ON AUTOMORPHIC DUALS AND ISOLATED REPRESENTATIONS; NEW PHENOMENA

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Abstract. In the paper we study automorphic duals of split classical groups in the non-archimedean case (defined in [Cl2]), and a relation between isolated points in the unitary and automorphic duals. Particular attention is devoted to the unramified unitary representations ([MuT]). In the unramified case, we study relation between the property of being automorphic (and isolated there), and intrinsic properties of representations. In the case of split classical groups we give combinatorial formulas for the number of isolated representations in the unramified unitary duals (these representations are also isolated representations in the automorphic duals), and for the number of so called strongly negative representations, which can be expected to be sets of isolated representations in the unramified automorphic duals. For the difference of special linear groups, we have plenty of both of these representations. We also discuss the cases of automorphic duals of general linear groups.

Introduction

Let $G(F)$ be the group of F-rational points of a semi simple group G defined over a local field F. One of big unsolved problems of harmonic analysis on $G(F)$ is classification of the set $\widehat{G(F)}$ of the equivalence classes of irreducible unitary representations of $G(F)$ (for general $G(F)$). This set is called the unitary dual of $G(F)$, and its classification turned out to be a surprisingly hard problem. If a maximal compact subgroup $K_{\max}$ in $G(F)$ is fixed, then we denote by $\widehat{G(F)}^{\mathbf{1}}$ the set of all classes in $\widehat{G(F)}$ which contain a non-zero vector invariant for the action of $K_{\max}$ (if F is non-archimedean, we shall always assume that maximal compact subgroup is special). This subset is called the unramified unitary dual of $G(F)$.

The unitary dual carries a natural topology, which is defined in terms of approximation of diagonal matrix coefficients on compact subsets (see the first section). Most mysterious part of $\widehat{G(F)}$ are isolated representations with respect to this topology (in the case of reductive G, a proper notion to consider is isolated representations modulo center; see the fourth section for the definition of this notion). Information about isolated representations is usually very important. For example, D. Kazhdan's result from [K], that the trivial representation of $G(F)$ is isolated, if the split rank of G is different from one, has very

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important arithmetic consequences. The main problem regarding isolated representations is that most standard constructions of the representation theory usually do not provide us with them, and that we do not know criteria to recognize from the representation itself, if it is isolated or not (without considering the neighborhood of the representation).

For a unitary representation Π, the support $\text{supp}(\Pi)$ of Π is defined to be the support of a measure that decomposes Π into a direct integral of elements of $\widehat{G(F)}$ (one can define also the support in terms of approximation of diagonal matrix coefficients on compact subsets; see the second section). Supports of particular representations may be very important, and we can again consider representations isolated in supports. For example, the support of $L^2(G(F))$ is called the reduced dual of $G(F)$. Significant part of the work of Harish-Chandra is related to the reduced duals. Here isolated representations are precisely square integrable casses (reduced dual consists of the tempered representations).

Let G be a semi simple group defined over a number field k (or more generally, over a global field k). Let v be a place of k, k_v the completion of k at v, and $\mathbb{A}_k$ the ring of adeles of k (recall that place of k is an equivalence class of absolute values on k; therefore in particular, v determines the number field k). The automorphic dual $\hat{G}_{v,\text{aut}}$ of G at v is defined to be the support of the representation of $G(k_v)$ on the space $L^2(G(k)\backslash G(\mathbb{A}_k))$ (by right translations). Unramified part of the automorphic dual will be denoted by $\hat{G}^{\mathbf{1}}_{v,\text{aut}}$. Following [BgLiSa], where the archimedean case was considered, we shall call unramified automorphic dual also Ramanujan dual. In this paper, when we say isolated automorphic representation, we shall mean isolated in the automorphic dual (it may not be isolated in the unitary dual).

Observe that we can have groups G and H defined over k which are not isomorphic, such that their k_v rational points are isomorphic, i.e. $G(k_v) \cong H(k_v)$. In principle, their automorphic duals $\hat{G}_{v,\text{aut}}$ and $\hat{H}_{v,\text{aut}}$ does not need to be the same, since in their definition different representations $L^2(G(k)\backslash G(\mathbb{A}_k))$ and $L^2(H(k)\backslash H(\mathbb{A}_k))$ of $G(k_v) \cong H(k_v)$ play role. Further, we can view group G defined over k, as a group defined over a finite extension k' of k. It can happen that we have places v of k and v' of k' such that $k_v \cong k'_{v'}$. Despite $G(k_v) \cong G(k'_{v'})$, in principle $\hat{G}_{v,\text{aut}}$ and $\hat{G}_{v',\text{aut}}$ does not need to be the same (as before, they are supports of different representations).

From the point of view of isolated representations in the unitary dual, automorphic duals are much more interesting then the reduced duals. For example, each isolated modulo center representation of $GL(n, k_v)$ is coming from the automorphic dual (clearly, it is isolated modulo center there), but in general, only very few isolated modulo center representations of $GL(n, k_v)$ belong to the reduced dual (see [T4] and [T5]).

Automorphic duals are very important objects already in the simplest cases, and we have not much complete answers even in such cases. For example, $\widehat{SL(2)}^{\mathbf{1}}_{\infty,\text{aut}}$ is not classified yet completely (∞ denotes the archimedean place of $\mathbb{Q}$). Neither we know the isolated points here. A. Selberg's $\frac{1}{4}$−conjecture suggests what the answers would be.

The primary goal of this paper is to recall what is known about unramified isolated representations in the unitary duals and in the automorphic duals of special linear groups at non-archimedean places, then what is not known but it is expected to hold for these groups, and to compare this with the corresponding information for classical split groups.

We shall see that there exists a sharp difference between these two cases. This paper may be viewed as a continuation of [MuT]. Very often we include proofs in the paper of facts related to the topology which should be known, but for which we could not find written proof.

Our experience suggests that, for study of automorphicity of representations in the unramified case, there is a very useful notion of negative and strongly negative representation. Negative and strongly negative representations have already shown very useful in some classifications: they are key representations in classifications in [Z], [Mu2] and [MuT]. We shall not present their technical definition here, but rather explain the property that they satisfy. The irreducible square integrable representations (modulo center) of a reductive group over a local field F can be characterized as irreducible representations with unitary central character, which satisfy (strict) inequalities of Casselman square integrability criterion in Theorem 4.4.6 of [Ca]. In [Mu2], G. Muić has defined strongly negative representations (resp. negative representations) as irreducible representations with unitary central character, which satisfy strict inequalities $>$ (resp. inequalities $\geq$), opposite to those ones which appear in the Casselman's square integrability criterion in Theorem 4.4.6 of [Ca].

We shall now consider the symplectic groups $Sp(2n)$ and the split odd-orthogonal groups $SO(2n+1)$ (see the first section for the definitions). If we fix a series of symplectic or odd-orthogonal groups, then the group of rank n will be denoted by G_n. We fix a number field k and a non-archimedean place v of k. The ring of integers of k_v will be denoted by $\mathcal{O}_{k_v}$. We shall view special linear groups and groups G_n as defied over k. In $SL(n,k_v)$ (resp. $G_n(k_v)$) we fix the maximal compact subgroup $SL(n,\mathcal{O}_{k_v})$ (resp. $G_n(\mathcal{O}_{k_v})$).

Now we shall start to compare the cases of special linear groups and classical groups.

(A) Number of known isolated unramified automorphic representations:

(1) The only presently known isolated unramified automorphic representation of $SL(n)$ at v, is the trivial representation $\mathbf{1}_{SL(n,k_v)}$ of $SL(n,k_v)$. So the number of presently known isolated unramified automorphic representations of $SL(n)$ is independent of the rank n (and equal to 1).

(2) The isolated unramified representations of $G_n(k_v)$ are classified in [MuT] (see Theorem 1.7 in the present paper). They are all automorphic (by Theorem 1.8 of G. Muić). We bring in this paper a combinatorial formula for the number of them. It is easy to see that the number of isolated automorphic unramified representations of groups G_n at v tends to the infinity as the rank n tends to the infinity.

To give an idea of the size of the sets of isolated unramified automorphic representations that one gets in this way, let us note that the number of isolated unramified representations of $SO(451,k_v)$ is 1 289 535 202 500. So we have in this case at least 1 289 535 202 500 isolated unramified automorphic representations.

Later we shall see how the following comparison is related to the automorphicity questions.

(B) Unramified isolated representations vs. strongly negative representations:

(1) The only isolated unramified representation of $SL(n,k_v)$ is the trivial representation $\mathbf{1}_{SL(n,k_v)}$, with exception $n \neq 2$ (then we do not have isolated unramified rep-

resentations). The only unramified strongly negative representation of $SL(n, k_v)$ is the trivial representation $\mathbf{1}_{SL(n,k_v)}$ of $SL(n, k_v)$. Therefore, these two sets coincide, except when $n = 2$.

(2) If $n \geq 1$, then the above two sets are always different for $G_n(k_v)$.

The set of isolated unramified representations is contained in the set of strongly negative representations. In general, isolated unramified representations can make very small portion of unramified strongly negative representations. For example, the number of unramified strongly negative representations of $SO(451, k_v)$ is 140 630 679 543 940. Therefore, the isolated unramified representations of the group $SO(451, k_v)$ form less than 1% of unramified strongly negative representations of this group.

Unramified irreducible strongly negative representations are interesting from the point of view of automorphicity for the following reason. In the case of $SL(n)$, the generalized Ramanujan conjecture implies that the only isolated unramified automorphic representation of $SL(n)$ at v would be the trivial representation (which is further the only irreducible unramified strongly negative representation). L. Clozel's "Arthur + ϵ" conjecture from [Cl2] (Conjecture 2 in [Cl2]) implies, after some considerations, that the set of isolated unramified automorphic representations of G_n at v, coincides with the set of irreducible unramified strongly negative representations in the case of groups $G_n(k_v)$. Therefore we can interpret now (B) in the following way:

(B′) Isolated unramified vs. isolated automorphic unramified representations (conjecturally):

(1) Assuming the generalized Ramanujan conjecture, the set of isolated unramified representations in the unitary dual of $SL(n, k_v)$ coincides with the set of isolated unramified automorphic representations of $SL(n)$ at v, except when $n = 2$.

(2) Assuming L. Clozel's "Arthur + ϵ" conjecture from [Cl2], the set of isolated unramified representations of groups G_n at v, forms in general a relatively small portion of the isolated unramified automorphic representations.

This implies, if we believe in the above two conjectures, that we may expect an intrinsic characterization of isolated representations in the unramified automorphic duals for the groups that we consider.

These are some of the topics that we consider in the paper. For these topics, it is important that the groups are split over k. One can get an idea of differences which happen in the non-split case from [Bd], [BdR], [Se] and [T5] ([T5] can be in a natural way generalized to the case of general linear groups over non-archimedean local division algebras).

Now we shall describe the content of the paper according to sections.

In the first section we introduce the notation that we use in the paper. We also present here the explicit classification of unramified unitary duals of classical split groups $SO(2n+1, F)$, $Sp(2n, F)$ and $O(2n, F)$ over a local non-archimedean field F, obtained in [MuT]. In the following section we shall use this classification. Presentation of the classification that we give here, is slightly different from [MuT] (this form may be more suitable for some applications in the theory of automorphic forms). In this section is also description of unramified isolated representations and strongly negative ones. These representations have a rather simple description (in terms of partitions). In particular, in this way we get a large families of isolated unramified automorphic representations, which have a pretty simple description.

The second section is devoted to general questions regarding unramified automorphic duals of classical split groups and isolated points there. First we observe that unramified negative representations are automorphic. Then we note that Clozel's "Arthur + ϵ" conjecture implies the opposite inclusion. Now assuming the Clozel's conjecture we can conclude that the isolated automorphic representations would be precisely the unramified strongly negative representations. At the end of this section we pose some natural questions.

In the third section we obtain combinatorial formulas for the numbers of unramified strongly negative representations, and of the isolated representations in the unitary duals of split classical groups over local non-archimedean field. Instead of presenting these formulas here, we present here a consequence of these formulas on a few examples. Below $\mathbf{I^1}(G(F))$ denotes the number of isolated unramified representations, and $\mathbf{SN^1}(G(F))$ denotes the number of unramified strongly negative representations of $G(F)$:

$G =$	$\mathbf{I^1}(G(F)) =$	$\mathbf{SN^1}(G(F)) =$	%
$Sp(4)$	1	2	50.00
$Sp(40)$	178	880	20.22
$Sp(340)$	11 322 187 942	586 385 730 874	1.93
$SO(5)$	2	3	66.67
$SO(41)$	266	1598	16.65
$SO(341)$	17 706 230 199	1 132 186 153 436	1.56
$O(340)$	10 859 296 005	559 481 612 686	1.94 .

In the third section one can also find lists of the numbers of isolated unramified representations and the numbers of unramified strongly negative representations up to rank 30. The description of unramified isolated representations and unramified strongly negative representations, and also the above table, suggest that it would be natural to expect

$$\lim_{n\to\infty} \frac{\mathbf{I^1}(G_n(F))}{\mathbf{SN^1}(G_n(F))} = 0.$$

In the fourth section we comment briefly the similar questions for the general linear groups (like in the second section for the split classical groups), with the difference that here we discuss the whole automorphic dual. Here we have much better understanding (but still some of these questions are extremely difficult even in the lowest ranks). Important notion in this section is the notion of rigid representation (all the exponents of the representations in the cuspidal support of such a representation, must be in $(1/2)\mathbb{Z}$; see the fourth section for precise definition). This notion, introduced by J. Bernstein, arises naturally in the work on unitarizability of representations of general linear groups (see [Bn]).

We expect similar situation in the archimedean case (the numbers that show up in this case for unramified representations are probably much lower).

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1. Notation and unramified unitary dual

In this section F denotes a local non-archimedean field of characteristic different from 2. The ring of integers in F will be denoted by $\mathcal{O}_F$. The uniformizing element of $\mathcal{O}_F$ will be denoted by ϖ_F. The normalized absolute value on F will be denoted by $|\ \ |_F$. Then $|\varpi_F|_F = \text{card}(\mathcal{O}_F/\varpi_F\mathcal{O}_F)^{-1}$.

We denote by ν the character $|\det|_F : GL(n,F) \to \mathbb{R}^\times$. Using the determinant homomorphism, we identify characters of $F^\times = GL(1,F)$ with characters of $GL(n,F)$. If φ is a character of $GL(n,F)$, then there exist a unique unitary character φ^u of $GL(n,F)$ and $e(\varphi) \in \mathbb{R}$ such that

$$\varphi = \nu^{e(\varphi)}\varphi^u.$$

Let π_i be a smooth representation of $GL(n_i,F)$, for $i = 1,2$. Let P be the maximal parabolic subgroup of $GL(n_1+n_2,F)$ containing upper triangular matrices, whose Levi subgroup is the group of quasi diagonal matrices $\{\text{quasi-diag}(g_1,g_2); g_i \in GL(n_i,F)\}$. The Levi subgroup is naturally isomorphic to $GL(n_1,F)\times GL(n_2,F)$. Then we denote by

$$\pi_1 \times \pi_2$$

the representation $\text{Ind}_P^{GL(n_1+n_2)}(\pi_1\otimes\pi_2)$ of $GL(n_1+n_2,F)$ parabolically induced by $\pi_1\otimes\pi_2$ from P (the induction that we consider is normalized, i.e. carries unitarizable representations to the unitarizable ones).

Denote by $J_n \in GL(n,F)$ the matrix $[\delta_{i,n+1-j}]_{1\leq i,j\leq n}$, where $\delta_{k,l}$ denotes the Kronecker symbol. Then the group of F-rational points of symplectic group of rank n is

$$Sp(2n,F) = \left\{ g \in GL(2n,F);\ g\cdot \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \cdot {}^t g = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\},$$

where ${}^t g$ denotes the transposed matrix of g (with respect to the main diagonal). Set

$$O(n,F) = \{g \in GL(n,F);\ {}^\tau g = g^{-1}\},$$

$$SO(n,F) = O(n,F) \cap SL(n,F),$$

where ${}^\tau g$ denotes the transposed matrix of g with respect to the second diagonal. We take $Sp(0,F)$ and $O(0,F)$ to be trivial groups, and consider their unique element formally as 0×0 matrix.

We shall fix one of the following three series of groups:

$$Sp(2n,F), n \geq 0,\ \ SO(2n+1,F), n\geq 0\ \text{ or }\ O(2n,F), n \geq 0,$$

and denote the corresponding group of split rank n by $G_n(F)$. We denote

$$n^* = \begin{cases} 2n+1 & \text{if } G_n(F) = Sp(2n,F); \\ 2n & \text{for other two series.} \end{cases}$$

By $N_\emptyset$ we denote the subgroup of all upper triangular unipotent matrices in $G_n(F)$. This is a maximal unipotent subgroup in $G_n(F)$.

In $GL(n,F)$ (resp. $SL(n,F)$) we fix the maximal compact subgroup $GL(n,\mathcal{O}_F)$ (resp. $SL(n,\mathcal{O}_F)$). We fix in $G_n(F)$ the maximal compact subgroup $K_{\max} = G_n(F) \cap GL(n',\mathcal{O}_F)$, where $n' = 2n+1$ if $G_n(F) = SO(2n+1,F)$ and $n' = 2n$ for other two series of groups. An irreducible representation (π, V) of $G_n(F)$, or $GL(n,F)$, or $SL(n,F)$, is called unramified if V contains a non-trivial vector invariant for the action of the maximal compact subgroup. Then the space of invariant vectors for the maximal compact subgroup is one dimensional.

For the group $G(F)$ of F-rational points of a reductive group defined over F, we denote the set of equivalence classes of irreducible smooth representations by $\widetilde{G(F)}$. The subset of unitarizable classes in $\widetilde{G(F)}$ is denoted by $\widehat{G(F)}$. If a maximal compact subgroup in G is fixed, then we denote by $\widetilde{G(F)}^{\mathbf{1}}$ the set of all unramified classes in $\widetilde{G(F)}$. We denote by $\widehat{G(F)}^{\mathbf{1}}$ the unramified classes in $\widehat{G(F)}$, and call it unramified unitary dual. The trivial (one-dimensional) representation of $G(F)$ will be denoted by $\mathbf{1}_{G(F)}$. The contragredient representation of π will be denoted by $\tilde{\pi}$. If π is a character, then $\tilde{\pi} \cong \pi^{-1}$.

Let $n_1, \ldots, n_l$ be positive integers and $m \geq 0$. Denote $n = n_1 + \cdots + n_l + m$. Let

$$M_{(n_1,\ldots,n_l,m)}(F) = \{\text{quasi-diag}(g_1,\ldots,g_l, s, {}^\tau g_l^{-1}, \ldots, {}^\tau g_1^{-1}); g_i \in GL(n_i,F), s \in G_m(F)\}.$$

This is a Levi subgroup in $G_n(F) = G_{n_1+\cdots+n_l+m}(F)$. We have obvious isomorphism

$$M_{(n_1,\ldots,n_l,m)}(F) \cong GL(n_1,F) \times \cdots \times GL(n_l,F) \times G_m(F). \tag{1.1}$$

For the maximal unipotent subgroup $N_\emptyset(F)$ in $G_{n_1+\cdots+n_l+m}(F)$, the group

$$P_{(n_1,\ldots,n_l,m)}(F) := M_{(n_1,\ldots,n_l,m)}(F)\, N_\emptyset(F)$$

is a parabolic subgroup in $G_{n_1+\cdots+n_l+m}(F)$, whose Levi factor is $M_{(n_1,\ldots,n_l,m)}(F)$.

Suppose that we have smooth representations π_i of $GL(n_i,F)$, $1 \leq i \leq l$, and a smooth representation σ of $G_m(F)$. Using the isomorphism (1.1), we can consider the representation $\pi_1 \otimes \cdots \otimes \pi_l \otimes \sigma$ as a representation of $M_{(n_1,\ldots,n_l,m)}(F)$. We shall denote by

$$\pi_1 \times \cdots \times \pi_l \rtimes \sigma$$

the representation $\mathrm{Ind}_{P_{(n_1,\ldots,n_l,m)}(F)}^{G_{n_1+\cdots+n_l+m}(F)}(\pi_1 \otimes \cdots \otimes \pi_l \otimes \sigma)$ of $G_{n_1+\cdots+n_l+m}(F)$ parabolically induced by $\pi_1 \otimes \cdots \otimes \pi_l \otimes \sigma$ from $P_{(n_1,\ldots,n_l,m)}(F)$.

From the other side, if we have a smooth representation π of $G_n(F)$, we denote by

$$\mathrm{Jacq}_{(n_1,\ldots,n_l,m)}(\pi)$$

the (normalized) Jacquet module of π with respect to $P_{(n_1,\ldots,n_l,m)}(F)$. It is a representation of $M_{(n_1,\ldots,n_l,m)}(F)$. If τ is an irreducible subquotient of $\mathrm{Jacq}_{(n_1,\ldots,n_l,m)}(\pi)$, using identification (1.1) we can write τ as $\tau_1 \otimes \cdots \otimes \tau_l \otimes \rho$, where τ_i are irreducible representations of $GL(n_i,F)$ and ρ is an irreducible representation of $G_m(F)$. Now we shall recall of some definitions from [Mu2]:

1.1. Definition. Let π be an irreducible unramified representation of $G_n(F)$. Then π is called negative if for any irreducible subquotient $\varphi = \varphi_1 \otimes \dots \varphi_n \otimes \mathbf{1}_{G_0(F)}$ of the Jacquet module $\mathrm{Jacq}_{(1,\dots,1,0)}(\pi)$ we have

$$\begin{aligned} &e(\varphi_1) \leq 0,\\ &e(\varphi_1) + e(\varphi_2) \leq 0,\\ &\qquad\vdots\\ &e(\varphi_1) + e(\varphi_2) + \ \dots\ + e(\varphi_n) \leq 0. \end{aligned}$$

Further, π will be called strongly negative if above we have always the strict inequalities.

1.2. Definition. (i) Let $m \in \mathbb{Z}_{>0}$. When $G_n(F) = SO(2n+1, F)$, we say that m has right parity if m is even. For remaining two series of groups we say that m has right parity if m is odd.
(ii) We call a pair

$$(m, \chi)$$

Jordan block if χ is an unramified character of $F^\times$ satisfying $\chi^2 \equiv 1$ (i.e., χ is selfdual - isomorphic to its contragredient), and $m \in \mathbb{Z}_{>0}$ has right parity.

By $\mathrm{Jord}_{\mathrm{sn}}(n)$ will be denoted the collection of all possible (finite) sets J which consist of Jordan blocks, such that

(1)
$$\sum_{(\chi,m)\in J} m = n^*;$$

(2) and additionally if $\chi \rtimes \mathbf{1}_{G_0(F)}$ reduces, the cardinality of

$$J(\chi) := \{m; (\chi, m) \ \in \ J\}$$

is even.

If we write $J(\chi)$ for $J \in \mathrm{Jord}_{\mathrm{sn}}(n)$, then χ will be always assumed to be unramified selfdual character of $F^\times$.

Denote by

$$\alpha_{\chi,1}$$

the unique non-negative real number such that $\nu^{\alpha_{\chi,1}}\chi \times \mathbf{1}_{G_0(F)}$ reduces. Define

$$J(\chi)' = \begin{cases} J(\chi), & \text{if } J(\chi) \text{ has even cardinality;} \\ J(\chi) \cup \{-2\alpha_{\chi,1} + 1\}, & \text{otherwise.} \end{cases}$$

To a character χ of $F^\times$ and $r_1, r_2 \in \mathbb{R}$ such that $r_2 - r_1 \in \mathbb{Z}$, we attach representation

$$\langle[\nu^{r_1}\chi, \nu^{r_2}\chi]\rangle := \nu^{(r_2+r_1)/2}\chi\ \mathbf{1}_{GL(r_2-r_1+1,F)}$$

if $r_2 \geq r_1$ (we use here Zelevinsky notation: $\langle[\nu^{r_1}\chi, \nu^{r_2}\chi]\rangle$ is characterized as a unique irreducible subrepresentation of $\nu^{r_1}\chi \times \nu^{r_1+1}\chi \times \cdots \times \nu^{r_2}\chi$). Otherwise, we take χ^{r_1,r_2} to be the trivial representation of the trivial group $GL(0,F)$ (we consider formally this group as 0×0 - matrices).

For $J \in \text{Jord}_{\text{sn}}(n)$ write $J(\chi)' = \{a_{2l_\chi}^{(\chi)}, a_{2l_\chi-1}^{(\chi)}, \dots, a_1^{(\chi)}\}$, where

$$a_{2l_\chi}^{(\chi)} > a_{2l_\chi-1}^{(\chi)} > \cdots > a_1^{(\chi)}$$

(if $J(\chi) = \emptyset$ we take $l_\chi = 0$). We define $\sigma(J)$ to be the unique irreducible unramified subquotient of

$$\left(\underset{\chi}{\times}\left(\underset{i=1}{\overset{l_\chi}{\times}} \langle[\nu^{-(a_{2i}^{(\chi)}-1)/2}\chi, \nu^{(a_{2i-1}^{(\chi)}-1)/2}\chi]\rangle\right)\right) \rtimes \mathbf{1}_{G_0(F)},$$

where the first product runs over (two) unramified selfdual characters of $F^\times$.

G. Muić in [Mu2] has proved the following explicit classifications of strongly negative and negative irreducible unramified representations:

1.3. Theorem (Mu2]). *(i) The mapping $J \mapsto \sigma(J)$ is a bijection from Jord$_{\text{sn}}(n)$ on the set of all strongly negative irreducible unramified representations of $G_n(F)$.*

(ii) Suppose $J \in$ Jord$_{\text{sn}}(m)$ and suppose that $\psi_1, \dots, \psi_l$ are unramified unitary characters of $GL(n_1,F), \dots, GL(n_l,F)$ respectively, such that $n_1 + \cdots + n_l + m = n$. Let π be the unique unramified irreducible subquotient (actually subrepresentation) of

$$\psi_1 \times \cdots \times \psi_l \rtimes \sigma(J).$$

Then π is an irreducible negative unramified representation of $G_n(F)$. Moreover, π determines J uniquely, and it determines characters $\psi_1, \dots, \psi_l$ up to a permutation and changes $\psi_i \leftrightarrow \psi_i^{-1}$. Further, each irreducible negative unramified representation of $G_n(F)$ is equivalent to some representation π as above.

1.4. Remark. For some purposes, it is sometimes more convenient the following description of $\text{Jord}_{\text{sn}}(n)$. Since there are exactly two selfdual unramified characters of $F^\times$, $\mathbf{1}_{F^\times}$ and $\mathbf{sgn}_{F^\times}$ (the non-trivial unramified character of order two), to $J \in \text{Jord}_{\text{sn}}(n)$ we attach the ordered pair

$$(J(\mathbf{1}_{F^\times}), J(\mathbf{sgn}_{F^\times})),$$

where we consider $J(\mathbf{1}_{F^\times})$ and $J(\mathbf{sgn}_{F^\times})$ as partitions. This pair determines J, and the partitions satisfy the following properties.

For a partition p of n into sum of k positive integers we shall write $|p| = n$ and $\text{card}(p) = k$. We shall write always members of partitions in descending order (not necessarily strictly).

In this way, $\text{Jord}_{\text{sn}}(n)$ (and irreducible unramified strongly negative representations of $G_n(F)$) are parameterized by pairs

$$(t, s),$$

where both t and s are partitions into different numbers of right parity, which satisfy $|t|+|s|=n^*$ and additionally

(1) for series $G_n(F)=O(2n,F)$, card(t), card$(s)\in 2\mathbb{Z}$;
(2) for series $G_n(F)=Sp(2n,F)$, card$(s)\in 2\mathbb{Z}$.

From [MuT] we get the following description of the unramified unitary dual:

1.5. Theorem. *(i) Let φ_i be unramified characters of $GL(n_i,F)$, such that $e(\varphi_i)>0$, for $i=1,\dots,m$, and let σ_{neg} be an irreducible negative unramified representation of $G_{n-n_1-\dots-n_m}(F)$ (we assume $n_1+\dots+n_m\le n$). Denote*

$$\pi=\varphi_1\times\dots\times\varphi_m\rtimes\sigma_{neg}.$$

For any φ showing up among $\varphi_1^u,\dots,\varphi_m^u$, denote by $\mathbf{e}_\pi(\varphi)$ the multiset of exponents $e(\varphi_i)$ for those i such that $\varphi_i^u\cong\varphi$, and suppose that the following conditions hold:

(1) $\mathbf{e}_\pi(\tilde\varphi)=\mathbf{e}_\pi(\varphi)$.
(2) If either $\varphi\not\cong\tilde\varphi$, or $\varphi=\tilde\varphi$ and $\nu^{\frac12}\varphi\rtimes\mathbf{1}_{G_0(F)}$ reduces, then $\alpha<\frac12$ for all $\alpha\in\mathbf{e}_\pi(\varphi)$.
(3) If $\tilde\varphi\cong\varphi$ and $\nu^{\frac12}\varphi\rtimes\mathbf{1}_{G_0(F)}$ is irreducible, then all exponents in $\mathbf{e}_\pi(\varphi)$ are <1. If we write $\mathbf{e}_\pi(\varphi)=\{\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_l\}$ in a way that

$$0<\alpha_1\le\dots\le\alpha_k\le\frac12<\beta_1\le\dots\le\beta_l<1,$$

then first $\beta_1<\dots<\beta_l$ (we can have $k=0$ or $l=0$). Further
(a) *$\alpha_i+\beta_j\ne 1$ for all $i=1,\dots,k$, $j=1,\dots,l$ and $\alpha_{k-1}\ne\frac12$ if $k>1$.*
(b) *card$\big(\{1\le i\le k:\alpha_i>1-\beta_1\}\big)$ is even if $l>0$.*
(c) *card$\big(\{1\le i\le k:1-\beta_j>\alpha_i>1-\beta_{j+1}\}\big)$ is odd for $j=1,\dots,l-1$.*
(d) *$k+l$ is even if $\varphi\rtimes\sigma_{neg}$ reduces.*

Then π is an irreducible unitarizable unramified representations of $G_n(F)$.

(ii) If we have an irreducible unitarizable unramified representation π of $G_n(F)$, then there exist $\varphi_1,\varphi_2,\dots,\varphi_m,\sigma_{neg}$ as in (i), which satisfy all the conditions in (i), such that

$$\pi\cong\varphi_1\times\dots\times\varphi_m\rtimes\sigma_{neg}.$$

Further, σ_{neg} and the multiset $(\varphi_1,\dots,\varphi_k)$ are uniquely determined by π up to equivalence.

To have an explicit classification, one needs to understand when $\nu^{\frac12}\varphi\rtimes\mathbf{1}_{G_0(F)}$ and $\varphi\rtimes\sigma_{neg}$ from above theorem reduce. Since in the above theorem φ is selfdual, we can write $\varphi=\langle[\nu^{-(p-1)/2}\chi,\nu^{(p-1)/2}\chi]\rangle$ where $p\in\mathbb{Z}_{>0}$ and χ is a selfdual unramified character of $F^\times$. Now the reducibility is described by the following results of G. Muić in [Mu2]:

1.6. Proposition. *Let*

$$\varphi = \langle [\nu^{-(p-1)/2}\chi, \nu^{(p-1)/2}\chi] \rangle,$$

where $p \in \mathbb{Z}_{>0}$ and χ is a selfdual unramified character of $F^\times$. Suppose that σ_{neg} is an (unramified) irreducible subrepresentation of some

$$\psi_1 \times \cdots \times \psi_s \rtimes \sigma(J),$$

where ψ_i are unitary unramified characters of general linear groups and $J \in Jord_{\mathrm{sn}}(q)$, $q \geq 0$. Then

(1) $\nu^{\frac{1}{2}}\varphi \rtimes \mathbf{1}_{G_0(F)}$ reduces if and only if $p+1$ has right parity;

(2) $\varphi \rtimes \sigma_{neg}$ reduces if and only if p has right parity, $(\chi, p) \notin J$ and $\varphi \notin \{\psi_1, \ldots, \psi_s\}$.

Theorem 1.5, together with above proposition, gives an explicit classification of unramified unitary duals of classical split groups. The expression in [MuT] is more direct. The expression that we present here is from an earlier version of our work, and may be more convenient for some purposes, in particular questions regarding automorphic forms.

The unitary dual $\widehat{G_n(F)}$ carries a natural topology: π is in the closure of X if and only if diagonal matrix coefficients of π on compact subsets can be approximated by finite sums of diagonal matrix coefficients from X (see [D] for more details, but also [T6] for other descriptions). Then the unramified part $\widehat{G_n(F)}^{\mathbf{1}}$ is an open subset of $\widehat{G_n(F)}$.

The isolated points in unitary dual are of particular importance (the most standard constructions of unitary representations usually do not produce isolated representations). They are very often crucial in construction of unitary duals.

From the classification Theorem 1.5 follows directly that if we have an isolated representation in the unramified unitary dual, then it is strongly negative. The isolated unramified representations in $G_n(F)$ are classified in [MuT]. They have the following simple description:

1.7. Theorem ([MuT]). *A representation π in the unitary dual of $G_n(F)$ is unramified and isolated, if and only if $\pi \cong \sigma(J)$ for some $J \in Jord_{\mathrm{sn}}(n)$, which satisfies for both unramified selfdual characters χ, the following conditions:*

(1) *$J(\chi) \cap \{2, 3\} = \emptyset$;*
(2) *no consecutive even or odd integers are contained in $J(\chi)$.*

At this point it is interesting to compare this result to the much more investigated (and understood) case of special linear groups. Here we fix maximal compact subgroup $SL(n, \mathcal{O}_F)$ in $SL(n, F)$. Then unramified representation $\pi \in SL(n, F)\hat{}$ is isolated if and only if $\pi \cong \mathbf{1}_{SL(n,F)}$ and $n \neq 2$.

One can define irreducible unramified strongly negative representations in a natural way for other reductive groups (converting the inequalities in Casselman square integrability criterion in Theorem 4.4.6 of [Ca]). Then unramified representation $\pi \in SL(n, F)\hat{}$ is strongly negative if and only if $\pi \cong \mathbf{1}_{SL(n,F)}$. Thus, in the case of special linear groups, being isolated and strongly negative is the same, with one exception. In the case of this exception, $1_{SL(2,F)}$ is also isolated, but in the (appropriate) automorphic dual (see the following section for definition of this notion). Therefore, for $SL(n,F)$ and unramified irreducible representations, the requirement of being strongly negative implies the requirement of being isolated in the automorphic dual.

At the end of this section we shall recall of a result of G. Muić ([Mu3]), which we shall need for understanding of automorphic duals in the following section.

1.8. Theorem ([Mu3]). *Assume char$(F) = 0$. We can take number field k and a place v such that the completion $k_v \cong F$. We consider G_n as a split group over k. Then every $\sigma = \sigma(J)$, $J \in Jord_{\mathrm{sn}}(n)$, is a local tensor factor of an irreducible subrepresentation of the representation of the adelic group $G_n(\mathbb{A}_k)$ in the space $L^2(G(k)\backslash G_n(\mathbb{A}_k))$.*

2. On unramified automorphic dual of classical groups and its isolated points

For a moment, let $\mathcal{G}$ be a locally compact group. One says that an irreducible unitary representation π of $\mathcal{G}$ is weakly contained in a unitary representation Π of $\mathcal{G}$ if diagonal matrix coefficients of π on compact subsets can be approximated by finite sums of diagonal matrix coefficients of Π (see [D] or [F] for more details).

Let now H be a semi-simple (or more generally, reductive) group defined over a global field k, and let v be a place of k. Denote by k_v the completion of k at v. The automorphic dual $\widehat{H}_{v,\mathrm{aut}}$ is defined to be the set of all equivalence classes of irreducible unitary representations of $H(k_v)$ which are weakly contained in the representation of $H(k_v)$ in the space $L^2(H(k)\backslash H(\mathbb{A}_k))$. If a maximal compact subgroup in $H(k_v)$ is fixed, then we denote the unramified representations in $\widehat{H}_{v,\mathrm{aut}}$ by $\widehat{H}^{\mathbf{1}}_{v,\mathrm{aut}}$.

For simplicity, we denote $F = k_v$. In the sequel, we shall restrict to the case

$$\mathrm{char}(k) = 0$$

(some of the results that we use latter, like Theorem 1.8 of G. Muić, are proved only for characteristic zero case).

Obviously, if π is isolated in $\widehat{H(F)}$ and $\pi \in \widehat{H}_{v,\mathrm{aut}}$, then π is isolated in $\widehat{H}_{v,\mathrm{aut}}$. L. Clozel has proved that in the case of simple connected semi-simple groups, $\mathbf{1}_{H(F)}$ is isolated in $\widehat{H}_{v,\mathrm{aut}}$ (for precise requirements on H see [Cl1], or 4.2 of [Cl2]).

Besides in [D], information regarding topological spaces $\widehat{H(F)}$ can be also found in [T1], [T6] and [T5]. Recall that the space $\widehat{H(F)}$ has countable basis of open sets ([D], 3.3.4), and that points in $\widehat{H(F)}$ are closed ([D], 4.4.1). By [T2], the topology of unitary duals of adelic reductive groups reduce to the topologies of unitary duals of local groups (for connected non-trivial groups in adelic case, we do not have isolated representations).

In the rest of this section, we shall consider only two out of the three series of groups introduced in the first section. We shall consider $Sp(2n)$ and $SO(2n+1)$, which we view as groups over number field k. We shall fix such a group and denote it by G_n. We shall often denote G_n also simply by G in the case that the rank of group does not play role in statements. We denote by

$$\widehat{G(F)}^{\mathbf{1}}_{\mathrm{neg}}$$

the set of all negative representations in the unramified dual $\widehat{G(F)}^{\mathbf{1}}$ of $G(F)$.

First we have the following proposition. The simple argument below is suggested by G. Muić (our arguing was more complicated).

2.1. Proposition.

$$\widehat{G(F)}^{\mathbf{1}}_{\mathrm{neg}} \subseteq \hat{G}^{\mathbf{1}}_{v,\mathrm{aut}}.$$

Proof. Let ψ_i be unramified unitary characters of $GL(n_i, F)$, and let σ be an irreducible strongly negative representation of $G_m(F)$. Each irreducible unramified negative representation of $G(F)$ is an irreducible subquotient of some $\psi_1 \times \psi_2 \times \cdots \times \psi_l \rtimes \sigma$, with ψ_i's and σ as above. Recall that by Theorem 1.8 of G. Muić, σ is a tensor factor of an irreducible subrepresentation of the representation of the adelic group in the space of square integrable automorphic forms.

Suppose for a moment that $\psi_1 \times \psi_2 \times \cdots \times \psi_l \rtimes \sigma$ is irreducible. Following Langlands description of automorphic spectra (Main Theorem of [A1]), one can associate to $\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_l \otimes \sigma$ extended to appropriate automorphic representation of the Levi subgroup over adeles, Eisenstein series. This will give a part of spectrum in the space of square integrable automorphic forms. This part of the spectrum is a direct integral, and in the support of this direct integral, considered as a representation of $G(F)$, is $\psi_1 \times \psi_2 \times \cdots \times \psi_l \rtimes \sigma$. Therefore, the last representation is automorphic.

If $\psi_1 \times \psi_2 \times \cdots \times \psi_l \rtimes \sigma$ is not irreducible, then we get that the irreducible unramified negative subquotient of it is automorphic from above observation and the fact that automorphic dual is closed (we can find irreducible representation $\psi_1' \times \psi_2' \times \cdots \times \psi_l' \rtimes \sigma$ as close to the reducible representation $\psi_1 \times \psi_2 \times \cdots \times \psi_l \rtimes \sigma$ as we want). □

2.2. Corollary. *If a negative representation is isolated in $\hat{G}^{\mathbf{1}}_{v,\mathrm{aut}}$, then it is strongly negative.*

Proof. Let $\pi \in \widehat{G(F)}^{\mathbf{1}}_{\mathrm{neg}}$. Suppose that π is not strongly negative. From the description of negative representations in (ii) of Theorem 1.3 and the reducibility criterion in (2) of Proposition 1.6 directly follow that we can find a sequence in $\widehat{G(F)}^{\mathbf{1}}_{\mathrm{neg}} \backslash \{\pi\}$ converging to π. Since by the previous proposition $\widehat{G(F)}^{\mathbf{1}}_{\mathrm{neg}} \subseteq \hat{G}^{\mathbf{1}}_{v,\mathrm{aut}}$, we see that π is automorphic, but not isolated in the automorphic dual. □

Now we shall recall very briefly of a conjecture of L. Clozel in the case of groups G_n (more details one can find in section 2.3 of [Cl2]). Let W_F be the Weil group of F, and let I_F be the inertia subgroup in W_F. Denote by frob_F a generator of $W_F/I_F \cong \mathbb{Z}$ (the quotient W_F/I_F naturally embeds in the Galois group of the algebraic closure of $\mathcal{O}_F/\varpi_F\mathcal{O}_F$ over $\mathcal{O}_F/\varpi_F\mathcal{O}_F$; then one can make the choice to take the inverse of $x \mapsto x^{\mathrm{card}(\mathcal{O}_F/\varpi_F\mathcal{O}_F)}$ for frob_F; but as L. Clozel notes in [Cl2], this is not important). If G is connected as an algebraic group, denote by ${}^LG^0$ the connected component of the dual group of G. Recall ${}^LSp(2n,F)^0 = SO(2n+1,\mathbb{C})$ and ${}^LSO(2n+1,F)^0 = Sp(2n,\mathbb{C})$. For $O(2n,F)$ define ${}^LO(2,F)^0 = O(2n,\mathbb{C})$.

Let π be an irreducible unramified representation of $G(F) = G_n(F)$. Then π is a subquotient of $\chi_1 \times \cdots \times \chi_n \rtimes \mathbf{1}_{G_0(F)}$ for some unramified characters χ_i of $F^\times$. Denote by
(2.1)
$$t_\pi = \operatorname{diag}(\chi_1(\varpi_F), \dots, \chi_n(\varpi_F), 1, \chi_n(\varpi_F)^{-1}, \dots, \chi_1({\varpi_F}^{-1})) \text{ if } G(F) = Sp(2n, F), \text{and}$$

$$t_\pi = \operatorname{diag}(\chi_1(\varpi_F), \dots, \chi_n(\varpi_F), \chi_n(\varpi_F)^{-1}, \dots, \chi_1({\varpi_F}^{-1})) \qquad \text{otherwise.}$$

Clearly $t_\pi \in {}^L G^0$. Further, t_π is determined by π up to a conjugation.

We recall now of Arthur unramified parameters. A homomorphism

$$\varphi : W_F \times SL(2, \mathbb{C}) \to {}^L G^0$$

is called unramified and isobaric of weight 0 if φ is trivial on I_F, if $\varphi | SL(2, \mathbb{C})$ is algebraic and if $\varphi(\mathrm{frob}_F)$ belongs to a maximal compact subgroup of ${}^L G^0$. Denote by

$$\widehat{G(F)}^{\mathbf{1}}_{\mathrm{Ar}}$$

the set of all classes π in $\widehat{G(F)}^{\mathbf{1}}$ for which there exists a parameter φ as above, such that

$$t_\pi = \varphi\left(\mathrm{frob}_F, \begin{bmatrix} |\varpi_F|_F^{1/2} & 0 \\ 0 & |\varpi_F|_F^{-1/2} \end{bmatrix}\right). \tag{2.2}$$

Now conjecture of L. Clozel (Conjecture 2 of [Cl2], which he calls "Arthur + ϵ") tells for our groups

2.3. Conjecture (L. Clozel).

$$\hat{G}^{\mathbf{1}}_{v,\mathrm{aut}} \subseteq \widehat{G(F)}^{\mathbf{1}}_{\mathrm{Ar}}.$$

We first observe that

2.4. Lemma.

$$\widehat{G(F)}^{\mathbf{1}}_{Ar} = \widehat{G(F)}^{\mathbf{1}}_{\mathrm{neg}}.$$

Proof. Theorem 0-4 of [MuT] implies that strongly negative unramified irreducible representations are in $\widehat{G(F)}^{\mathbf{1}}_{Ar}$. Let us comment this in more detail. Consider a strongly negative unramified irreducible representation $\pi = \sigma(s, t)$ (we use the notation as in Remark 1.4). Recall that we have the following parities: members of partitions t and s are even if we deal with odd-orthogonal groups, and odd otherwise. Further, card(s) is even if we deal with symplectic groups. Introduce t' and s' as in the introduction. Further, write $t' = (a_1, a_2, \dots, a_{2i})$, $a_1 > a_2 > \cdots > a_{2i}$, and $s' = (b_1, b_2, \dots, b_{2j})$, $b_1 > b_2 > \cdots > b_{2j}$, also as in the introduction.

Then $\sigma(t,s)$ is the unique irreducible unramified subquotient of the parabolically induced representation

$$\Big(\times_{l=1}^{i}\ \langle[\nu^{-(a_{2l}-1)/2}\mathbf{1}_{F^\times},\nu^{(a_{2l-1}-1)/2}\mathbf{1}_{F^\times}]\rangle\Big)\times \Big(\times_{l=1}^{j}\ \langle[\nu^{-(b_{2l}-1)/2}\mathbf{sgn}_{F^\times},\nu^{(b_{2l-1}-1)/2}\mathbf{sgn}_{F^\times}]\rangle\Big)\rtimes\mathbf{1}_{G_0(F)}.$$

Denote

$$\psi=(\bigoplus_{l\in t}(1_{W_F}\otimes E_l))\ \oplus\ (\bigoplus_{l\in s}(\mathbf{sgn}_{W_F}\otimes E_l)), \tag{2.3}$$

where E_l denotes the l-dimensional irreducible algebraic representation of $SL(2,\mathbb{C})$ and $\mathbf{sgn}_{W_F}$ denotes the unique character of order 2, which is trivial on I_F (observe that $\mathbf{sgn}_{W_F}(\mathrm{frob}_F)=-1$). Recall that (in a suitable basis)

$$E_l\left(\begin{bmatrix}|\varpi_F|_F^{1/2} & o\\ 0 & |\varpi_F|_F^{-1/2}\end{bmatrix}\right)=\mathrm{diag}(|\varpi_F|_F^{(l-1)/2},|\varpi_F|_F^{(l-1)/2-1},\dots,|\varpi_F|_F^{-(l-1)/2}). \tag{2.4}$$

Let $G(F)=SO(2n+1,F)$. Then one gets t' (resp. s') from t (resp. s) attaching 0 whenever $\mathrm{card}(t)$ (resp. $\mathrm{card}(s)$) is odd. Observe that in this case $\sigma(t,s)$ is the unique irreducible unramified subquotient of

$$\Big(\times_{l=1}^{2i}\ \langle[\nu^{1/2}\mathbf{1}_{F^\times},\nu^{(a_l-1)/2}\mathbf{1}_{F^\times}]\rangle\Big)\times \Big(\times_{l=1}^{2j}\ \langle[\nu^{1/2}\mathbf{sgn}_{F^\times},\nu^{(b_l-1)/2}\mathbf{sgn}_{F^\times}]\rangle\Big)\rtimes\mathbf{1}_{G_0(F)}. \tag{2.5}$$

Further, observe that the terms which are in t' but not in t, and in s' but not in s (i.e. 0's) do not have any contribution to the formula. Now the formula (2.1) for t_π, (2.5) and (2.4) imply that ψ from (2.3) applied to $\left(\mathrm{frob}_F,\begin{bmatrix}|\varpi_F|_F^{1/2} & o\\ 0 & |\varpi_F|_F^{-1/2}\end{bmatrix}\right)$, is conjugate of $t_{\sigma(t,s)}$.

Let $G(F)=Sp(2n,F)$. Here $s'=s$ and one gets t' from t attaching -1 to t. In this case $\sigma(t,s)$ is the unique irreducible unramified subquotient of

$$\Big(\times_{l=2}^{i}\ \big(\langle[\nu\mathbf{1}_{F^\times},\nu^{(a_{2l-1}-1)/2}\mathbf{1}_{F^\times}]\rangle\times\langle[\mathbf{1}_{F^\times},\nu^{(a_{2l}-1)/2}\mathbf{1}_{F^\times}]\rangle\big)\Big)\times\langle[\nu\mathbf{1}_{F^\times},\nu^{(a_2-1)/2}\mathbf{1}_{F^\times}]\rangle\times\Big(\times_{l=1}^{j}\ \big(\langle[\nu\,\mathbf{sgn}_{F^\times},\nu^{(b_{2l-1}-1)/2}\mathbf{sgn}_{F^\times}]\rangle\times\langle[\,\mathbf{sgn}_{F^\times},\nu^{(b_{2l}-1)/2}\mathbf{sgn}_{F^\times}]\rangle\big)\Big)\rtimes\mathbf{1}_{G_0(F)}. \tag{2.6}$$

One can now easily see that ψ from (2.3) applied to $\left(\mathrm{frob}_F,\begin{bmatrix}|\varpi_F|_F^{1/2} & o\\ 0 & |\varpi_F|_F^{-1/2}\end{bmatrix}\right)$, is conjugate of $t_{\sigma(t,s)}$.

So we have seen that each irreducible unramified strongly negative representation is coming from an Arthur parameter (in a sense of (2.2)).

Now we consider general negative unramified irreducible representations and how to get Arthur parameter for it. Let for the beginning π be a strongly negative representation as above, and let φ be an unramified unitary character of $GL(m,F)$. Denote by π' the unique irreducible unramified subquotient (actually a subrepresentation) of $\varphi \rtimes \pi$. Then, in a sense of (2.2), π' corresponds to the parameter

$$\varphi \otimes E_m \quad \oplus \quad \varphi^{-1} \otimes E_m \quad \oplus \quad \psi. \tag{2.7}$$

Continuing this procedure, we get that each irreducible unramified negative representation is coming from some Arthur parameter in the above sense. This proves one inclusion in the lemma.

To get opposite inclusion, one applies the reverse procedure. One takes a representation $\pi \in \widehat{G(F)}^{\mathbf{1}}$ and suppose that it corresponds to an Arthur parameter ψ'. Then using the fact that ψ' is selfdual, one makes decompositions like in (2.7) until it is possible (φ must be unitary). When we cannot perform more this step, we shall remain with a parameter corresponding to a strongly negative representation. In this way one gets that π must be negative. This completes the proof of the lemma. $\square$

Observe that Proposition 2.1 and above lemma imply

$$\widehat{G(F)}^{\mathbf{1}}_{Ar} \subseteq \hat{G}^{\mathbf{1}}_{v,\mathrm{aut}}. \tag{2.8}$$

Conjecture 2.3 of L. Clozel would imply equality. Further, assuming the same conjecture we can tell much more than in Corollary 2.2:

2.5. Proposition. *Conjecture 2.3 implies that the set of isolated unramified representations in the automorphic dual $\hat{G}^{\mathbf{1}}_{v,\mathrm{aut}}$ is equal to the set of all equivalence classes of irreducible unramified strongly negative representation of G.*

Proof. Let us recall a simple fact. A general fact is that all the unramified representations form an open subset of the unitary dual ([D], 3.3.2; see also 3.3.3 there).

We shall assume in the proof that Conjecture 2.3 holds. This conjecture implies that $\widehat{G(F)}^{\mathbf{1}}_{\mathrm{neg}} = \hat{G}^{\mathbf{1}}_{v,\mathrm{aut}}$. Now from Corollary 2.2 follows that an isolated representation in $\hat{G}^{\mathbf{1}}_{v,\mathrm{aut}}$ must be strongly negative. To complete the proof of the proposition, it is enough to prove that each strongly negative representation in $\widehat{G(F)}^{\mathbf{1}}_{\mathrm{neg}} = \hat{G}^{\mathbf{1}}_{v,\mathrm{aut}}$ is isolated in this set.

Suppose that there exists an irreducible strongly negative representation π which is not isolated in the automorphic dual. Then there exists a sequence of automorphic unramified unitary representations π_n converging to π, such that for all n we have $\pi_n \not\cong \pi$. First suppose that sequence π_n contains infinitely many strongly negative representations. Since there exists only finitely many such representations of $G(F)$, we would get that π_n contains a stationary subsequence converging to some strongly negative π'. This would imply $\pi \cong \pi'$ since a sequence can have at most one unramified limit. This is a contradiction.

This implies that the sequence π_n contains infinitely many irreducible negative representations, which are not strongly negative. Consider the subsequence consisting of all such representations. It is easy to see that in this subsequence, we can pass to a subsequence which converges to an unramified representation, which is negative, but not strongly negative. So again we get a contradiction, since a sequence can have only one unramified limit. The proof is now complete. □

At this point it may be natural to ask the following questions (some of which may be extremely hard), and check compatibility with L. Clozel Conjecture 2.3 (and [KLs]). The answers to the questions may not be positive (and some answers may be already known), but we hope that work on these questions may result with progress.

2.6. Questions. Let G be a simple Chevalley group, let p be a fixed prime and let $K_{\max} = G(\mathbb{Z}_p)$.

(1) Is each isolated representation in $\widehat{G(\mathbb{Q}_p)}^{\mathbf{1}}$ automorphic (i.e. in $\hat{G}_{p,\mathrm{aut}}$)?
(2) Is each negative representation in $\widehat{G(\mathbb{Q}_p)}^{\mathbf{1}}$ automorphic (in particular, is each strongly negative representation in $\widehat{G(\mathbb{Q}_p)}^{\mathbf{1}}$ automorphic)?
(3) Is each representation in $\widehat{G}^{\mathbf{1}}_{p,\mathrm{aut}}$ negative?
(4) Is each isolated representation in $\widehat{G}^{\mathbf{1}}_{p,\mathrm{aut}}$ strongly negative?
(5) Is each strongly negative representation isolated in $\widehat{G}^{\mathbf{1}}_{p,\mathrm{aut}}$?
(6) Is each isolated representation in $\widehat{G(\mathbb{Q}_p)}$ automorphic?

Condition that group is split, seems to be important for (6).

3. Numbers of unramified irreducible strongly negative and isolated representations of classical groups

First we shall introduce some simple combinatorial functions, which we shall use in this section. For $n \in \mathbb{Z}$ denote by

$$p(n)$$

the number of (unordered) partitions of n into positive integers. Each (unordered) partition can be write in a unique way as a sequence of decreasing (not necessarily strictly) integers. We shall always assume that partitions are written in this way. Fix $k \geq 0$. Let

$$p_k(n)$$

be the number of partitions of n into k pieces. By

$$p_{\leq k}(n)$$

we denote the number of partitions of n in at most k pieces. Clearly $p(n) = \sum_{i=0}^{n} p_i(n)$ and $p_{\leq k}(n) = \sum_{i=0}^{k} p_i(n)$.

For $j \geq 0$ define

$$p^{(j)}(n)$$

to be the number of all possible l-tuples $(m_1, \ldots, m_l) \in (\mathbb{Z}_{>0})^l$ for all $l \geq 0$, such that $m_i \geq m_{i+1} + j$ for $1 \leq i \leq l-1$ and $n = \sum_{i=1}^{l} m_i$. Note that $p^{(0)} = p$, and further, $p^{(1)}$ are partitions into different pieces.

For $d \geq 1$ define $p^{(j),d}(n)$ in the same way as $p^{(j)}(n)$, but requiring additionally that all $m_i \geq d$. Clearly $p = p^{(0),1}$ and $p^{(1)} = p^{(1),1}$.

Denote by

$$p_k^{(j)}(n) \text{ and } p_k^{(j),d}(n)$$

the number of partitions entering in the definitions of $p^{(j)}(n)$ and $p^{(j),d}(n)$, respectively, but into exactly k pieces. Analogously we define

$$p_{\leq k}^{(j)}(n) \text{ and } p_{\leq k}^{(j),d}(n)$$

(i.e., the numbers of corresponding partitions into at most k pieces)

We shall denote by

$$\mathbf{SN^1}(G_n(F))$$

the number of irreducible unramified strongly negative representations of $G_n(F)$, and by

$$\mathbf{I}^1(G_n(F))$$

the number of isolated representations in the unramified unitary dual (n is ≥ 0). Now we shall give combinatorial formulas for these numbers.

Case of $G_n(F) = SO(2n+1,F)$.

In this case, $\mathbf{SN}^1(SO(2n+1,F))$ is equal to the sum, when k runs from 0 to n, of the number of pairs (a_k, a_{n-k}) where a_i is a partition of $2i$ into different even positive integers.

Dividing members of partitions a_i by 2, we get a partition of i into different pieces, and we get all such partitions in this way. Therefore, the number of all such partitions a_i, which we shall denote by $\#\{a_i\}$, is $p^{(1)}(i)$. Thus

$$\mathbf{SN}^1(SO(2n+1,F)) = \sum_{k=0}^{n} p^{(1)}(k)p^{(1)}(n-k). \tag{3.1}$$

To get $\mathbf{I}^1(SO(2n+1,F))$, we need to sum, when k runs from 0 to n, the number of pairs (b_k, b_{n-k}) where b_i is a partition of $2i$ into even integers ≥ 4, such that all differences are at least 4. Denote the number of all such partitions b_i by $\#\{b_i\}$. Dividing members of such partitions b_i by 2, we get $\#\{b_i\} = p^{(2),2}(i)$. Thus

$$\mathbf{I}^1(SO(2n+1,F)) = \sum_{k=0}^{n} p^{(2),2}(k)p^{(2),2}(n-k).$$

Now we shall use that $p^{(2),2}(n) = \sum_{k=0}^{n} p_k^{(2),2}(n)$. Further, we subtract 1 from the last member of partitions corresponding to $p_k^{(2),2}(n)$, 2 from one member before, 3 from the following member etc. We get that $p_k^{(2),2}(n) = p_k^{(1),1}(n-k(k+1)) = p_k^{(1)}(n-k(k+$

1)). Now we do similar procedure as before, subtracting 0, 1, 2 etc., in reverse order, from partitions corresponding to $p_k^{(1),1}(n-k(k+1))$, and get that $p_k^{(1),1}(n-k(k+1)) = p_k^{(0)}(n-k(k+1)/2-k(k-1)/2) = p_k(n-k^2)$. Therefore $p_k^{(2),2}(n) = p_k(n-k^2)$. Thus $p^{(2),2}(n) = \sum_{k=0}^{n} p_k(n-k^2)$ (observe that $p_k(n-k^2) > 0$ implies $n-k^2 \geq k$; therefore $p^{(2),2}(n) = \sum_{k=0}^{[(-1+\sqrt{1+4n})/2]} p_k(n-k^2)$, where $[x]$ denotes the biggest integer not exceeding real number x).

The above calculation implies

$$\mathbf{I}^1(SO(2n+1,F)) = \sum_{k=0}^{n} \Big(\sum_{j=0}^{k} p_j(k-j^2) \Big) \Big(\sum_{j=0}^{n-k} p_j(n-k-j^2) \Big). \tag{3.2}$$

3.1. *Remark.* (i) The implementation of formula (3.1) in Mathematica program can be given by:

Table[Sum[PartitionsQ[i] PartitionsQ[$n-i$], $\{i,0,n\}$], $\{n,0,m\}$]

This will list numbers of unramified strongly negative representations of $SO(1,F)$,$SO(3,F)$, . . . , $SO(2m+1,F)$.

(ii) The implementation of formula (3.2) in Mathematica program can be given by:

Table[Sum[Sum[Length[IntegerPartitions[$n-k-j^2$, $\{j\}$]], $\{j,0,n-k\}$]
Sum[Length[IntegerPartitions[$k-j^2$, $\{j\}$]], $\{j,0,k\}$], $\{k,0,n\}$], $\{n,0,m\}$]

This will give numbers of unramified isolated representations of $SO(1,F)$, $SO(3,F)$, . . . , $SO(2m+1,F)$.

(iii) The following table gives comparison of the first 30 values of $\mathbf{SN}^1(SO(2n+1,F))$ and $\mathbf{I}^1(SO(2n+1,F))$ (starting from rank 1; in rank 0 both values are 1):

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\mathbf{SN}^1$	2	3	6	9	14	22	32	46	66	93	128	176	238	319	426	562	736
$\mathbf{I}^1$	0	2	2	3	4	7	8	13	16	23	28	40	48	66	82	107	132

and further

18	19	20	21	22	23	24	25	26	27	28	29	30
960	1242	1598	2048	2608	3306	4175	5248	6570	8198	10190	12622	15589
171	208	266	324	406	494	614	740	912	1098	1338	1604	1945.

Case of $O(2n, F)$.

Here $\mathbf{SN}^1(O(2n, F))$ is equal to the sum, when k runs from 0 to n, of the number of pairs (c_k, c_{n-k}) where c_i is a partition of $2i$ into even number of different odd positive integers.

Write such a partition of $2i$ into $2j$ pieces, where $0 \leq j \leq i$, in descending order. Subtract from the last term 1, one before 3, then 5 etc. Divide all obtained terms by 2. Then we get partitions of $i - 2j^2$ into at most $2j$ terms. From this easily follows that the number of all possible c_i's as above, which we denote by $\#\{c_i\}$, is $\#\{c_i\} = \sum_{j=0}^{i} p_{\leq 2j}(i - 2j^2)$. This implies

$$\mathbf{SN}^1(O(2n, F)) = \sum_{k=0}^{n} \Big(\sum_{j=0}^{k} p_{\leq 2j}(k - 2j^2) \Big) \Big(\sum_{j=0}^{n-k} p_{\leq 2j}(n - k - 2j^2) \Big). \tag{3.3}$$

To get $\mathbf{I}^1(O(2n, F))$ we need to sum, when k runs from 0 to n, the number of pairs (d_k, d_{n-k}) where d_i is a partition of $2i$ into even number of odd positive integers different from 3, such that all differences are at least 4. We shall split these partitions into two disjoint subsets: the ones containing 1, and the rest.

From a partition of $2i$ into $2j$ pieces from the first group, subtract $1, 5, 9, \dots, 1+4(2j-1)$ in reverse. Divide all the terms by 2. This is a partition of $i - 4j^2 + j$ into $\leq 2j - 1$ pieces.

From a partition of $2i$ into $2j$ from the second group, subtract $5, 9, \dots, 1 + 8j$ in reverse order. Divide all the terms by 2. This is a partition of $i - 4j^2 - 3j$ into $\leq 2j$ pieces.

Thus the number of all possible such d_i's (denoted by $\#\{d_i\}$) is

$$\#\{d_i\} = \sum_{j=1}^{i} p_{\leq 2j-1}(i - 4j^2 + j) + \sum_{j=0}^{i} p_{\leq 2j}(i - 4j^2 - 3j). \tag{3.4}$$

Therefore, we get

$$\begin{aligned} \mathbf{I}^1(O(2n, F)) = \sum_{k=0}^{n} & \Big(\sum_{j=1}^{k} p_{\leq 2j-1}(k - 4j^2 + j) + \sum_{j=0}^{k} p_{\leq 2j}(k - 4j^2 - 3j) \Big) \\ & \Big(\sum_{j=1}^{n-k} p_{\leq 2j-1}(n - k - 4j^2 + j) + \sum_{j=0}^{n-k} p_{\leq 2j}(n - k - 4j^2 - 3j) \Big). \end{aligned} \tag{3.5}$$

3.2. *Remark.* (i) The implementation of formula (3.3) in Mathematica program can be given by:

Table[Sum[Sum[Length[IntegerPartitions[$k - 2j^2, 2j$]], {$j, 0, k$}]
Sum[Length[IntegerPartitions[$n - k - 2j^2, 2j$]], {$j, 0, n - k$}], {$k, 0, n$}], {$n, 0, m$}]

This will list numbers of unramified strongly negative representations of groups $O(0, F)$, $O(2, F)$, $O(4, F)$, ... , $O(2m, F)$.

(ii) The implementation of formula (3.4) in Mathematica program can be given by:

Table[Sum[(Sum[Length[IntegerPartitions[$k - 4j^2 + j, 2j - 1$]], $\{j, 1, k\}$]+
Sum[Length[IntegerPartitions[$k - 4j^2 - 3j, 2j$]], $\{j, 0, k\}$])
(Sum[Length[IntegerPartitions[$n - k - 4j^2 + j, 2j - 1$]], $\{j, 1, n - k\}$]+
Sum[Length[IntegerPartitions[$n - k - 4j^2 - 3j, 2j$]], $\{j, 0, n - k\}$]), $\{k, 0, n\}$], $\{n, 0, m\}$]

It will list numbers of unramified isolated representations of $O(0, F)$, $O(2, F)$, $O(4, F)$, . . . , $O(2m, F)$.

(iii) The following table gives comparison of the first 30 values of $\mathbf{SN}^1(O(2n, F))$ and $\mathbf{I}^1(O(2n, F))$ (starting from rank 1; in rank 0 both values are 1):

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\mathbf{SN}^1$	0	2	2	5	6	11	14	24	30	46	60	87	112	157	202	275	352
$\mathbf{I}^1$	0	0	2	2	2	3	6	7	10	13	18	23	30	40	50	64	82

and further

18	19	20	21	22	23	24	25	26	27	28	29	30
469	596	781	986	1272	1598	2037	2540	3206	3976	4972	6132	7608
103	128	161	198	246	302	370	452	553	668	810	978	1177.

Case of $Sp(2n, F)$.

In the symplectic case, $\mathbf{SN}^1(Sp(2n, F))$ is equal to the sum, when k runs from 0 to n, of the number of pairs (c_k, e_{n-k}) where c_i is a partition of $2i$ into even number of different odd positive integers and e_i is a partition of $2i + 1$ into odd positive integers (clearly, such a partition must have odd number of pieces). From the previous case we know $\#\{c_i\} = \sum_{j=0}^{i} p_{\leq 2j}(i - 2j^2)$.

Write a partition of $2i + 1$ into $2j + 1$ different odd positive integers in a descending order, where $0 \leq j \leq i$. Subtract from the last term 1, one before 3, then 5 etc. Divide all obtained terms by 2. Now we get a partitions of $i-2j^2-2j$ into at most $2j+1$ terms. Thus the number of all possible such e_i's is $\#\{e_i\} = \sum_{j=0}^{i} p_{\leq 2j+1}(i - 2j^2 - 2j)$. This implies

$$\mathbf{SN}^1(Sp(2n, F)) = \sum_{k=0}^{n} \Bigg(\sum_{j=0}^{k} p_{\leq 2j}(k - 2j^2) \Bigg) \Bigg(\sum_{j=0}^{n-k} p_{\leq 2j+1}(n - k - 2j^2 - 2j) \Bigg). \tag{3.6}$$

To get $\mathbf{I}^1(Sp(2n, F))$, we need to sum, when k runs from 0 to n, the number of pairs (d_k, f_{n-k}) where d_i is a partition of $2i$ into even number of odd positive integers different from 3, such that all differences are at least 4 and and f_i is a partition of $2i + 1$ into odd positive integers (clearly, such a partition must have odd number of pieces) such that the differences are at least 4, and 3 does not show up in the partition. By (3.4) we have formula for $\#\{d_i\} = \sum_{j=1}^{i} p_{\leq 2j-1}(i - 4j^2 + j) + \sum_{j=0}^{i} p_{\leq 2j}(i - 4j^2 - 3j)$.

Now we shall consider partitions f_i. As before, we split these partitions into two disjoint subsets: the ones containing 1, and the rest.

From a partition of $2i+1$ into $2j+1$ terms from the first group, with $0 \leq j \leq i$, subtract $1, 5, 9, \dots, 1+8j$ in reverse order. Divide all obtained terms by 2. This is a partition of $i - 4j^2 - 3j$ into $\leq 2j$ pieces.

From a partition of $2i+1$ into $2j+1$ pieces from the second group, subtract $5, 9, \dots, 5+8j$ in reverse order. Divide all obtained terms by 2. This is a partition of $i - 4j^2 - 7j - 2$ into $\leq 2j+1$ pieces. Thus $\#\{f_i\} = \sum_{j=0}^{i} p_{\leq 2j}(i - 4j^2 - 3j) + \sum_{j=0}^{i} p_{\leq 2j+1}(i - 4j^2 - 7j - 2)$. Therefore, we get

$$\mathbf{I}^1(Sp(2n, F)) = \sum_{k=0}^{n} \Big(\sum_{j=1}^{k} p_{\leq 2j-1}(k - 4j^2 + j) + \sum_{j=0}^{k} p_{\leq 2j}(k - 4j^2 - 3j) \Big) \Big(\sum_{j=0}^{n-k} p_{\leq 2j}(n - k - 4j^2 - 3j) + \sum_{j=0}^{n-k} p_{\leq 2j+1}(n - k - 4j^2 - 7j - 2) \Big). \tag{3.7}$$

3.3. *Remark.* (i) The implementation of formula (3.6) in Mathematica program can be given by:

```
Table[Sum[Sum[Length[IntegerPartitions[k − 2j^2, 2j]], {j, 0, k}]
Sum[Length[IntegerPartitions[n − k − 2j^2 − 2j, 2j + 1]], {j, 0, n − k}], {k, 0, n}], {n, 0, m}]
```

This will list numbers of unramified strongly negative representations of $Sp(0, F)$, $Sp(2, F)$, $Sp(4, F)$, ... , $Sp(2m, F)$.

(ii) The implementation of formula (3.7) in Mathematica program is:

```
Table[Sum[(Sum[ Length[IntegerPartitions[k − 4j^2 + j, 2j − 1]], {j, 1, k}]+
Sum[Length[IntegerPartitions[k − 4j^2 − 3j, 2j]], {j, 0, k}])
(Sum[Length[IntegerPartitions[n − k − 4j^2 − 3j, 2j]], {j, 0, n − k}]+
Sum[ Length[IntegerPartitions[n − k − 4j^2 − 7j − 2, 2j + 1]], {j, 0, n − k}]),
{k, 0, n}], {n, 0, m}]
```

It will list numbers of unramified isolated representations of $Sp(0, F)$, $Sp(2, F)$, $Sp(4, F)$, ... , $Sp(2m, F)$.

(iii) The following table gives comparison of the first 30 values of $\mathbf{SN}^1(Sp(2n, F))$ and $\mathbf{I}^1(Sp(2n, F))$ (starting from rank 1; in rank 0 both values are 1):

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\mathbf{SN}^1$	1	2	3	6	8	13	18	27	37	53	71	100	132	179	235	313	405
$\mathbf{I}^1$	0	1	2	2	3	4	7	8	12	15	21	26	35	44	56	72	91

and further

18	19	20	21	22	23	24	25	26	27	28	29	30
531	681	880	1119	1429	1801	2280	2852	3575	4444	5529	6827	8436
114	143	178	219	273	333	409	499	609	735	892	1073	1292.

4. On automorphic dual of general linear group and its isolated points

In this section k is a number field, v is a non-archimedean place of k and $F = k_v$.

Let π be an irreducible representation of $GL(n,F)$. For a multiset $(\rho_1,\dots,\rho_l)$ of irreducible cuspidal representations of general linear groups we shall say that it is a cuspidal support of π, if π is a subquotient of $\rho_1\times\cdots\times\rho_l$. Each ρ_i can be written uniquely as $\nu^{e(\rho_i)}\rho_i^u$, where $e(\rho_i)\in\mathbb{R}$ and ρ_i^u are unitarizable (equivalently, have unitary central characters). The following definition is compatible with J. Bernstein definition of the rigid case in [Bn]:

4.1. Definition. We shall say that π is rigid if all $e(\rho_i)\in(1/2)\mathbb{Z}$.

We shall denote by $\widehat{GL(n,F)}_{rig}$ the set of all rigid representations in the unitary dual $\widehat{GL(n,F)}$.

Let us recall very briefly of the unitary dual of general linear groups (for more details see [T4]). To an irreducible square integrable modulo center representation δ of $GL(m,F)$, $m\ge 1$, and $n\ge 1$, we attach representation $u(\delta,n)$ of a general linear group, which is the unique irreducible quotient of the parabolically induced representation

$$\nu^{(n-1)/2}\delta\times\nu^{(n-1)/2-1}\delta\times\cdots\times\nu^{-(n-1)/2}\delta.$$

Consider further for $0<\alpha<1/2$ the representation

$$\pi(u(\delta,n),\alpha):=\nu^{\alpha}u(\delta,n)\times\nu^{-\alpha}u(\delta,n).$$

Then parabolically inducing tensor products of representations $u(\delta,n)$'s and $\pi(u(\delta,n),\alpha)$'s, we'll get all the (equivalence classes of) irreducible unitarizable representations of general linear groups.

Now it is obvious that $\widehat{GL(n,F)}_{rig}$ is the set of all classes of irreducible unitary representation which one gets parabolically inducing only tensor products of representations $u(\delta,n)$'s (and not $\pi(u(\delta,n),\alpha)$'s).

For an irreducible unitarizable cuspidal representation ρ of a general linear group and $m\ge 1$, the representation

$$\nu^{(m-1)/2}\rho\times\nu^{(m-1)/2-1}\rho\times\cdots\times\nu^{-(m-1)/2}\rho$$

has a unique irreducible subrepresentation. We denote it by

$$\delta(\rho,m).$$

This representation is square integrable modulo center, and one gets all such representations in this way.

In [T5] (and [T6]) we have defined notion of representation isolated modulo center in the unitary dual of a reductive group H: π is isolated modulo center if the set $\chi\pi$, when χ runs over the set $\mathrm{Unr}_u(H(F))$ of all unramified unitary characters of $H(F)$, is open in the unitary dual of $H(F)$. Denote by ω_π the central character of π. Let $\widehat{H(F)}_{\omega_\pi}=\{\tau\in$

$\widehat{H(F)}; \omega_\tau = \omega_\pi\}$. Then π is isolated modulo center if and only if π is isolated point of $\widehat{H(F)}_{\omega_\pi}$ (Lemma 6.2 of [T6]).

In the same way one defines isolated modulo center representations in the automorphic dual.

We have proved in [T5] that an irreducible unitary representation π of a general linear group is isolated modulo center in the unitary dual, if and only if it is isomorphic to some $u(\delta(\rho, n), m)$ as above, where $n \neq 2$ and $m \neq 2$. It follows from [J] that all representations $u(\delta, n)$ are automorphic. Therefore, all isolated modulo center representations in the unitary dual are also isolated modulo center in the automorphic duals. A result of W. Luo, Z. Rudnick and P. Sarnak, which W. Müller and B. Speh realized that holds in a grater generality, give us a possibility to extend this result in one direction:

4.2. Proposition. *Let ρ be a unitarizable irreducible cuspidal representation of $GL(d_\rho,F)$. Then the representation $u(\delta(\rho, m), l)$ is isolated modulo center in the automorphic dual $\widehat{GL(mld_\rho)}_{v,\text{aut}}$ for all $m \geq 1$ and $l \geq 1$.*

Proof. First we shall recall of an old fact regarding topology of unitary dual of the group of rational points $H(F)$ of a connected reductive group over a non-archimedean field F. Let ρ be an irreducible cuspidal representation of a Levi factor $M(F)$ of a parabolic subgroup $P(F)$ of $H(F)$. The set $\widetilde{H(F)}(\rho)$ of all irreducible subquotients of $\text{Ind}_{P(F)}^{H(F)}(\chi\rho)$, when χ runs over the set of all unramified characters of $M(F)$, is called a connected component (of $\widetilde{H(F)}$). Then $\widetilde{H(F)}(\rho) \cap \widehat{H(F)}$ is an open and closed subset of $\widehat{H(F)}$ ([T1]).

Therefore, if a sequence π_n converges to some π in $\widehat{H(F)}$, and π is a subquotient of some $\text{Ind}_{P(F)}^{H(F)}(\rho)$, where ρ is an irreducible cuspidal representation of a Levi factor $M(F)$ of a parabolic subgroup $P(F)$ of $H(F)$, then after finitely many indexes, all the terms of the sequence belong to the connected component $\widetilde{H(F)}(\rho)$.

To simplify notation, denote in this proof $G = GL(mld_\rho)$. Observe that for proving that $\pi \in \widehat{G(F)}$ is isolated modulo center in the automorphic dual, it is enough to prove that for each sequence π_n in $\hat{G}_{v,\text{aut}}$ converging to π, infinitely many π_n's belong to $\text{Unr}_u(G)\pi$. In this way, we shall prove that $u(\delta(\rho, m), l)$ is isolated modulo center.

Suppose that a sequence σ_n, $n \in \mathbb{Z}_{>0}$, contained in $\hat{G}_{v,\text{aut}}$, converges to $u(\delta(\rho, m), l)$. Passing to a subsequence, we can suppose that all the terms of the sequence belong to the connected component of $u(\delta(\rho, m), l)$. Write $\sigma_n = \sigma_n^{(1)} \times \sigma_n^{(2)} \times \cdots \times \sigma_n^{(r_n)}$, where $\sigma_n^{(i)}$ are representations of the form $u(\delta', p)$ or $\pi(u(\delta', p), \alpha')$. After passing to a subsequence, we can assume that all r_n are the same, and equal to some r.

Suppose that a sequence $\sigma_n^{(1)}$, $n \in \mathbb{Z}_{>0}$, contains infinitely many terms of the form $u(\delta', p)$. Passing to a subsequence of the sequence σ_n, we can assume that all the terms $\sigma_1^{(n)}$ are of that form. Write $\sigma_n^{(1)} = u(\delta(\rho_n, q_n), p_n)$. After passing to a subsequence of the sequence σ_n, we can assume that all p_n are the same, and all q_n are the same. Denote $p = p_n$ and $q = q_n$.

Further, since we are in the same connected component as $u(\delta(\rho, m), l)$, then $\rho_n \cong \chi_n\rho$ for some unramified character χ_n. Passing to a subsequence of the sequence σ_n, we can assume that the sequence ρ_n converges to some ρ' (it is enough to chose a convergent subsequence of the sequence χ_n). Denote $\sigma^{(1)} = u(\delta(\rho', q), p)$.

If the sequence $\sigma_n^{(1)}$, $n \in \mathbb{Z}_{>0}$, does not contain infinitely many terms of the form $u(\delta', p)$, then passing to a subsequence of the sequence σ_n, we can assume that all the terms are of the form $\pi(u(\delta', p), \alpha')$. Write $\sigma_n^{(1)} = \pi(u(\delta(\rho_n, q_n), p_n), \alpha_n)$. In the same way as above we chose p, q and ρ'. Passing to a subsequence of the sequence σ_n we can suppose that the sequence α_n converges to some α'. Clearly $0 \leq \alpha' \leq 1/2$. Denote in this case $\sigma^{(1)} = \nu^{\alpha'} u(\delta(\rho', q), p) \times \nu^{-\alpha'} u(\delta(\rho', q), p)$.

We continue this procedure with terms $\sigma_n^{(2)}$, and further with $\sigma_n^{(3)}$ etc. In this way we shall pass to a subsequence of the sequence σ_n, and get $\sigma^{(2)}, \sigma^{(3)}, \dots, \sigma^{(r)}$ in analogous way as $\sigma_n^{(1)}$. Now each limit of the sequence σ_n must be a subquotient of $\sigma^{(1)} \times \sigma^{(2)} \times \cdots \times \sigma^{(r)}$ (see more details regarding the topology in [T5]).

Since $u(\delta(\rho, m), l)$ is a limit, we get directly that $r = 1$. Now we have two possibilities. The first is that the sequence $\sigma_n = \sigma_n^{(1)}$, $n \in \mathbb{Z}_{>0}$, consists of terms of the form $u(\delta', p)$. Then $\sigma_n \cong u(\delta(\chi_n \rho, m), l)$ for some unramified characters χ_n, which must be unitary. Therefore, in this case we have obtained that the initial sequence, which converges to $u(\delta(\rho, m), l)$, contains infinitely many terms which are in the set of all $\chi u(\delta(\rho, m), l)$, when χ runs over all unitary unramified characters.

It remains to consider the case when the sequence $\sigma_n = \sigma_n^{(1)}$, $n \in \mathbb{Z}_{>0}$, consists of terms of the form $\pi(u(\delta', p), \alpha')$ (with finitely many exceptions). Then $u(\delta(\rho, m), l)$ must be a subquotient of $\sigma^{(1)} = \nu^{\alpha'} u(\delta(\rho', q), p) \times \nu^{-\alpha'} u(\delta(\rho', q), p)$. First observe that α' cannot be 0. Further, we cannot have $0 < \alpha' < 1/2$. So, the only remaining possibility is $\alpha' = 1/2$. Therefore, there exists a sequence $\pi(u(\delta(\chi_n \rho, q), p), \alpha_n)$ in the automorphic dual such that χ_n is a sequence of unramified unitary characters converging to the trivial character and α_n converges to $1/2$ (recall $0 < \alpha_n < 1/2$).

For proving that representations $u(\delta(\rho, m), l)$ are isolated modulo center in the automorphic dual, it would be enough to prove that representations $\pi(u(\delta(\chi_n \rho, q), p), \alpha_n)$ cannot be automorphic when we come close enough to $1/2$ with α_n. By Langlands description of automorphic spectra (Main Theorem of [A1]), global automorphic representation come in two ways: as representations in the discrete spectrum and as Eisenstein series from discrete spectra of proper Levi subgroups.

First suppose that infinitely many of $\pi(u(\delta(\chi_n \rho, q), p), \alpha_n)$'s are coming from the closure of the local factors of discrete part of the spectrum. But observe that representations $\pi(u(\delta(\chi_n \rho, q), p), \alpha_n)$, with n big enough (which implies that α_n are close to $1/2$), cannot be in the closure of such representations, since by Propositions 3.3 and 3.4 of [MüSp], absolute values of exponents coming with local factors of the form $u(\delta', \ell)$, can go up to some fixed constant strictly smaller then $1/2$. Because of this, infinitely many of $\pi(u(\delta(\chi_n \rho, q), p), \alpha_n)$'s cannot be in the closure of the local factors of discrete part of the spectrum.

It remains to consider the case when infinitely many of the (complementary series) representations $\pi(u(\delta(\chi_n \rho, q), p), \alpha_n)$ are in the closure of the continuous spectrum. By Langlands description of continuos spectrum, we obtain this part by integrating Eisenstein series associated to discrete spectra of proper Levi subgroups. To this spectra we can apply again Propositions 3.3 and 3.4 of [MüSp] (see above). It follows from this that infinitely many of $\pi(u(\delta(\chi_n\rho, q), p), \alpha_n)$'s cannot be in the closure of the local factors of the continuous part of the spectrum. This ends the proof. □

A direct consequence of the above lemma is the following fact: if we have a rigid representation which is isolated modulo center in the automorphic dual, then it is isomorphic to some $u(\delta, n)$.

4.3. *Remarks.* Let us recall two related facts interesting for study of isolated representations in automorphic duals.

(i) By Lemma 4 of [V] we have

$$\widehat{GL(n,F)}_{rig} \subseteq \widehat{GL(n)}_{v,\mathrm{aut}}. \tag{4.1}$$

(ii) The trivial representation of $SL(n, F)$ is isolated in the unramified automorphic dual (if $n \neq 2$, this follows from [K], and for $n = 2$ this follows from [Cl1]).

In the moment we do not know much more regarding isolated points in the automorphic duals. But we can tell much more if we assume the **generalized Ramanujan conjecture** (for general linear groups), which tells that all the local tensor factors of irreducible cuspidal automorphic representations should be tempered.

4.4. Lemma. *Assume that the generalized Ramanujan conjecture holds. Then*

(1)
$$\widehat{GL(n,F)}_{rig} = \widehat{GL(n)}_{v,\mathrm{aut}}.$$

(2) $\pi \in \widehat{GL(n,F)}$ *is isolated modulo center in* $\widehat{GL(n)}_{v,\mathrm{aut}} \iff \pi$ *is isomorphic to* $u(\delta, m)$ *for some* δ *and* m.

(3) *Unramified* $\pi \in \widehat{SL(n,F)}$ *is isolated in* $\widehat{SL(n)}_{v,\mathrm{aut}} \iff \pi \cong \mathbf{1}_{SL(n,F)}$.

Recall that without assuming generalized Ramanujan conjecture, we know that in (1) holds inclusion $\subseteq$, and in (2) and (3) implications from the right to the left (the last implication, among others, follows from [Cl1]; see also [K]).

Proof. We need to prove remaining inclusion and implications.

(1) The comment after Lemma 4 in [V] tells that generalized Ramanujan conjecture would imply equality (this is a consequence of Langlands description of automorphic spectra).[1]

(2) One directly sees that if a representation in $\widehat{GL(n,F)}_{rig}$ is not of the form $u(\delta, n)$, then it is not isolated modulo center (in $\widehat{GL(n,F)}_{rig}$). This and (1) give required implication.

(3) We shall briefly outline the proof of remaining implication in this case (we need only to consider the case $n \geq 2$). Suppose that $\pi \not\cong \mathbf{1}_{SL(n,F)}$ is in the unramified automorphic

[1] The generalized Ramanujan conjectures would follow from equalities in (1) at all places (including archimedean).

dual of $SL(n,F)$. Let $A(F)$ be the subgroup of diagonal matrices in $GL(n,F)$, and let $Z(F)$ be the center of $GL(n,F)$. Take an unramified character χ of $A(F)\cap SL(n,F)$ such that π is a subquotient of $\mathrm{Ind}_{P^{SL}_{\min}(F)}^{SL(n,F)}(\chi)$ ($P^{SL}_{\min}(F)$ denotes the minimal parabolic subgroup in $SL(n,F)$ consisting of upper triangular matrices).

We can extend χ to an unramified character χ_1 of $A(F)$. Let φ' be the unramified character of $F^\times$ satisfying $\varphi'(\varpi_F)^n = \chi_1^{-1}(\mathrm{diag}(\varpi_F,\varpi_F,\dots,\varpi_F))$. Denote by $\varphi_1=\varphi'\circ\det : A(F)\to\mathbb{C}^\times$ and $\chi^\# = \chi_1\varphi_1$. Then $\chi^\#$ extends χ, and $\chi^\#(\mathrm{diag}(\varpi_F,\varpi_F,\dots,\varpi_F)) = 1$. Therefore, $\chi^\#$ is unitary on $Z(F)$.

Let $\pi^\#$ be an irreducible unramified subquotient of $\mathrm{Ind}_{P^{GL}_{\min}(F)}^{GL(n,F)}(\chi^\#)$ ($P^{GL}_{\min}(F)$ denotes the minimal parabolic subgroup in $GL(n,F)$ consisting of upper triangular matrices). By the choice of $\chi^\#$, $\pi^\#$ has a unitary central character. Recall a well known fact $\mathrm{Ind}_{P^{GL}_{\min}(F)}^{GL(n,F)}(\chi^\#)|SL(n,F)\cong \mathrm{Ind}_{P^{SL}_{\min}(F)}^{SL(n,F)}(\chi)$ (the isomorphism is given by restriction). Recall that $\pi^\#|SL(n,F)$ is a direct sum of finitely many irreducible representations (see [T7] among others). It obviously contains an irreducible unramified representation of $SL(n,F)$. Since π is a subquotient of $\mathrm{Ind}_{P^{SL}_{\min}(F)}^{SL(n,F)}(\chi)$, and π is the only unramified irreducible subquotient, we conclude that $\pi\hookrightarrow\pi^\#|SL(n,F)$ (the last representation is a subquotient of $\mathrm{Ind}_{P^{GL}_{\min}(F)}^{GL(n,F)}(\chi^\#)|SL(n,F)\cong \mathrm{Ind}_{P^{SL}_{\min}(F)}^{SL(n,F)}(\chi)$). Now by (i) of Proposition 2.7 in [T7], $\pi^\#$ is unitarizable (recall that $\pi^\#$ has unitary central character).

Now by Remark 1 of [V], $\pi^\#$ is automorphic representation of $GL(n,F)$. Since $\pi^\#$ is unramified, we know that $\pi^\#$ is fully induced (from a parabolic subgroup) by a tensor product of unramified unitary characters and of complementary series starting with unramified unitary characters (see [T3]). Since we assume the generalized Ramanujan conjecture, we can use (1) of Lemma 4.1. Therefore, $\pi^\#$ is fully induced by a tensor product of unramified unitary characters only. Thus $\pi^\#\cong\chi_1\times\chi_2\times\cdots\times\chi_\ell$, where $\ell\geq 1$ and χ_i is an unramified unitary character of $GL(n_i,F)$, $n_i\geq 1$, $1\leq i\leq\ell$. If $\ell=1$, then $\pi\cong\mathbf{1}_{SL(n,F)}$. Since π is non-trivial, $\ell\geq 2$.

Let ϕ_i, $i\geq 1$, be a sequence of non-trivial unramified unitary characters of $GL(n_1,F)$ converging to $\mathbf{1}_{GL(n_1,F)}$. Let π_i be an irreducible unramified subquotient of the representation $\big((\phi_i\chi_1)\times\chi_2\times\cdots\times\chi_\ell\big)|SL(n,F)$. Then π_i are automorphic by Remark 1 of [V] and Lemma 4 of [V] (which is (i) of Remarks 4.3). Further, the sequence π_i converges to π. Since the sequence π_i contains infinitely many nonequivalent representations, π is not isolated in the automorphic dual. This completes the proof. □

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