}} \;-\; \sum_{\alpha\in\Sigma_+(\mathcal{Q})} \epsilon_\alpha\, e_\alpha\, d_\alpha\ , \quad \text{with} \quad \epsilon_\alpha \in \{0,1\}\ . \tag{8.4e}$$

In particular, $\mathrm{skew}_{W(\mathcal{Q})}(f_{\rho_{\mathcal{Q}}})$ occurs with coefficient 1. The difference of $\mathrm{skew}_{W(\mathcal{Q})}(f_{\rho_{\mathcal{Q}}})$ and $(-1)^{\mathrm{card}(\Sigma_+(\mathcal{Q}))}\Delta_{\mathcal{Q}}$ is then both divisible by $\Delta_{\mathcal{Q}}$ and in the span of $\mathrm{skew}_{W(\mathcal{Q})}(f_v)$'s with v as in (8.4) and the condition at least one $\epsilon_\alpha \neq 0$. This forces the difference to be the zero function.

To prove (ii), as previously noted, if h is a $W(\mathcal{Q})$-skew function, then the quotient $h/\Delta_{\mathcal{Q}}$ is $W(\mathcal{Q})$-invariant and regular. Thus $\mathrm{span}\left\{ \mathrm{skew}_{W(\mathcal{Q})}(\,-\,,\,\beta\,) \mid \beta \in M_\emptyset \right\} \subset \Delta_{\mathcal{Q}}\ \mathcal{R}[\Psi(M_\emptyset)]^{W(\mathcal{Q})}$. The opposite inclusion is obvious and so the two are equal.

The proof of (iii) is immediate from (ii). $\square$

8.5. Case $\rho_{\mathcal{Q}} \notin \Lambda$: The real vector space $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ contains both Λ and $\rho_{\mathcal{Q}}$. It is elementary the union

$$\widetilde{\Lambda} \;:=\; \Lambda \cup (\,\rho_{\mathcal{Q}} + \Lambda\,) \tag{8.5a}$$

is a $\mathrm{Stab}(\mathcal{Q})$-invariant lattice of $\mathbb{R}\otimes_{\mathbb{Z}}\Lambda$ and $[\widetilde{\Lambda}:\Lambda]=2$. Set

$$\psi_{\widetilde{\Lambda}} \ := \ \text{character of order two on } \widetilde{\Lambda} \text{ with kernel } \Lambda\ . \tag{8.5b}$$

The inclusion $\iota:\Lambda\to\widetilde{\Lambda}$ leads to a double covering map

$$\widetilde{\Psi} := \mathrm{Hom}(\widetilde{\Lambda},\mathbb{C}^{\times}) \xrightarrow{\ \pi\ } \Psi(M_{\emptyset}) = \mathrm{Hom}(\Lambda,\mathbb{C}^{\times})\ , \tag{8.5c}$$

with $\Upsilon=\mathrm{Ker}(\pi):=\{1\,,\,\psi_{\widetilde{\Lambda}}\}$. Let

$$\mathrm{sgn}_{\widetilde{\Lambda}} \ := \ \text{character of order two of } \mathrm{Ker}(\pi)\ . \tag{8.5d}$$

It is evident the actions of $\mathrm{Stab}(\mathcal{Q})$ and $\Upsilon=\mathrm{Ker}(\pi)$ on $\widetilde{\Psi}$ and $\mathcal{R}[\widetilde{\Psi}]$ commute with one another. For any subgroup $\Gamma\subset\mathrm{Stab}(\mathcal{Q})$, let $\Gamma\Upsilon$ denote the subgroup of $\mathcal{R}[\widetilde{\Psi}]$ automorphisms generated by Γ and Υ. Obviously, $\Gamma\Upsilon$ is isomorphic to the direct product of Γ and Υ. If κ and τ are two one-dimensional characters of Γ and Υ respectively, let $\kappa\cdot\tau$ denote the tensor product character of $\Gamma\Upsilon$.If κ is a character of Γ. For convenience, we also denote by κ the character of $\Gamma\Upsilon$ which is the tensor product of κ and the trivial character of Υ, and similarly when τ is a character of Υ. With the obvious extrapolation of the notation of section 7.10, it is elementary we have inclusion isomorphisms $\mathcal{R}[\Psi(M_{\emptyset})]^{\Gamma}\xrightarrow{\ \pi^*\ }\mathcal{R}[\widetilde{\Psi}]^{\Gamma\Upsilon}$ and $\mathcal{R}[\Psi(M_{\emptyset})]^{\Gamma,\mathrm{sgn}}\xrightarrow{\ \pi^*\ }\mathcal{R}[\widetilde{\Psi}]^{\Gamma\Upsilon,\mathrm{sgn}}$, and

$$\begin{aligned}
\mathcal{R}[\widetilde{\Psi}]^{\Gamma} \ &= \ \mathcal{R}[\widetilde{\Psi}]^{\Gamma\Upsilon}\oplus\mathcal{R}[\widetilde{\Psi}]^{\Gamma\Upsilon,\mathrm{sgn}_{\widetilde{\Lambda}}}\ , \quad \text{where}\\
\mathcal{R}[\widetilde{\Psi}]^{\Gamma\Upsilon,\mathrm{sgn}_{\widetilde{\Lambda}}} \ &:= \ \{\, f\in\mathcal{R}[\widetilde{\Psi}]^{\Gamma} \mid f(\,\psi_{\widetilde{\Lambda}}\cdot\psi\,) = -f(\,\psi\,)\,\}\ ,\\
\mathcal{R}[\widetilde{\Psi}]^{\Gamma,\mathrm{sgn}} \ &= \ \mathcal{R}[\widetilde{\Psi}]^{\Gamma\Upsilon,\mathrm{sgn}}\oplus\mathcal{R}[\widetilde{\Psi}]^{\Gamma\Upsilon,\,\mathrm{sgn}\cdot\mathrm{sgn}_{\widetilde{\Lambda}}}\ , \quad \text{where}\\
\mathcal{R}[\widetilde{\Psi}]^{\Gamma\Upsilon,\mathrm{sgn}\cdot\mathrm{sgn}_{\widetilde{\Lambda}}} \ &:= \ \{\, f\in\mathcal{R}[\widetilde{\Psi}]^{\Gamma,\mathrm{sgn}} \mid f(\,\psi_{\widetilde{\Lambda}}\cdot\psi\,) = -f(\,\psi\,)\,\}\ .
\end{aligned} \tag{8.5e}$$

All of the spaces in (8.5e) are $\mathcal{R}[\Psi(M_{\emptyset})]^{\Gamma}$-modules.

For $v\in\widetilde{\Lambda}$, extrapolate (7.7e) to define

$$\begin{aligned}
\tilde{f}_v \ :\ \widetilde{\Psi} &\longrightarrow \mathbb{C}\\
\psi &\longrightarrow \psi(v) \quad .
\end{aligned} \tag{8.5f}$$

It is clear if $v\in\Lambda$, then $\tilde{f}_v = f_v\circ\pi$, and so $\mathrm{skew}_{W(\mathcal{Q})}(\tilde{f}_v) = \mathrm{skew}_{W(\mathcal{Q})}(f_v)\circ\pi$ and $\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_v) = \mathrm{sym}_{W(\mathcal{Q})}(f_v)\circ\pi$. Similarly, if $v\in(\rho_{\mathcal{Q}}+\Lambda)$, then $\tilde{f}_v\in\mathcal{R}[\widetilde{\Psi}]^{\Upsilon,\mathrm{sgn}_{\widetilde{\Lambda}}}$, and so $\mathrm{skew}_{W(\mathcal{Q})}(\tilde{f}_v)\in\mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q})\Upsilon,\mathrm{sgn}\cdot\mathrm{sgn}_{\widetilde{\Lambda}}}$ and $\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_v)\in\mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q})\Upsilon,\mathrm{sgn}_{\widetilde{\Lambda}}}$.

We extrapolate (7.7t) to

$$\widetilde{\Delta}_{\mathcal{Q}} \ := \ \tilde{f}_{-\rho_{\mathcal{Q}}} \prod_{\alpha\in\Sigma_+(\mathcal{Q})} (1-\tilde{f}_{d_\alpha}^{\,e_\alpha})\ . \tag{8.5g}$$

Then, the analogue of the first two parts of Proposition 8.4a is

Proposition 8.5h.
(i) $\mathrm{skew}_{W(\mathcal{Q})}(\tilde{f}_{\rho_{\mathcal{Q}}}) \;=\; (-1)^{\mathrm{card}(\Sigma_+(\mathcal{Q}))}\widetilde{\Delta}_{\mathcal{Q}}$
(ii)

$$\mathrm{span}\left\{\, \mathrm{skew}_{W(\mathcal{Q})}(\,-\,,\,\beta\,)\circ\pi \;\middle|\; \beta\in M_{\emptyset}\,\right\} \;=\; \mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q})\Upsilon,\mathrm{sgn}} \tag{8.5i}$$

and

$$\mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q})\Upsilon,\mathrm{sgn}} \;=\; \widetilde{\Delta}_{\mathcal{Q}}\; \mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q})\Upsilon,\mathrm{sgn}_{\widetilde{\Lambda}}}\;. \tag{8.5j}$$

Proof. Easy adaptation of the proof of Proposition 8.4a. $\square$

We compare the two cases $\rho_{\mathcal{Q}}\in\Lambda$ and $\rho_{\mathcal{Q}}\notin\Lambda$. When $\rho_{\mathcal{Q}}\in\Lambda$, the space of $W(\mathcal{Q})$-skew functions $\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q}),\mathrm{sgn}}$ is $\Delta_{\mathcal{Q}}$ times the space $\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}$ of $W(\mathcal{Q})$-invariant functions. When $\rho_{\mathcal{Q}}\notin\Lambda$, the $W(\mathcal{Q})$-skew functions $\mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q}),\mathrm{sgn}\cdot\mathrm{sgn}_{\widetilde{\Lambda}}}$ is $\widetilde{\Delta}_{\mathcal{Q}}$ times the $W(\mathcal{Q})$-invariant functions $\mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q}),\mathrm{sgn}_{\widetilde{\Lambda}}}$. Each of these spaces of $W(\mathcal{Q})$-invariant functions is a $\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}$-module. The fundamental difference between the two cases is that while $\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}$, as a $\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}$-module, is obviously free and rank one, the module $\mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q}),\mathrm{sgn}_{\widetilde{\Lambda}}}$ need not even be free.

The set $(\,\overline{C}(\mathcal{Q})\,\cap\,\Lambda\,)$ is a semi-group and addition defines an action of this semi-group on the set $\overline{C}(\mathcal{Q})\,\cap\,(\,\rho_{\mathcal{Q}}+\Lambda\,)$. It is easily seen that we can choose finitely many $v_1\,,\,\ldots\,,v_m\in\overline{C}(\mathcal{Q})\,\cap\,(\,\rho_{\mathcal{Q}}+\Lambda\,)$ so that

$$\overline{C}(\mathcal{Q})\,\cap\,(\,\rho_{\mathcal{Q}}+\Lambda\,) \;=\; \bigcup_{1\le i\le m} v_i+(\,\overline{C}(\mathcal{Q})\,\cap\,\Lambda\,)\;. \tag{8.5k}$$

Proposition 8.5l. $\mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q})\Upsilon,\mathrm{sgn}_{\widetilde{\Lambda}}} \;=\; \sum_{1\le i\le m}\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_{v_i})\,\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}$ *, and so*

$$\mathrm{span}\left\{\, \mathrm{skew}_{W(\mathcal{Q})}(\,-\,,\,\beta\,)\circ\pi \;\middle|\; \beta\in M_{\emptyset}\,\right\} \;=\; \sum_{1\le i\le m}\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_{v_i})\;\widetilde{\Delta}_{\mathcal{Q}}\;\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}\;. \tag{8.5m}$$

Proof. The space $\mathcal{R}[\widetilde{\Psi}]^{W(\mathcal{Q})\Upsilon,\mathrm{sgn}_{\widetilde{\Lambda}}}$ is spanned by $\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_v)$ as v runs over the left side of (8.5k). By an elementary 'highest weight' induction argument, the span when v is restricted to run over the coset $v_i+(\,\overline{C}(\mathcal{Q})\cap\Lambda\,)$ is $\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_{v_i})\,\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}$. $\square$

Corollary 8.5n. *The span of the product functions* $\mathrm{skew}_{W(\mathcal{Q})}(-,\gamma_0)$ $\mathrm{skew}_{W(\mathcal{Q})}(-,\gamma)$, *as* γ_0 *and* γ *vary over* $M_{\emptyset}$ *is*

$$\sum_{1\le i,j\le m}\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_{v_i})\,\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_{v_j})\;\widetilde{\Delta}^2_{\mathcal{Q}}\;\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}\;. \tag{8.5o}$$

In particular, it is an ideal of $\mathcal{R}[\Psi(M_{\emptyset})]^{W(\mathcal{Q})}$

Proof. Obvious from Propositions 8.5l and 8.5h. $\square$

8.6 Example for $\mathrm{SL}(4)(F)$**.** Suppose $G = \mathrm{SL}(4)(F)$, and $\mathcal{Q} = \Psi(M_\emptyset)$ is the unramified component. This is the Iwahori Bernstein component. Then $\Lambda = \{\, (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 = 0 \,\}$, and

$$\rho_{\mathcal{Q}} \ = \ (\, \frac{3}{2} \,,\, \frac{1}{2} \,,\, -\frac{1}{2} \,,\, -\frac{3}{2} \,) \ . \tag{8.6a}$$

The set in (8.5k) can be taken to be
(8.6b)

$$v_1 := (\, \frac{3}{2} \,,\, -\frac{1}{2} \,,\, -\frac{1}{2} \,,\, -\frac{1}{2} \,) \ ,\ v_2 := (\, \frac{1}{2} \,,\, \frac{1}{2} \,,\, -\frac{1}{2} \,,\, -\frac{1}{2} \,) \ ,\ \text{and} \ \ v_3 := (\, \frac{1}{2} \,,\, \frac{1}{2} \,,\, \frac{1}{2} \,,\, -\frac{3}{2} \,) \ .$$

For $1 \leq i, j \leq 3$, let $\mathcal{J}$ denote the ideal of $\mathcal{R}[\Psi(M_\emptyset)]^{W(\mathcal{Q})}$ which is generated by the products $\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_{v_i})\,\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_{v_j})$, It is evident the zero variety of the ideal $\mathcal{J}$ equals the zero variety of the three elements $\mathrm{sym}_{W(\mathcal{Q})}(\tilde{f}_{v_i})$, $1 \leq i \leq 3$. It can be shown the zero variety of these three elements is non-empty and has finitely many points. Therefore the ideal $\mathcal{J}$ cannot be a principal ideal and consequently, the ideal of (8.5o) also cannot be a principal ideal.

8.7. We summarize and apply the results of the sections 8.4 and 8.5.

Proposition 8.7a. *Suppose $\mathcal{Q}$ is a $\Psi(M_\emptyset)$-orbit of characters of $M_\emptyset$, and* $\mathrm{Stab}(\mathcal{Q}) = W(\mathcal{Q})$*. Then, the space $\mathcal{FT}(\mathcal{Q})$ is the following:*
(i) If $\rho_{\mathcal{Q}} \in \Lambda$, then

$$\mathcal{FT}(\mathcal{Q}) \ = \ \frac{1}{\mu_{G|M_\emptyset}} \ \Delta^2_{\mathcal{Q}} \ \mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q})} \ . \tag{8.7b}$$

In particular, $\mathcal{FT}(\mathcal{Q})$ is a principal ideal of $\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q})}$.
(ii) If $\rho_{\mathcal{Q}} \notin \Lambda$, then

$$\mathcal{FT}(\mathcal{Q}) \ = \ \frac{1}{\mu_{G|M_\emptyset}} \sum_{1 \leq i,j \leq m} \mathrm{sym}_{\mathrm{Stab}(\mathcal{Q})}(\tilde{f}_{v_i})\,\mathrm{sym}_{\mathrm{Stab}(\mathcal{Q})}(\tilde{f}_{v_j}) \ \widetilde{\Delta}^2_{\mathcal{Q}} \ \mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q})} \ . \tag{8.7c}$$

Proof. Apply Proposition 8.4a and Corollary 8.5n. $\square$

Remark 8.7d. In particular, Proposition 8.7a covers the case when $\mathcal{Q}$ is the unramified component $\Psi(M_\emptyset)$ as well as the case when $\mathrm{Stab}(\mathcal{Q}) = \{\,1\,\}$. If $\mathrm{Stab}(\mathcal{Q}) = \{\,1\,\}$, then $\mathcal{FT}(\mathcal{Q}) = \mathcal{R}[\mathcal{Q}]$.

8.8. Here, we consider the situation when the component $\mathcal{Q}$ contains a character σ which is fixed by $\mathrm{Stab}(\mathcal{Q})$.

Proposition 8.8a. *Suppose $G = \mathsf{G}(F)$ is a split F-group, and $\mathcal{Q}$ is any $\Psi(M_\emptyset)$-orbit of characters of $M_\emptyset$. Then, there exists a character $\sigma \in \mathcal{Q}$, which is fixed by* $\mathrm{Stab}(\mathcal{Q})$.

Proof. Let $W_\emptyset$ denote the Weyl group of $M_\emptyset$. Recall the maximal bounded subgroup ${}^{\circ}M_\emptyset$ of $M_\emptyset$ is Weyl invariant. All the characters of $\mathcal{Q}$ have the same restriction to ${}^{\circ}M_\emptyset$. Let σ_0 denote this character. The hypothesis G is split means there is a complementary

Weyl invariant subgroup $L \subset M_\emptyset$ to ${}^\circ M_\emptyset$, i.e., L is Weyl invariant, and $M_\emptyset = L \times {}^\circ M_\emptyset$. Indeed, set $Y := \mathrm{Hom}_F(\mathrm{GL}(1), M_\emptyset)$, and let ϖ be a uniformizing element of F. Then

$$L \;:=\; \{\, \phi(\varpi) \mid \phi \in Y \,\} \tag{8.8b}$$

is a Weyl-invariant subgroup which is complementary to ${}^\circ M_\emptyset$.

The character σ_0 can obviously be extended trivially on L to become a character $\sigma \in \mathcal{Q}$. It is clear σ is $\mathrm{Stab}(\mathcal{Q})$-invariant. □

The existence of a $\mathrm{Stab}(\mathcal{Q})$-fixed character $\sigma \in \mathcal{Q}$, means the map (7.8b) is a $\mathrm{Stab}(\mathcal{Q})$-equivariant bijection, and it provides a $\mathrm{Stab}(\mathcal{Q})$-equivariant identification of $\mathcal{R}[\Psi(M_\emptyset)]$ and $\mathcal{R}[\mathcal{Q}]$. In this situation, the following is elementary, but crucial: The function $\Delta_{\mathcal{Q}}$ (7.7t), when $\rho_{\mathcal{Q}} \in \Lambda$ and the function $\widetilde{\Delta}_{\mathcal{Q}}$ (8.5g), when $\rho_{\mathcal{Q}} \notin \Lambda$ are both $W(\mathcal{Q})$-skew and $R(\mathcal{Q})$-invariant.

Consider the sgn character of $W_\emptyset$ restricted to $R(\mathcal{Q})$. The restriction can be extended trivially across $W(\mathcal{Q})$ to give a character ϵ of $\mathrm{Stab}(\mathcal{Q})$. We have:

$$\epsilon = \begin{cases} \text{sgn} & \text{on} \quad R(\mathcal{Q}) \\ \text{trivial} & \text{on} \quad W(\mathcal{Q})\ . \end{cases} \tag{8.8c}$$

Proposition 8.8d. *Suppose the component $\mathcal{Q}$ admits a* $\mathrm{Stab}(\mathcal{Q})$*-fixed character σ. Then, under the identification of $\mathcal{R}[\Psi(M_\emptyset)]$ and $\mathcal{R}[\mathcal{Q}]$ by (7.8b):*

(i) If $\rho_{\mathcal{Q}} \in \Lambda$, then

$$\mathrm{span}\left\{\, \mathrm{skew}_{\mathrm{Stab}(\mathcal{Q})}(\,-\,,\,\beta\,) \mid \beta \in M_\emptyset \,\right\} \;=\; \Delta_{\mathcal{Q}}\ \mathcal{R}[\Psi(M_\emptyset)]^{\mathrm{Stab}(\mathcal{Q}),\epsilon}\ . \tag{8.8e}$$

(ii) If $\rho_{\mathcal{Q}} \notin \Lambda$, then with $\tilde{f}_{v_i}$ as in Proposition 8.5l,

$$\begin{aligned} &\mathrm{span}\left\{\, \mathrm{skew}_{\mathrm{Stab}(\mathcal{Q})}(\,-\,,\,\beta\,)\circ\pi \mid \beta \in M_\emptyset \,\right\} \;= \\ &\qquad \sum_{1\le i\le m} \mathrm{sym}_{\mathrm{Stab}(\mathcal{Q})}(\tilde{f}_{v_i})\ \widetilde{\Delta}_{\mathcal{Q}}\ \mathcal{R}[\Psi(M_\emptyset)]^{\mathrm{Stab}(\mathcal{Q}),\epsilon}. \end{aligned} \tag{8.8f}$$

Here, we have identified $\mathcal{R}[\widetilde{\Psi}(M_\emptyset)]^{\mathrm{Stab}(\mathcal{Q})\Upsilon,\epsilon}$ and $\mathcal{R}[\Psi(M_\emptyset)]^{\mathrm{Stab}(\mathcal{Q}),\epsilon}$.

Proof. The left side space is $\mathcal{R}[\Psi(M_\emptyset)]^{\mathrm{Stab}(\mathcal{Q}),\mathrm{sgn}}$ (resp. $\mathcal{R}[\widetilde{\Psi}(M_\emptyset)]^{\mathrm{Stab}(\mathcal{Q})\Upsilon,\mathrm{sgn}}$). The function $\Delta_{\mathcal{Q}}$ (resp. $\widetilde{\Delta}_{\mathcal{Q}}$) is $W(\mathcal{Q})$-skew and $R(\mathcal{Q})$-invariant, and it divides any $W(\mathcal{Q})$-skew function. Therefore, for all $\beta \in M_\emptyset$, the ratio $\mathrm{skew}_{\mathrm{Stab}(\mathcal{Q})}(\,-\,,\,\beta\,)/\Delta_{\mathcal{Q}}$ (resp. $(\,\mathrm{skew}_{\mathrm{Stab}(\mathcal{Q})}(\,-\,,\,\beta\,)\circ\pi)/\widetilde{\Delta}_{\mathcal{Q}}$ lies in $\mathcal{R}[\Psi(M_\emptyset)]^{\mathrm{Stab}(\mathcal{Q}),\epsilon}$ (resp. $\mathcal{R}[\widetilde{\Psi}(M_\emptyset)]^{\mathrm{Stab}(\mathcal{Q})\Upsilon,\epsilon\cdot\mathrm{sgn}_{\widetilde{\Lambda}}}$), and furthermore, as β varies over $M_\emptyset$, one gets exactly these two spaces. It follows the two sides of (8.8e) and (8.8f) are equal. □

Theorem 8.8g. *Under the hypothesis and identification of Proposition 8.8d, the space $\mathcal{FT}(\mathcal{Q})$ is the following:*

(i) If $\rho_{\mathcal{Q}} \in \Lambda$, then

$$\mathcal{FT}(\mathcal{Q}) \;=\; \frac{1}{\mu_{G|M_\emptyset}}\;\Delta^2_{\mathcal{Q}}\;\left(\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q}),\epsilon}\right)^2 . \tag{8.8h}$$

(ii) If $\rho_{\mathcal{Q}} \notin \Lambda$, then

$$\mathcal{FT}(\mathcal{Q}) \;=\; \frac{1}{\mu_{G|M_\emptyset}}\;\sum_{1\le i,j\le m} \mathrm{sym}_{\mathrm{Stab}(\mathcal{Q})}(\tilde{f}_{v_i})\,\mathrm{sym}_{\mathrm{Stab}(\mathcal{Q})}(\tilde{f}_{v_j})\;\widetilde{\Delta}^2_{\mathcal{Q}}\;\left(\mathcal{R}[\mathcal{Q}]^{\mathrm{Stab}(\mathcal{Q}),\epsilon}\right)^2 . \tag{8.8i}$$

Proof. Apply (8.8e) and (8.8f). □

It is unknown to the authors whether, in the general situation, $\mathcal{Q}$ has a Stab($\mathcal{Q}$)-fixed character. If a Stab($\mathcal{Q}$)-fixed character exists, then Theorem 8.8g provides a very explicit description of the space $\mathcal{FT}(\mathcal{Q})$.

8.9. We make some final remarks.

(1) The definition of $\mathcal{D}_{\mathcal{Q}}$, $\rho_{\mathcal{Q}}$, and $D_{\mathcal{Q}}$ is based on the considerations of §7.7; in particular, on the root system $\Sigma(\mathcal{Q})$ and the semi-direct product $\mathrm{Stab}(\mathcal{Q}) = R(\mathcal{Q}) \ltimes W(\mathcal{Q})$. Instead of $\Sigma(\mathcal{Q})$, and $W(\mathcal{Q})$, we can also consider

$$\Sigma'(\mathcal{Q}) \;:=\; \{\ \alpha \in \Sigma_{\mathrm{red}}(A_\emptyset)\ \big|\ w_\alpha \in \mathrm{Stab}(\mathcal{Q})\ \} . \tag{8.9a}$$

As in §7.7, for $w \in W_\emptyset$, we have $w\,w_\alpha\,w^{-1} = w_{w(\alpha)}$; so, $\Sigma'(\mathcal{Q})$ is a Stab($\mathcal{Q}$)-invariant set of roots. Note that $\Sigma(\mathcal{Q}) \subset \Sigma'(\mathcal{Q})$. The objects defined in §7.7 for the root system $\Sigma(\mathcal{Q})$ have analogues for the root system $\Sigma'(\mathcal{Q})$. In particular, $\mathrm{Stab}(\mathcal{Q}) = R'(\mathcal{Q}) \ltimes W'(\mathcal{Q})$, with $R(\mathcal{Q}) \supset R'(\mathcal{Q})$ and $W(\mathcal{Q}) \subset W'(\mathcal{Q})$. The generalizations $\mathcal{D}'_{\mathcal{Q}}$, $D'_{\mathcal{Q}}$, and $\Delta'_{\mathcal{Q}}$ are divisible respectively by $\mathcal{D}_{\mathcal{Q}}$, $D_{\mathcal{Q}}$, and $\Delta_{\mathcal{Q}}$. Under the assumption that $\mathcal{Q}$ has a Stab($\mathcal{Q}$)-fixed character, which is true when G is F-split, we have analogues of Proposition 8.8d and Theorem 8.8g.

An example of this is $G = \mathrm{Sp(2n)}(F)$, with the standard realization of G as $2n \times 2n$ matrices. Let $d(c_1,\dots,c_n) \in G$ be the diagonal matrix with (i,i)-entry equal to c_i, let $\chi_2 \in \widehat{F^\times}$ be a ramified character of order two, and let $\mathcal{Q} \subset \widetilde{A_\emptyset}$ be the Bernstein component containing the character

$$d(c_1,\dots,c_n) \;\to\; \chi_2(\,c_1\cdot\ldots\cdot c_n\,) . \tag{8.9b}$$

Then, $\mathrm{Stab}(\mathcal{Q}) = W_\emptyset$. Here, $\Sigma'(\mathcal{Q}) = \Sigma(A_\emptyset)$, so $W'(\mathcal{Q}) = W_\emptyset$, and $\Sigma\,(\mathcal{Q})$ is the subset of short roots of $\Sigma(A_\emptyset)$. In particular, $W(\mathcal{Q})$ is isomorphic to the Weyl group of SO(2n) and it is of index two in $W_\emptyset$.

(2) Recall example 8.6 for $G = \mathrm{SL}(4)(F)$ is a situation in which $\rho_{\mathcal{Q}} = (\,\frac{3}{2}\,,\,\frac{1}{2}\,,\,-\frac{1}{2}\,,\,-\frac{3}{2}\,) \notin \Lambda$. The same is true for $\mathrm{GL}(4)(F)$, but in this situation we note there is an element, e.g., $z \in (\,\frac{1}{2}\,,\,\frac{1}{2}\,,\,\frac{1}{2}\,,\,\frac{1}{2}\,) \in \widetilde{\Lambda}$, which is fixed by Stab($\mathcal{Q}$) so that $\rho_{\mathcal{Q}} + z = (\,2\,,\,1\,,\,0\,,\,-1\,) \in \Lambda$.

More generally, suppose $\mathcal{Q}$ is a component satisfying the condition there exists $z \in \widetilde{\Lambda}$, which is fixed by $\mathrm{Stab}(\mathcal{Q})$, so that $\rho_{\mathcal{Q}} + z \in \Lambda$. Then, in (8.4d), we can replace $\rho_{\mathcal{Q}}$ by $\rho_{\mathcal{Q}} + z$ to define a function $\Xi_{\mathcal{Q}} \in \mathcal{R}[\Psi(M_\emptyset]$ which is $W(\mathcal{Q})$-skew, i.e., analogous to $\Delta_{\mathcal{Q}}$. Then, $\Xi_{\mathcal{Q}}^2$ is associate to $D_{\mathcal{Q}}$, and provided $\mathcal{Q}$ has a $\mathrm{Stab}(\mathcal{Q})$-fixed character, we can conclude the space $\mathcal{FT}(\mathcal{Q})$ is a principal ideal equal to $\mathcal{D}_{\mathcal{Q}}$ divided by the Plancherel density.

8. Notation index

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