REDUCIBILITY AND DISCRETE SERIES IN THE CASE OF CLASSICAL p-ADIC GROUPS; AN APPROACH BASED ON EXAMPLES

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Dedicated to Professor Takayuki Oda on the occasion of his 60th birthday.

Abstract. In this paper, we first present the basic ideas of the method of determining reducibility or irreducibility of parabolically induced representations of classical p-adic groups using Jacquet modules. After that we explain the construction of irreducible square integrable representations by considering characteristic examples. We end with a brief presentation of the classification of irreducible square integrable representations of these groups modulo cuspidal data, which was obtained jointly with C. Mœglin.

Contents

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Introduction

Professor Oda invited me in 2007 to the University of Tokyo, to give several lectures presenting the main ideas of my work on reducibility of parabolically induced representations and discrete series of classical p-adic groups. This paper is based on the notes of lectures given during that visit. We are very thankful to Professor Oda for the interest that he expressed in our work, and for explaining to us the relationship of our work to his.

This paper has two objectives. The first objective of this paper is to explain how Jacquet modules can be used for analyzing reducibility of parabolically induced representations of classical groups over non-archimedean local fields, based on examples. The second objective is to explain, also based on examples, ideas in the classification of irreducible square integrable representations of these groups modulo cuspidal data. This classification has been obtained jointly with C. Mœglin.

Although this paper deals with problems from the representation theory of p-adic groups, the questions that we consider have at least some number-theoretic relevance. Namely, non-archimedean local fields show up as completions of number fields (in characteristic zero), and we shall say something regarding this a little bit later. In general, it is well known that harmonic analysis on the groups that we consider in this paper have a very strong relationship to some aspects of number theory. For example, Plancherel measures are expressed by the local L-functions etc.

The full power of the methods and results that we are presenting in this paper can be found in a number of already published ones. The method of applying Jacquet modules for determining reducibility of parabolically induced representations and their composition series was introduced in [39]. This method has been significantly improved by C. Jantzen (he studied the degenerate principal series; see [14] and several of his other papers) and by G. Muić (he studied generalized principal series in [27] and standard representations in [28]). The classification of irreducible square integrable representations of classical p-adic groups modulo cuspidal data is covered by [21] and [24] (we also have several other papers giving more precise information about some classes of square integrable representations). In general, these papers are very technical. Rather then trying to present these technically very complicated aspects of results in general case, we put an emphasis on important characteristic examples, which are relatively simple in comparison with the general case, but still illustrate the methods and results, sometimes better than the final results. Our experience is that we often understand new topics much more easily in this way. This is the reason that this paper may be useful not only to those working in the area of representation theory of reductive p-adic groups, but also to those whose primary interest is automorphic forms.

For examining reducibility questions (in particular related to the tempered setting) and square integrability, there are other (mainly analytic) methods. We shall not study them in this paper, but shall briefly comment on some of them on a few occasions. These methods are less elementary than the methods that we present in this paper. They are based on standard intertwining operators (and their normalizations), Plancherel measures (and L-functions), R-groups etc. (see [12], [25], [26], [31], [32] and [33] among others). They play a very important role in C. Mœglin's paper [21] on the exhaustion of square integrable representations, which is closely related to the topic of our paper.

This paper has three main parts. In the first part "Representations of classical p-adic groups and structures built on them" (sections 1. to 14.)

we present general facts related to the study of reducibility of parabolically induced representations of classical p-adic groups, and to the construction of irreducible square integrable representations of these groups. We start the second part "Jacquet modules methods of determining reducibility of parabolically induced representations" (sections 15. to 27.) with relatively simple examples of determining reducibility or irreducibility of specific parabolically induced representations. Later on, we consider more complicated cases, and study their composition series. In this way we obtain some square integrable subquotients. We further study parabolically induced representations related to the square integrable representations that we have obtained. In this way we notice some phenomena which are used in the third part "On irreducible square integrable representations and their parameters" (sections 28. to 35.), where invariants (defined by C. Mœglin) are attached to irreducible square integrable representations which classify them modulo cuspidal data (and a natural assumption). In this part of the paper we present characteristic examples of irreducible square integrable representations (we emphasize that these invariants classify them modulo cuspidal data and a natural assumption). We end the paper with a general formulation of the classification of irreducible square integrable representations modulo cuspidal data.

The reader familiar with general facts of representation theory of reductive p-adic groups can skip the first 8 sections, and go directly to section 9. Those who are also familiar with the Bernstein-Zelevinsky theory can go directly to section 14. Very basic ideas regarding studying reducibility of parabolically induced representations are contained in sections 15. - 22. The reader more interested in reducibility questions can read the remaining sections 23. - 27. for additional information and examples. The reader more interested in square integrability can skip these five sections and go directly to section 28.

We are thankful to C. Jantzen for a very careful reading of the paper, and a number of suggestions which have significantly improved the style of the paper. I. Matić has also passed through the whole paper, and made a numerous corrections. The referee gave a number of useful comments. Discussions with M. Hanzer, A. Moy and G. Muić on some aspects of this paper, helped us to better understand some topics and explain them in the paper. We are thankful to all of them. The first version of this paper was written when we were visiting the University of Tokyo. We are thankful to the Univiersity for its hospitality.

Representations of classical p-adic groups and structures built on them

1. Historical observations

Induced representation are the main object of the study in this paper. Let us recall that they were present in representation theory from the very beginning in the development of the subject. Namely, the notion of a representation of a finite group was introduced by Frobenius in 1897 (the notion of a character of a representation had already existed). The following year Frobenius introduced induced representations (and Frobenius reciprocity, which plays a very important role in our paper).

A representation of a group G is a homomorphism π of G into the group of linear isomorphisms of a complex vector space V (we shall consider only complex representations here; representations over other fields are also very important). For a representation (σ, U) of a subgroup P of a finite group G, the representation of G **induced** by σ is the representation of G on the space $\mathrm{Ind}_P^G(\sigma)$ of all functions $f : G \to U$ which satisfy

$$f(pg) = \sigma(p)f(g), \qquad \forall p \in P, \forall g \in G. \tag{1.1}$$

The group G acts by the right translations $R_g f(x) = f(xg)$. The most basic fact regarding induced representations is **Frobenius reciprocity**:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \cong \mathrm{Hom}_P(\pi|_P, \sigma), \tag{1.2}$$

where π is a representation of G ($\pi|_P$ denotes the restriction to P of the representation π).

Induced representations play a very important role in the construction of irreducible representations ((π, V) is irreducible if $\{0\}$ and V are the only subspaces of V invariant for all $\pi(g)$, $g \in G$). For example, one can get from (simple) one-dimensional representations of a subgroup, such as the trivial representation, interesting non-trivial representations of the group.

It is interesting to note that induction did not play a role in the construction of irreducible representations of compact Lie groups (induced representations here are too big). Also, it did not play a role in the case of commutative groups, since irreducible representations are one-dimensional in this case. We note that in both cases induced representations were studied (at least induced representations by trivial representations of subgroups).

Induction played a very important role in the next phase of development - harmonic analysis on groups, especially harmonic analysis on reductive groups over local fields. What is particularly important here is a special case of induction, parabolic induction. I. M. Gelfand and M. A. Naimark were the ones who were first aware of the importance of these representations for harmonic analysis in this case. They obtained the first important results about them, which were later extended to more general cases by Harish-Chandra and others.

2. Smooth representations and the unitary dual

The fundamental object for harmonic analysis on a locally compact group G is the **(unitary) dual** $\hat{G}$ of G, i.e., the set of equivalence classes of (topologically) irreducible unitary representations (on Hilbert spaces) of G (a unitary representation is a continuous homomorphism π of G into the group of unitary operators on a Hilbert space H; a unitary representation is irreducible if $\{0\}$ and H are the only closed subspaces of H invariant for all $\pi(g)$, $g \in G$). See [38] for more discussion regarding harmonic analysis on groups.

We shall consider (linear) reductive groups over non-archimedean local fields. Thanks (mostly) to the work of Harish-Chandra and J. Bernstein, the problem of studying the irreducible unitary representations can be algebraized (and we can forget Hilbert space representations and continuity conditions): it reduces to smooth representations. We shall now recall the definition.

Reductive groups over non-archimedean local fields are locally compact and totally disconnected. A locally compact totally disconnected group G has a neighborhood of the identity consisting of open compact subgroups. A representation (π, V) of G is called **smooth** if the stabilizer of v in G is open, for each $v \in V$. The set of all equivalence classes of irreducible smooth representations of G will be denoted by $\tilde{G}$, and called the **non-unitary dual** of G (the term non-unitary that we have used here is to stress that we do not require unitarity in the definition of $\tilde{G}$; it does not mean that $\tilde{G}$ consists of non-unitary representations only). Another (maybe better) choice would be to call $\tilde{G}$ smooth dual (or admissible) dual of G.

In what follows, by a representation, we shall mean a smooth representation.

A smooth representation is called **unitarizable** if there exists a G-invariant inner product on V. If π is irreducible, such an inner product is unique up to a positive scalar (this follows by Schur's lemma). Denote by $\hat{G}$ the subset of all unitarizable classes in $\tilde{G}$. Then $\hat{G}$, defined in this way is in a canonical bijection with the unitary dual of G defined above.

The above reduction is enabled by the following fundamental fact proved in [2] by J. Bernstein: each irreducible unitary representation of G on a Hilbert space H is admissible, i.e. for each open compact subgroup K of G, the K-invariants H^K form a finite dimensional (complex) vector space.

All representations in $\tilde{G}$ are also admissible (this fact was proved by H. Jacquet). A smooth representation which is admissible will be called an admissible representation. It is easy to show that unitarizable admissible representations are always semisimple.

3. The non-unitary dual and unitary dual

Attempts to get the classification of $\hat{G}$ directly did not produce results (for general G as above) and it was much easier to construct elements of $\tilde{G}$. These facts motivated the following strategy for classifying $\hat{G}$, due to Harish-Chandra. The strategy has two steps:

- classify $\tilde{G}$ (the **problem of non-unitary dual**);
- identify unitarizable classes in $\tilde{G}$, i.e., determine the subset $\hat{G} \subset \tilde{G}$ (the **unitarizability problem**).

In these lectures we shall talk only about the first problem, the problem of the non-unitary dual.

A general framework for handling the first problem is offered by the Langlands classification. Some definitions are necessary to explain this classification. Square integrable representations and parabolic induction play crucial role in it.

4. Square integrable representations

Let G be a reductive group over a local non-archimedean field F. One important example is $G = GL(n, F)$, and $F = \mathbb{Q}_p$, i.e., the field of p-adic numbers.

Let (π, V) be a smooth representation of G. On the space V' of linear forms on V there exists a natural representation: $(\pi'(g)v')(v) = v'(\pi(g^{-1})v)$. The set of all linear forms with open stabilizer is a (smooth) subrepresentation, which we denote by $(\tilde{\pi}, \tilde{V})$, and call the **contragredient** of (π, V). We denote by $\bar{\pi}$ the complex conjugate representation of a representation π.

Functions of the form $g \mapsto \tilde{v}(\pi(g)v), G \to \mathbb{C}$, for $v \in V, \tilde{v} \in \tilde{V}$, are called **matrix coefficients** of the representation (π, V).

Denote by $Z(G)$ the center of G. The group $G/Z(G)$ (as well as G) is unimodular. When we talk about integrability, we mean with respect to the invariant measure. Schur's lemma implies that for each smooth irreducible representation (π, V) there is a character ω_π of $Z(G)$ such that $\pi(z) = \omega_\pi(z)\,\mathrm{id}_V$ for all $z \in Z(G)$. The character ω_π is called the **central character** of π.

Definition. *An irreducible representation (π, V) of G is called* **square integrable modulo center** *if*

(1) *the central character of π is a unitary character;*
(2) *the absolute values of all the matrix coefficients of π are square integrable functions on $G/Z(G)$.*

If the center of G is compact, square integrable modulo center representations will be simply called square integrable. Sometimes, if the center is not compact, we also abbreviate square integrable modulo center representations by square integrable.

It is easy to show that these representations are unitarizable (the G-invariant inner product is given by $(v_1, v_2) \mapsto \int_{G/Z(G)} \tilde{v}(\pi(g)v_1)\overline{\tilde{v}(\pi(g)v_2)}dg$, where $\tilde{v} \in \tilde{V}\backslash\{0\}$ is fixed). We denote by $D_u(G)$ the set of all such classes (of representations) in $\hat{G}$.

A smooth representation (π, V) of G is called **essentially square integrable** if it becomes square integrable modulo center after twisting by some character of G. The set of all such representations will be denoted by $D(G)$.

Square integrable representations play a very important role in a number of questions, from several points of view (not only in harmonic analysis).

For example, irreducible square integrable representations are very distinguished representations from the functional-analytic point: if the center of G is compact, they can be characterized as irreducible subrepresentations of $L^2(G)$ (the representation of G on $L^2(G)$ defined by right translations).

On the other side, irreducible essentially square integrable representations of reductive groups are also particularly interesting to understand from the point of view of the modern theory of automorphic forms. Namely, the **local Langlands correspondence** for a split semi simple group G predicts the existence of a natural partition of $\tilde{G}$ into finite sets of representations

$$\tilde{G} = \bigcup_{\varphi} \Pi_{\varphi},$$

where φ runs over all conjugacy classes of continuous homomorphisms of $W_F \times SL(2,\mathbb{C})$, where W_F denotes the **Weil group** of F [1], into the complex **dual Langlands L-group** ${}^LG^0$ (see below), such that φ maps elements of the first factor to semi simple elements and is algebraic on the second factor (such homomorphisms will be called **admissible homomorphisms**). Elements of Π_{φ} are called **L-packets** (since all representations in an L-packet should have the same L-functions). The mapping which sends $\pi \in \Pi_{\varphi}$ to φ will be denoted by Φ_G, or simply by Φ when it is evident which correspondence we consider. This mapping is called the **local Langlands correspondence** for G.

For establishing local Langlands correspondences, the case of square integrable representations is crucial. For them, there should exist a partition

$$D(G) = \cup_{\varphi} \Pi_{\varphi},$$

where φ runs over all admissible homomorphisms whose images are not contained in any proper Levi subgroup (such homomorphisms will be called **discrete admissible homomorphisms**).

Further, elements in a square integrable L-packet Π_{φ} should be parameterized (roughly) by the equivalence classes of irreducible representations of the **component group**

$$\mathrm{Cent}_{{}^LG^0}(\varphi(W_F \times SL(2,\mathbb{C})))/\mathrm{Cent}_{{}^LG^0}(\varphi(W_F \times SL(2,\mathbb{C}))^0\ Z({}^LG^0).$$

Above, $\mathrm{Cent}_{{}^LG^0}X$ denotes the centralizer of $X \subseteq {}^LG^0$, while $(\mathrm{Cent}_{{}^LG^0}X)^0$ denotes the connected component of the identity.

Instead of going into the general definition of ${}^LG^0$ (which involves root data and dual root data), let us indicate what ${}^LG^0$ is in the cases of interest

[1] The Weil group is a dense subgroup of the absolute Galois group G_F of F (i.e. of the Galois group over F of its separable algebraic closure). More precisely, denote by $\mathcal{O}_F$ the maximal compact subring of F and by $\mathfrak{p}_F$ its unique maximal ideal. Consider the natural homomorphism $r : G_F \to G_{\mathcal{O}_F/\mathfrak{p}_F}$. Let $<\mathrm{Frob}>$ be the subgroup generated by the Frobenius automorphism $\mathrm{Frob} \in G_{\mathcal{O}_F/\mathfrak{p}_F}$. Then $W_F = r^{-1}(<\mathrm{Frob}>)$. The topology on W_F is that for which (the inerta group) $\mathrm{Ker}(r)$ gets the topology from G_F, and $\mathrm{Ker}(r)$ is open in W_F.

of us:

$$
\begin{aligned}
{}^{L}GL(n,F)^0 &= GL(n,\mathbb{C}),\\
{}^{L}Sp(2n,F)^0 &= SO(2n+1,\mathbb{C}),\\
{}^{L}SO(2n+1,F)^0 &= Sp(2n,\mathbb{C}).
\end{aligned}
$$

In the first case, the component groups are trivial, while in the remaining two cases the component groups are isomorphic to direct sums of copies of $\mathbb{Z}/2\mathbb{Z}$ (therefore, L-packets should have cardinalities which are always a power of 2). In the case of $GL(n,F)$, discrete admissible homomorphisms are just continuous irreducible representations of $W_F \times SL(2,\mathbb{C})$ which are semi simple on the first factor and algebraic on the second factor.

Later on, we shall also see the importance of square integrable representations for the classification of non-unitary duals.

5. Parabolic subgroups

We shall fix a **maximal split torus** $A_\emptyset$ in G and a **minimal parabolic subgroup** $P_\emptyset$ in G containing $A_\emptyset$. Subgroups containing $P_\emptyset$ will be called **standard parabolic subgroups**. Let P be a standard parabolic subgroup. The Levi decomposition $P = MN$ (with N unipotent) will be called a **standard Levi decomposition** if M contains $A_\emptyset$.

In the case of $G = GL(n,F)$, we shall take for $P_\emptyset$ (resp., $A_\emptyset$) the subgroup of upper triangular (resp., diagonal) matrices in G. Let $\alpha = (n_1,\dots,n_l)$ be an ordered partition of n. Denote by M_α^{GL} (resp., N_α^{GL}) block-diagonal (resp., unipotent block-upper triangular) matrices of type $(n_1,\dots,n_l)$ in G. Then $P_\alpha^{GL} := M_\alpha^{GL}N_\alpha^{GL}$ is a standard parabolic subgroup, and this is its standard Levi decomposition. In this way we obtain a bijection of ordered partitions of n onto standard parabolic subgroups of G. The Levi subgroup M_α^{GL} is isomorphic to $GL(n_1,F)\times\cdots\times GL(n_l,F)$ in a natural way.

When we consider the classical groups $Sp(2n,F)$ or $SO(n,F)$, we fix maximal split tori and minimal parabolic subgroups obtained by intersection with the corresponding subgroups from general linear groups. A similar situation occurs with standard parabolic subgroups and standard Levi decompositions.

6. Parabolic induction, tempered representations and the Langlands classification

Let $P = MN$ be a parabolic subgroup of G. Denote by δ_P the **modular character** of P. Fix a smooth representation (σ, U) of M. On the space $\mathrm{Ind}_P^G(\sigma)$ of all locally constant functions $f : G \to U$ satisfying

$$
f(mng) = \delta_P(m)^{1/2}\sigma(m)f(g), \quad \forall m\in M, \forall n\in N, \forall g\in G, \tag{6.1}
$$

the group G acts by right translations:

$$
(R_g f)(x) = f(xg).
$$

This defines a smooth representation of G, which is called the representation of G **parabolically induced** by σ from P. Parabolic induction carries unitarizable representations to unitarizable ones (this is the reason to have the modular character $\delta_P^{1/2}$ in the requirement (6.1)).

Definition. *An irreducible smooth representation τ of G is called* **tempered** *if there exists a parabolic subgroup $P = MN$ and an irreducible square integrable modulo center representation δ of M such that*

$$\tau \hookrightarrow Ind_P^G(\delta).$$

The **Langlands classification** (or maybe more appropriately, the Langlands parameterization), starts with a triple (P, τ, χ), where $P = MN$ is a standard parabolic subgroup of G, τ is an irreducible tempered representation of M, where $P = MN$ is standard Levi decomposition of P, and χ is a positive valued character of M satisfying a certain positivity condition (which we shall illustrate in several examples). To such a triple one attaches the irreducible quotient of $\mathrm{Ind}_P^G(\chi\tau)$ (the irreducible quotient is unique and it has multiplicity one in the whole induced representation). In this way $\tilde{G}$ is parameterized by triples (P, τ, χ) as above.

Example - The case of general linear groups. *We shall fix on F the absolute value satisfying $d(ax) = |a|_F dx$ (i.e. the* **modulus character***), where dx is an invariant measure on the additive group of F. We shall denote by ν the character*

$$\nu : g \mapsto |det(g)|_F$$

of $GL(n, F)$.

Define

$$D = \cup_{n=1}^{\infty} D(GL(n, F)), \quad D_u = \cup_{n=1}^{\infty} D_u(GL(n, F)).$$

Let $M(D)$ be the set of all finite multisets in D (in multisets, multiplicities greater than 1 are allowed). For $\delta \in D$ there exists a unique $e(\delta) \in \mathbb{R}$ and $\delta^u \in D_u$ such that

$$\delta \cong \nu^{e(\delta)} \delta^u.$$

For

$$d = (\delta_1, \dots, \delta_l) \in M(D),$$

take a permutation p of $\{1, \dots, l\}$ such that $e(\delta_{p(1)}) \geq \cdots \geq e(\delta_{p(l)})$. Then the representation

$$Ind(\delta_{p(1)} \otimes \cdots \otimes \delta_{p(l)})$$

(of corresponding general linear group, parabolically induced from the corresponding standard parabolic subgroup) has a unique irreducible quotient, which we denote by

$$L(d).$$

This is the Langlands classification for general linear groups:

$$d \mapsto L(d) \text{ is a bijection of } M(D) \text{ onto } \cup_{n=0}^{\infty} (GL(n, F))\tilde{}.$$

The key fact that enables the above description of the Langlands classification, is the irreducibility of tempered parabolic induction for general linear groups (proved by H. Jacquet, among others). The above description of the Langlands classification is very much in the spirit of the local Langlands correspondences for general linear groups.

We have a similar descriptions of the Langlands classification for other classical groups (we shall give that description later).

Note that the Langlands classification reduces the problem of classifying the non-unitary dual to the following two problems:

(i) classification of irreducible square integrable modulo center representations of Levi factors;

(ii) understanding reducibility of representations parabolically induced by irreducible square integrable modulo center representations of Levi factors.

Having in mind (the formulation of) the Langlands classification, it is important to understand reducibility of much more general (and non-unitarizable) parabolically induced representations then in (ii) (at least in some very important situations).

These two problems - classification of irreducible square integrable representations of classical groups and understanding of irreducibility/reducibility of parabolically induced representations - will be the main topic of the rest of our paper.

Remark. *We shall not discuss the unitarizability problem in this paper. Let us note that having a good understanding of both of the above problems is very important for unitarizability. For working on unitarizability one needs to have a more explicit understanding of non-unitary dual than the general form of the Langlands classification. Reducibility of parabolically induced representations is related to the size of the unitary dual. For one thing, it is related to the existence of complementary series.*

Before we discuss how one can deal with the above problems, we recall some basic facts that we shall need for dealing with that problems.

7. Parabolic induction - basic facts

We take a moment to recall some general facts regarding parabolic induction (one can find more information in [3] and [4]):

(1) Fix a parabolic subgroup P of G and its Levi subgroup M. The process of attaching the parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ of G to a representation σ of M lifts in a natural way to a functor, which is called the **functor of parabolic induction**. This functor is **exact**, and it carries representations of finite length of M to the representations of finite length of G.

(2) Parabolic induction commutes with contragredients, i.e.,

$$\mathrm{Ind}_P^G(\sigma)\tilde{}\cong \mathrm{Ind}_P^G(\tilde{\sigma}).$$

(3) **Induction in stages**: If $P_1 = M_1N_1$ and $P_2 = M_2N_2$ are standard parabolic subgroups with standard Levi decompositions such that $P_1 \subseteq P_2$, and if σ is a smooth representation of M_1, then for parabolically induced representations we have

$$\mathrm{Ind}_{P_1}^G(\sigma) \cong \mathrm{Ind}_{P_2}^G(\mathrm{Ind}_{P_1\cap M_2}^{M_2}(\sigma)).$$

(4) **Induction from associate representations and associate parabolic subgroups**: If $P_1 = MN_1$ and $P_2 = MN_2$ are two (not necessarily standard) parabolic subgroups and σ is a smooth representation of M of finite length, then $\mathrm{Ind}_{P_1}^G(\sigma)$ and $\mathrm{Ind}_{P_2}^G(\sigma)$ have the same composition series.

Two parabolic subgroups $P_1 = M_1N_1$ and $P_2 = M_2N_2$ are called **associate** if M_1 is a conjugate of M_2.

8. Jacquet modules

Frobenius reciprocity for parabolically induced representations holds in the same form as for the finite groups. However, there is a big problem: that the restriction is in general a very big representation of the subgroup (usually of infinite length). From the other side, parabolic induction is induction from P, but with the representation **trivial on** N. Using this fact, we easily get the form of Frobenius reciprocity below.

For a representation (π, V) of a group G and a parabolic subgroup $P = MN$ of G, let

$$V(N) = \mathrm{span}_{\mathbb{C}}\{\pi(n)v - v; v \in V, n \in N\}.$$

Since M normalizes N, this space is M-invariant. Denote by

$$r_M^G(\pi)$$

the quotient representation of M on $V/V(N)$, twisted by $(\delta_P|_M)^{-1/2}$. Then $r_M^G(\pi)$ is called **Jacquet module** of π with respect to $P = MN$.

Now, **Frobenius reciprocity** becomes

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \cong \mathrm{Hom}_M(r_M^G(\pi), \sigma) \tag{8.1}$$

for representations π and σ of G and M respectively.

The fundamental properties of Jacquet modules are listed below (see [3] and [4] for more information):

(1) Jacquet modules lift in a natural way to a functor, which is called the **Jacquet module functor**. This functor is **exact** and it carries representations of finite length of G into representations of finite length of M.

(2) **Transitivity of Jacquet modules**: If $P_1 = M_1N_1$ and $P_2 = M_2N_2$ are standard parabolic subgroups with standard Levi decompositions, such that $P_1 \subseteq P_2$, and π a smooth representation of G, we have

$$r^G_{M_1}(\pi) \cong r^{M_2}_{M_1}(r^G_{M_2}(\pi)).$$

(3) **Jacquet module of the contragredient representation**: Let π be a smooth representation of G of finite length, and let $P = MN$ be a standard parabolic subgroup (with standard Levi decomposition). Denote by $\bar{P} = M\bar{N}$ the opposite parabolic subgroup (this is a parabolic subgroup satisfying $P \cap \bar{P} = M$). Then $r^G_M(\tilde{\pi})$ is isomorphic to the contragredient representation of the Jacquet module of π with respect to $\bar{P} = M\bar{N}$.

Suppose that for an irreducible representation π of G we have $r^G_M(\pi) \neq \{0\}$ for some (standard) proper parabolic subgroup $P = MN$ of G. Then Frobenius reciprocity implies that π embeds into $\mathrm{Ind}^G_P(\sigma)$ for some irreducible representation σ of M. Taking P minimal with property $r^G_M(\pi) \neq \{0\}$, we would get that π embeds into $\mathrm{Ind}^G_P(\sigma)$, where σ satisfies

$$r^M_{M'}(\sigma) = \{0\}$$

for all proper parabolic subgroups $P' = M'N'$ of M.

Definition. *An irreducible representation π of G for which*

$$r^G_M(\pi) = \{0\}$$

holds for all proper parabolic subgroups $P = MN$ of G is called **cuspidal**. *Otherwise, we say that representation is non-cuspidal.*

This definition obviously opens the following strategy for classifying $\tilde{G}$ in two stages:

(1) first classify the irreducible cuspidal representations of Levi subgroups;
(2) then classify the non-cuspidal irreducible representations, i.e., the irreducible subquotients (or subrepresentations) of representations of G parabolically induced by irreducible cuspidal representations from proper (standard) parabolic subgroups.

Langlands classification may be viewed as a refinement of this strategy: it reduces the second problem to the problem of classifying of all irreducible tempered subquotients. As we know, this question splits into the following problems:

(a) classifying the irreducible square integrable modulo center subquotients of representations parabolically induced by irreducible cuspidal ones;
(b) understanding representations parabolically induced by square integrable modulo center representations.

The matrix coefficients of irreducible cuspidal representations are compactly supported modulo center (and the converse also holds; [4]). Therefore, irreducible cuspidal representations are essentially square integrable.

In this paper we shall deal with the problem of constructing and classifying all the non-cuspidal irreducible square integrable modulo center subquotients of parabolically induced representations. We shall not go into the problem of constructing irreducible cuspidal representations.

9. The geometric lemma

For a non-cuspidal representation, it is very useful to know its Jacquet modules explicitly. Since we are interested in irreducible subquotients of parabolically induced representations, it is important for us to know the Jacquet modules of induced representations, at least to a certain level. The complete structure of these Jacquet modules is usually hard to understand, at least for the most interesting cases (already for $SL(2,F)$: for example it is not that easy to tell if the Jacquet module of $\mathrm{Ind}^{SL(2,F)}(\chi)$ from the minimal parabolic subgroup is semi simple when $\chi^2 \equiv 1$, which is in the unitary situation; see [4]). However, there is a very useful result in that direction. It does not completely describe the Jacquet modules of parabolically induced representations, but rather some filtrations. Moreover, these subquotients of Jacquet modules of parabolically induced representations are described as representations parabolically induced by certain Jacquet modules of the inducing representation. This (technical) description is very useful, and for the sake of completeness, we shall give it here.

Fix a parabolic subgroup $P = MN$ of G and a smooth representation σ of M. For $x \in G$, $x^{-1}\sigma$ will denote the representation of $x^{-1}Mx$ given by $(x^{-1}\sigma)(x^{-1}mx) = \sigma(m)$, $m \in M$. In other words, $(x^{-1}\sigma)(m') = \sigma(xm'x^{-1})$, $m' \in x^{-1}Mx$.

Denote by W the Weyl group of $A_\emptyset$ (the quotient group of the normalizer of $A_\emptyset$ by the centralizer). Let $P = MN$ and $Q = LU$ be standard parabolic subgroups, with standard Levi decompositions. Define

$$W^{M,L} = \{w \in W; w(M \cap P_\emptyset)w^{-1} \subseteq P_\emptyset, w^{-1}(L \cap P_\emptyset)w \subseteq P_\emptyset\}.$$

Now by J. Bernstein, A.V. Zelevinsky ([3]), and by W. Casselman ([4]), we have:

Geometric lemma. *Let σ be a smooth admissible representation of M. Then one can enumerate the elements $w_1, w_2, \dots, w_m$ of $W^{M,L}$ in such a way that there exists a filtration*

$$\{0\} = \tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_m = r_L^G(\mathrm{Ind}_P^G(\sigma))$$

of $r_L^G(\mathrm{Ind}_P^G(\sigma))$ such that for $1 \le i \le m$,

$$\tau_i/\tau_{i-1} \cong \mathrm{Ind}^L_{wMw^{-1}\cap L}(w(r^M_{M\cap wLw^{-1}}(\sigma))).$$

For more details one should consult the original papers (the Geometric lemma in terms of roots may be more suitable for some calculations; see [4]).

As we shall soon see, already for the analysis of a single induced representation, we need to consider various Jacquet modules of several induced representations and their filtrations. This requires (among other things) frequent computations of $W^{M,L}$ and the groups $M\cap w^{-1}Lw$, $N' = wMw^{-1}\cap L$. These computations may take a considerable time even for a single representation. Therefore, it would be convenient to have a simple way (or algorithm) for obtaining the above filtrations.

Surprisingly, for classical groups there is very elegant solution to this problem. In a way, this problem is solved by "algebraization" of the Geometric lemma (in this case). Before we go on to explain it, let us derive some direct consequences of the Geometric lemma.

10. Some general consequences

Suppose below that σ is an irreducible cuspidal representation of M.

(1) A simple case of the Geometric lemma implies that the composition series of the Jacquet module $r_M^G(\mathrm{Ind}_P^G(\sigma))$ consist of all $w\sigma$ such that w runs over all representatives of $W_M\backslash W_G/W_M$ which normalize M (W_G and W_M denote the Weyl groups of G and M respectively with respect to $A_\emptyset$). From this, one can get a number of interesting general facts.

Conjugating an induced representation and applying induction from associate parabolic subgroups (see (4) of section 7.), for a standard parabolic subgroup $P' = M'N'$ associate to P, one gets irreducible subquotients of $r_{M'}^G(\mathrm{Ind}_P^G(\sigma))$ by conjugating the irreducible subquotients of $r_M^G(\mathrm{Ind}_P^G(\sigma))$ by the corresponding element (thus, they are all cuspidal).

Further, if a standard parabolic subgroup P' does not contain parabolic subgroup associate to P, then $r_{M'}^G(\mathrm{Ind}_P^G(\sigma))$ is trivial (i.e., equal to $\{0\}$). In the case that the parabolic subgroup strictly contains a parabolic subgroup associate to P, the Jacquet modules are non-trivial, and all irreducible subquotients are non-cuspidal.

(2) The description of composition series implies that if $r_{M'}^G(\mathrm{Ind}_P^G(\sigma))$ has a cuspidal subquotient for a standard parabolic subgroup $P' = M'N'$, then P and P' are associate. Further, if σ' is an irreducible cuspidal subquotient, then σ and σ' are conjugated.

More generally, let τ be an irreducible admissible representation of M and ρ', ρ'' irreducible cuspidal subquotients of $r_{M'}^G\left(\mathrm{Ind}_P^G(\tau)\right)$ and $r_{M''}^G\left(\mathrm{Ind}_P^G(\tau)\right)$ respectively, where $P' = M'N'$ and $P'' = M''N''$ are standard parabolic subgroups (with standard Levi decompositions). Then P' and P'' are associate and ρ', ρ'' are conjugate.

Similarly, if $P' = M'N'$, $P'' = M''N''$ denote two parabolic subgroups of G, and σ', σ'' denote irreducible cuspidal representations of M' and M'' respectively, such that $\mathrm{Ind}_P^G(\sigma')$ and $\mathrm{Ind}_{P''}^G(\sigma'')$ have a common composition factor, then P' and P'' must be associate, while σ' and σ'' are conjugate.

(3) The description of Jacquet modules in (1), together with some simple representation theory, implies that if π is an irreducible subquotient of $\mathrm{Ind}_P^G(\sigma)$ (as above, σ is irreducible cuspidal), then there exists $w \in G$ which normalizes M such that π is isomorphic to a subrepresentation of $\mathrm{Ind}_P^G(w\sigma)$. Now Frobenius reciprocity implies that $r_M^G(\pi)$ is non-zero (and each irreducible subquotient is cuspidal). Further, if $P' = M'N'$ is a parabolic subgroup such that $r_{M'}^G\left(\mathrm{Ind}_P^G(\sigma)\right)$ has an irreducible cuspidal subquotient, then $r_{M'}^G(\pi) \neq \{0\}$.

If τ is an irreducible admissible representation of M, and π an irreducible subquotient of $\mathrm{Ind}_P^G(\tau)$ such that $r_{M'}^G\left(\mathrm{Ind}_P^G(\tau)\right) \neq \{0\}$ for some standard parabolic subgroup $P' = M'N'$ of G, then from the transitivity of Jacquet modules, we get $r_{M'}^G(\pi) \neq \{0\}$.

In the above considerations, we have used only a small portion of the information coming from the Geometric lemma, namely, only the information coming from the calculation of the minimal non-trivial Jacquet modules. Much more information is contained in the Jacquet modules with respect to the other parabolic subgroups. We shall soon explain how one can use this information to understand irreducibility/reducibility of parabolically induced representations in some important cases. Before we explain this, we shall show how one can get good control of these other Jacquet modules for series of classical groups. But first, we shall say a few words about

11. The case of maximal parabolic subgroups

In this section, we shall see what the Geometric lemma gives in the simplest case, when we induce from an irreducible cuspidal representation of a maximal parabolic subgroup, like $P_\emptyset$ in $SL(2,F)$. We start with the case $G = SL(2,F)$ and $P = P_\emptyset$.

We identify the subgroup of diagonal matrices $A_\emptyset$ in $G = SL(2,F)$ with $F^\times$ using the identification $\mathrm{diag}(a, a^{-1}) \leftrightarrow a$. Let χ be a character of $F^\times$. Now the Geometric lemma implies that χ, χ^{-1} is the composition series of $r_M^G\left(\mathrm{Ind}_P^G(\chi)\right)$. This implies that the length of $\mathrm{Ind}_P^G(\chi)$ is at most two.

Suppose that χ is unitary. Then if $\chi \neq \bar{\chi}$, the unitarizability of $\mathrm{Ind}_P^G(\chi)$ and Frobenius reciprocity imply the irreducibility of $\mathrm{Ind}_P^G(\chi)$.

Suppose that χ is not unitary, and that $\mathrm{Ind}_P^G(\chi)$ reduces. Then one irreducible subquotient is a square integrable representation by Casselman's square integrability criterion (which will be discussed later). Since square integrable representations are unitarizable, which implies that they are equivalent to their Hermitian contragredients (i.e., the complex conjugates of the contragredients), $\mathrm{Ind}_P^G(\chi)$ and $\mathrm{Ind}_P^G((\tilde{\chi})^-)$ have an irreducible subquotient in common. Now, general properties regarding principal series (discussed in section 10.) imply that $\chi \cong (\tilde{\chi})^{-}$ or $\tilde{\chi} \cong (\tilde{\chi})^{-}$. Since χ is not unitary, we must have $\chi = \bar{\chi}$.

Thus, here the Geometric lemma gives irreducibility in the case of non-real valued characters, but does not tell us anything about reducibility points (in the most interesting case) of real valued characters (observe that we know here that $\mathrm{Ind}_P^G(|\ |_F^{1/2})$ reduces since 1_G is subquotient; here 1_G denotes the representation of G on one-dimensional vector space, with the trivial action of G). Summing up, Jacquet modules provide us only qualitative information.

Quantitative information (which is the most interesting), i.e. where exactly reducibility points are, is obtained using analytic methods (from which one can also deduce the above qualitative information).

Therefore, it is not surprising that for a long time Jacquet modules were not used much in treating more general reducibility problems (and construction of square integrable representations). In the case of general linear groups, Bernstein and Zelevinsky used something much stronger - Gelfand-Kazhdan derivatives - and got complete answers modulo cuspidal representations (we shall describe briefly these answers later).

For other (classical) groups, the same problems are much, much harder, while the analogous powerful tool does not exist in this case. Therefore, it was natural to try to use analytic methods there. But analytic methods, which are not even very simple for $SL(2)$, became quite involved in more general cases.

For example, R. Gustafson needed considerable work to settle (degenerate principal series) representations of $Sp(2n, F)$ parabolically induced by unramified characters from the Siegel parabolic subgroup (see [8]). S.S. Kudla and S. Rallis needed additional significant work to settle the case of remaining ramified characters (see [18]). The case of Siegel parabolic subgroup fits well with their methods since the unipotent radical is commutative (which is not the case with other maximal parabolic subgroups). After these two relatively simple cases, the question which arises is what should we expect in more complicated cases, and what kind of methods should we use to handle these problems. It seems that the best is a combination of both methods. When one induces from an irreducible cuspidal representation of a maximal parabolic subgroup, one is forced to use the analytic tools (simply put, Jacquet modules do not give much). In other cases, Jacquet module methods happen to be surprisingly powerful. For example, although not very useful in the simple case of $SL(2, F) = Sp(2, F)$, the use of Jacquet module methods gives complete answers (i.e. reducibility points, lengths of representations and Langlands parameters of irreducible subquotients) for all degenerate principal series of all $Sp(2n, F)$ or $SO(2n+1, F)$, $n \geq 2$, induced from maximal parabolic subgroups (with the help of the Bernstein-Zelevinsky theory; see [14]). Roughly, the power of Jacquet module methods for bigger groups lies in the fact that in this case we have more standard parabolic subgroups, and by comparing information coming from different parabolic subgroups we can obtain surprising amount in this case.

The same type of analysis which we have described for $SL(2, F)$ can be applied to a maximal parabolic subgroup $P = MN$ of a connected reductive F-group and an irreducible cuspidal representation σ of M. For simplicity, we shall assume that the center of G is compact. Suppose that $\mathrm{Ind}_P^G(\sigma)$ reduces. Then general facts on induced representations (in section 7.) imply that there exists $w \in W$ which is not in M, but normalizes M. Now $\{\sigma, w\sigma\}$ is the composition series of $r_M^G\left(\mathrm{Ind}_P^G(\sigma)\right)$. If σ is unitarizable, then Frobenius reciprocity (and semi simplicity) implies irreducibility if $\sigma \ncong w\sigma$ (regular case).

Suppose that σ is not unitarizable, and that $\mathrm{Ind}_P^G(\sigma)$ reduces. Then the length of $\mathrm{Ind}_P^G(\sigma)$ is two and one irreducible subquotient is again square integrable by the Casselman criterion. In the same way as above, we get that $\mathrm{Ind}_P^G(\sigma)$ and $\mathrm{Ind}_P^G((\tilde{\sigma})^{-})$ have an irreducible subquotient in common. Now section 10. implies $\sigma \cong (\tilde{\sigma})^{-}$ or $w\sigma \cong (\tilde{\sigma})^{-}$, and further non-unitarity implies $w\sigma \cong (\tilde{\sigma})^{-}$.

The hard part here is to understand when we have reducibility in the case $w\sigma \cong (\tilde{\sigma})^{-}$. This is a very hard problem, and it requires analytic methods (see [31] and [32]). We shall say few words about this later.

12. Hopf algebras in the case of general linear groups

In this section we recall the notation and structures that have been introduced by J. Bernstein and A.V. Zelevinsky. They have introduced an algebraic structure in which they incorporated some very basic properties of the representation theory of general linear groups (like induction in stages, transitivity of Jacquet modules, the Geometric lemma etc.). It is interesting that they did not use this structure much in their work, but rather pointed it out as an interesting curiosity (later Zelevinsky used it in the finite field case; we have used this structure in a substantial way in our work on representation theory of general linear groups over p-adic division algebras in [35]).

For a reductive F-group G, we denote by

$$\mathcal{R}(G)$$

the free $\mathbb{Z}$-module with basis $\tilde{G}$ (this is the Grothendieck group of the category of all smooth finite length representations of G). The multiplicity of $\tau \in \tilde{G}$ in a finite length representation π will be denoted by $m(\tau : \pi)$. We denote the semi simplification of π by

$$\mathrm{s.s.}(\pi) = \sum_{\tau \in \tilde{G}} m(\tau : \pi)\tau \in \mathcal{R}(G).$$

There is a natural partial ordering $\leq$ on $\mathcal{R}(G)$ (the positive cone consists of finite (formal) sums of elements of $\tilde{G}$). When we write $\leq$ between representations (of finite length), this will actually mean inequality among their semi simplifications.

For a parabolic subgroup $P = MN$, the functors r_M^G and Ind_P^G factor in a natural way to homomorphisms of Grothendieck groups (from $\mathcal{R}(G)$ into $\mathcal{R}(M)$ and from $\mathcal{R}(M)$ into $\mathcal{R}(G)$ respectively). These homomorphisms are also denoted by the same symbol as the functors. They are homomorphisms of ordered groups (i.e., they also respect orderings).

In what follows we shall often use the fact that for two reductive groups G_1 and G_2 we have

$$\mathcal{R}(G_1 \times G_2) \cong \mathcal{R}(G_1) \otimes \mathcal{R}(G_2)$$

(this is a consequence of the fact that representations in $\tilde{G}_i$ are admissible; see [4]).

Now we return to the general linear groups and recall the Bernstein-Zelevinsky notation. If π_1 and π_2 are representations of $GL(n_1, F)$ and $GL(n_2, F)$, let

$$\pi_1 \times \pi_2 = \mathrm{Ind}_{P^{GL}_{(n_1,n_2)}}^{GL(n_1+n_2,F)}(\pi_1 \otimes \pi_2).$$

Then induction by stages implies

$$\pi_1 \times (\pi_2 \times \pi_3) \cong (\pi_1 \times \pi_2) \times \pi_3.$$

Set

$$R_n = \mathcal{R}(GL(n,F)) \quad \text{and} \quad R = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} R_n.$$

Lift $\times$ to a multiplication on R, and factor $\times : R \times R \to R$ through a mapping

$$m : R \otimes R \to R.$$

In this way R becomes a **commutative ring** (the commutativity follows from induction from associate parabolic subgroups: $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ have the same composition series).

For $\pi \in GL(n,F)\tilde{}$ consider s.s. $\left(r_{M^{GL}_{(k,n-k)}}^{GL(n,F)}(\pi)\right) \in R_k \otimes R_{n-k}$ and define

$$m^*(\pi) = \sum_{k=0}^{n} \text{s.s.} \left(r_{M^{GL}_{(k,n-k)}}^{GL(n,F)}(\pi)\right) \in \sum_{k=0}^{n} R_k \otimes R_{n-k} \hookrightarrow R \otimes R\ .$$

Lift m^* to additive mapping $m^* \colon R \to R \otimes R$, which will be called **comultiplication**. In this way R becomes **coalgebra** (coassociativity follows from the transitivity of Jacquet modules). Moreover, with m and m^*, R is a **Hopf algebra** over $\mathbb{Z}$, i.e., the comultiplication is multiplicative:

$$m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \times m^*(\pi_2), \tag{12.1}$$

which follows from the Geometric lemma.

Observe that the above formula enables us to get, in a very simple and direct way, composition factors of Jacquet modules (for maximal parabolic subgroups) of induced representations, as representations induced by Jacquet modules of inducing representations. Therefore, the most important requirement of the Hopf algebra structure, the Hopf axiom (12.1), can be viewed as an algebraization of the Geometric lemma in this case.

The Hopf algebra R is commutative, but not cocommutative.

By construction, the structure of a Hopf algebra contains in itself a number of the most important basic properties of representation theory, applied to general linear groups: induction in stages and induction from associate parabolic subgroups, transitivity of Jacquet modules and the Geometric lemma. There is one more remarkable property of R: it is a **polynomial algebra** over D (this follows from the fact that tempered induction for general linear groups is irreducible, and from basic properties of the Langlands classification).

Remark. *Although the definition of the Hopf algebra R involves only Jacquet modules for maximal parabolic subgroups, the transitivity of Jacquet modules enables one to compute other Jacquet modules using the Hopf algebra structure of R. For example, let $\rho_1, \dots, \rho_l$ be irreducible cuspidal representations of general linear groups. Using the Hopf algebra structure of R, one can prove by simple induction the following fact: if σ is an irreducible cuspidal subquotient of a Jacquet module of $\rho_1 \times \cdots \times \rho_l$ with respect to a standard parabolic subgroup, then there exists a permutation p of $\{1, \dots, l\}$ such that $\sigma \cong \rho_{p(1)} \otimes \cdots \otimes \rho_{p(l)}$; further, each $\rho_{p(1)} \otimes \cdots \otimes \rho_{p(l)}$ with p as above, shows up as a subquotient of the Jacquet module with respect to some standard parabolic subgroup.*

Later on, we shall show how to extend the above algebraic approach to other classical groups. Before that, we shall describe some very basic facts from the Bernstein-Zelevinsky theory.

13. Square integrable representations of general linear groups

The set of all equivalence classes of irreducible cuspidal representations of all $GL(n, F)$, $n \geq 1$, will be denoted by

$$\mathcal{C}.$$

A set of the form $\{\rho, \nu\rho, \dots, \nu^k\rho\}$, where $\rho \in \mathcal{C}, k \in \mathbb{Z}_{\geq 0}$, will be called a **segment in** $\mathcal{C}$, and denoted by $[\rho, \nu^k\rho]$. The set of all such segments will be denoted by

$$\mathcal{S}.$$

The geometric lemma implies that if $\rho \times \rho'$ reduces ($\rho, \rho' \in \mathcal{C}$), then $\rho' \cong \nu^\alpha\rho$ for some $\alpha \in \mathbb{R}$. A fundamental fact, proved by J. Bernstein and A.V. Zelevinsky, using Gelfand-Kazhdan derivatives, is that $\rho \times \rho'$ reduces if and only if $\rho' \cong \nu^{\pm 1}\rho$, i.e., precisely when $\alpha = \pm 1$ (later on, this was proved by F. Shahidi using L-functions).

One can attach to each segment $\Delta = [\rho, \nu^k \rho] = \{\rho, \nu\rho, \dots, \nu^k \rho\} \in \mathcal{S}$ the unique irreducible subrepresentation $\delta(\Delta)$ of

$$\nu^k \rho \times \nu^{k-1}\rho \times \cdots \times \nu\rho \times \rho.$$

The representation $\delta(\Delta)$ is essentially square integrable.

The irreducible representation $\delta(\Delta)$ is characterized by the fact that $\nu^k\rho \otimes \nu^{k-1}\rho \otimes \cdots \otimes \nu\rho \otimes \rho$ is in its Jacquet module (for the corresponding standard Jacquet module). From this, the following simple formula follows easily

$$m^*(\delta([\rho, \nu^k\rho])) = \sum_{i=-1}^{k} \delta([\nu^{i+1}\rho, \nu^k\rho]) \otimes \delta([\rho, \nu^i\rho]), \tag{13.1}$$

which is very useful in applications (we take formally $GL(0, F)$ to be the trivial group; the representation of the trivial group on one-dimensional vector space is denoted by 1; in (13.1) we take formally $\delta(\emptyset) = 1$). The mapping $\Delta \mapsto \delta(\Delta)$ is a bijection from $\mathcal{S}$ onto D (the set of all the equivalence classes of irreducible essentially square integrable representations of general linear groups $GL(n, F)$, $n \geq 1$).

The above classification of J. Bernstein and A.V. Zelevinsky reduced the problem of establishing local Langlands correspondences for general linear groups to the case of cuspidal representations in the following way. For a positive integer n and an irreducible cuspidal representation ρ of a general linear group, let

$$\delta(\rho, n) = \delta([\nu^{-(n-1)/2}\rho, \nu^{(n-1)/2}\rho]).$$

Let E_n be (the unique up to an equivalence) irreducible algebraic representation of $SL(2, \mathbb{C})$ of dimension n. Denote by Φ the local Langlands correspondences for general linear groups. Then

$$\Phi(\delta(\rho, n)) = \Phi(\rho) \otimes E_n.$$

Therefore, it is enough to establish the local Langlands correspondences for irreducible cuspidal representations, which was done later, first by G. Laumon, M. Rapoport and U. Stuhler in positive characteristic ([19]), and after that by M. Harris and R. Taylor ([9]), and also G. Henniart ([13]) in characteristic 0.

We say that $\Delta_1, \Delta_2 \in \mathcal{S}$ are linked, if

$$\Delta_1 \cup \Delta_2 \in \mathcal{S} \backslash \{\Delta_1, \Delta_2\}.$$

The representation $\delta(\Delta_1) \times \delta(\Delta_2)$ reduces if and only if Δ_1 and Δ_2 are linked, and then

$$\{L(\delta(\Delta_1), \delta(\Delta_2)), \delta(\Delta_1 \cup \Delta_2) \times \delta(\Delta_1 \cap \Delta_2)\}$$

is the composition series of $\delta(\Delta_1) \times \delta(\Delta_2)$.

Let $a = (\Delta_1, \dots, \Delta_l) \in M(\mathcal{S})$. If there are $i, j \in \{1, \dots, l\}$ such that Δ_i and Δ_j are linked, then replacing the pair Δ_i, Δ_j by the pair

$$\Delta_i \cup \Delta_j, \Delta_i \cap \Delta_j$$

in a, gives a multiset a'. We write $a' \prec a$. Using $\prec$ we generate partial ordering $\leq$ on $M(\mathcal{S})$. Then for $a = (\Delta_1, \dots, \Delta_l), a' = (\Delta'_1, \dots, \Delta'_{l'}) \in M(\mathcal{S})$, $L(\delta(\Delta'_1), \dots, \delta(\Delta'_{l'}))$ is a subquotient of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_l)$ if and only if $a' \leq a$. Further, if a' is minimal among those satisfying $a' \leq a$, then the multiplicity is one.

In particular, if among $\Delta_1, \dots, \Delta_l$ we do not have linked pairs of segments, then the representation

$$\delta(\Delta_1) \times \cdots \times \delta(\Delta_l)$$

is irreducible. One can find all these results in [46], and much more.

We denote by 1_G the trivial representation (on a one-dimensional space) of a group G. The trivial representation of the trivial group will be denoted simply by 1.

In the Grothendieck group, $\delta([\nu^{-(n-1)/2} 1_{F^\times}, \nu^{(n-1)/2} 1_{F^\times}])$ is an alternated sum of

$$\text{Ind}_{P_\alpha}^{GL(n,F)}(\delta_{P_\alpha}^{-1/2}|_{M_\alpha}), \tag{13.2}$$

where $\alpha = (n_1, \dots, n_l)$ runs over all ordered partitions of n.

Remark. *The notion of a Steinberg representation comes from the finite field case. In that setting, this representation is an irreducible representation whose dimension is equal to the cardinality of a p-Sylow subgroup, where p is the characteristic of the finite field, and it is alternated sum of* $Ind_P^G(1_M)$ *where P runs over standard parabolic subgroups. Since the last property holds for* $\delta([\nu^{-(n-1)/2} 1_{F^\times}, \nu^{(n-1)/2} 1_{F^\times}])$ *(it is an alternated sum of representations (13.2)), this representation is called the Steinberg representation of* $GL(n, F)$.

Our representations $\delta([\rho, \nu^k \rho])$ *have similar properties to the Steinberg representations. This is the reason that they are called generalized Steinberg representations. They can also be related to the Steinberg representations using Hecke algebra isomorphisms.*

In general, for generalized Steinberg representations all the non-trivial Jacquet modules are irreducible essentially square integrable representations, and the inducing representation is regular (this notion will be explained later). These properties hold only for a small portion of square integrable representations of other classical groups. This is the reason that we shall later call irreducible square integrable representations which satisfy these two properties square integrable representations of Steinberg type.

The definition of the Steinberg representation for general connected reductive group G over a local field is the following:

Definition. *The representation* $Ind_{P_\emptyset}^G(\delta_{P_\emptyset}^{1/2}|_{M_\emptyset})$ *contains a unique square integrable subquotient. This subquotient is called the Steinberg representation of* G *and denoted by* St_G.

14. Other classical groups

In this section we shall recall the definitions of other split classical groups, and introduce notation which will again incorporate some very basic properties of the representation theory of these groups. We shall work with symplectic and split odd orthogonal groups in this paper, but we could work in a similar way with other classical groups (see [24]).

Let $J_n \in GL(n,F)$ be the matrix having entries 1 on the second diagonal, and 0 in the remaining places, and let I_n be the identity matrix in $GL(n,F)$. Denote by ${}^tS$ (resp., ${}^\tau S$) the transposed matrix of S (resp., the transposed matrix of S with respect to the second diagonal). The symplectic group is

$$Sp(2n,F) = \left\{ S \in GL(2n,F);\ {}^tS \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} S = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\},$$

and the split odd-orthogonal group is

$$SO(2n+1,F) = \{S \in SL(2n+1,F);\ {}^\tau S\, S = I_{2n+1}\}.$$

We could also introduce these groups geometrically.

In what follows, sometimes we denote $GL(n,F)$, $Sp(2n,F)$, or $SO(2n+1,F)$, simply by $GL(n)$, $Sp(2n)$, or $SO(2n+1)$, respectively (i.e. if we do not specify which rational points we consider, then it will always mean that we consider F-rational points).

We shall fix a series of symplectic or odd-orthogonal groups, and denote by S_n corresponding group of rank n from the series (i.e. $Sp(2n,F)$ or $SO(2n+1,F)$). We can parameterize the standard parabolic subgroups of S_n by ordered partitions $\alpha = (n_1,\dots,n_k)$ of integers m, where $0 \le m \le n$, in the following way. For $S_n = Sp(2n,F)$ (resp., $S_n = SO(2n+1,F)$) let

$$\alpha' = (n_1,...,n_k,2n-2m,n_k,...,n_1)$$

$$(\text{resp., } \alpha' = (n_1,...,n_k,2n+1-2m,n_k,...,n_1)).$$

Set

$$P_\alpha = P_{\alpha'}^{GL} \cap S_n, \quad M_\alpha = M_{\alpha'}^{GL} \cap S_n, \quad N_\alpha = N_{\alpha'}^{GL} \cap S_n.$$

Now $(g_1,\dots,g_k,h,\ {}^\tau g_k^{-1},\dots,\ {}^\tau g_1^{-1}) \mapsto \text{quasi-diag}(g_1,\dots,g_k,h)$ is an isomorphism of the product $GL(n_1,F) \times \cdots \times GL(n_k,F) \times S_{n-m}$ onto M_α. We shall use this identification in the sequel.

In particular, for $m \leqslant n$, the bijection $(g,h) \leftrightarrow \text{quasi-diag}(g,h,\ {}^\tau g^{-1})$ gives an identification of $GL(m,F) \times S_{n-m}$ with $M_{(m)}$. For admissible representations π and σ of finite length of $GL(m,F)$ and S_{n-m} respectively, define

$$\pi \rtimes \sigma = \text{Ind}_{P_{(m)}}^{S_n}(\pi \otimes \sigma)$$

Then induction in stages implies

$$(\pi_1 \times \pi_2) \rtimes \sigma \cong \pi_1 \rtimes (\pi_2 \rtimes \sigma).$$

Let

$$R(S) = \underset{n \in \mathbb{Z}_{\geq 0}}{\oplus} \mathcal{R}(S_n).$$

One lifts $\rtimes$ to a biadditive mapping

$$R \times R(S) \to R(S).$$

The lift is also denoted by $\rtimes$. In this way $R(S)$ becomes an **R-module**. In this module we have

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma,$$

which follows from the fact that parabolically induced representations from associate parabolic subgroups (and representations) have the same composition series. The mapping $\rtimes : R \times R(S) \to R(S)$ factors through $R \otimes R(S)$ by

$$\mu : R \otimes R(S) \to R(S)$$

in a natural way.

For $\pi \in \tilde{S}_n$ define

$$\mu^*(\pi) = \sum_{k=0}^{n} \text{s.s.} \left(r_{M_{(k)}}^{S_n}(\pi) \right) \in \sum_{k=0}^{n} R_k \otimes R(S_{n-k}) \hookrightarrow R \otimes R(S).$$

Extend μ^* to an additive mapping $\mu^* \colon R(S) \to R \otimes R(S)$. We call it **comultiplication** on $R(S)$, and $R(S)$ is a **comodule** over R (coassociativity again follows from the transitivity of Jacquet modules).

Up to now we have observed that $R(S)$ is a module and a comodule over R. It is easy to see that it is not a Hopf module, but it is not far from Hopf module as we shall see soon. Let

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^* : R \to R \otimes R, \tag{14.1}$$

where 1 denotes the identity mapping (on R), $\sim$ the contragredient mapping and s the transposition mapping $\sum x_i \otimes y_i \mapsto \sum y_i \otimes x_i$. Then

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma) \tag{14.2}$$

($R \otimes R(S)$ is an $R \otimes R$-module in a natural way). We say that $R(S)$ is an **M^*-Hopf module** over R (in this terminology m^*-Hopf module corresponds to usual Hopf module).

The above formula incorporates the Geometric lemma in the structure of $R(S)$ over R, i.e., algebraizes the Geometric lemma. Again, this is a simple formula, which enables us to get, in a simple way, compositions factors of Jacquet modules of parabolically induced representations of classical groups.

There exists a distinguished Jacquet module in the sense that it requires only the GL-theory to handle it (and that it is non-trivial): let π be a non-trivial subquotient of $\pi' \rtimes \sigma$, where π is a representation of some $GL(p, F)$

and σ is an irreducible cuspidal representation of some S_q. Then we define

$$s_{GL} = r^{S_{p+q}}_{M_{(p)}}(\pi).$$

This will be called the **Jacquet module of π of GL-type**.

Let ρ be an irreducible cuspidal representation of a general linear group and $x, y \in \mathbb{R}$ such that $y - x \in \mathbb{Z}_{\geq 0}$. In what follows, we shall use formula for $M^*\big(\delta([\nu^x\rho, \nu^y\rho])\big)$ several times. One gets the following directly from (13.1) and (14.1):

(14.3) $$M^*\big(\delta([\nu^x\rho, \nu^y\rho])\big)$$
$$= \sum_{i=x-1}^{y} \sum_{j=i}^{y} \delta([\nu^{-i}\tilde{\rho}, \nu^{-x}\tilde{\rho}]) \times \delta([\nu^{j+1}\rho, \nu^y\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]),$$

where $y - i, y - j \in \mathbb{Z}_{\geq 0}$ in the above sums.

Jacquet module methods of determining reducibility of parabolically induced representations

15. Reducibility - irreducibility

Reducibility of parabolically induced representations is closely related to construction of non-cuspidal irreducible square integrable representations. This fact is already shown by the simple example of induction from a maximal parabolic subgroup by a non-unitarizable irreducible cuspidal representation. In general, the induction from a maximal parabolic subgroup by a non-cuspidal non-unitarizable irreducible square integrable representation also quite often gives square integrable subquotient(s). Nevertheless, in the case of general linear groups, this does not happen too often (it happens if and only if corresponding linked segments are disjoint).

We shall explore the idea of constructing non-cuspidal irreducible square integrable representations of classical groups starting from non-cuspidal irreducible square integrable representations of general linear groups. For this, it is necessary to have an efficient method of controlling reducibility of parabolically induced representations when it will be necessary (for square integrable subquotients we must have reducibility of considered parabolically induced representation).

We shall now try to explain how one can have a fairly good control of these reducibilities. Before that, we shall recall briefly of the Casselman's

16. Square integrability criterion

We shall recall of the Casselman square integrability criterion only in the case of classical groups that we consider. We consider $\mathbb{R}^n$ supplied with the standard inner product $((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_i x_i y_i$. Let

$$\beta_i = (\underbrace{1, 1, \dots, 1}_{i \text{ times}}, 0, 0, \dots, 0) \in \mathbb{R}^n, \ 1 \leq i \leq n.$$

Fix a non-cuspidal irreducible admissible representation π of S_n and take $\alpha = (n_1, \dots, n_\ell)$ with $r^{S_n}_{M_\alpha}(\pi) \neq \{0\}$, such that there exists an irreducible cuspidal subquotient of this Jacquet module. Then all irreducible subquotients are cuspidal. Denote such a subquotient by σ. We can write σ as $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_\ell \otimes \rho$ where $\rho_i \in GL(n_i, F)\tilde{}$, $\rho \in \tilde{S}_{n-(n_1+\cdots+n_\ell)}$. Denote $m = n_1 + \cdots + n_\ell$. Define

$$e_*(\sigma) = (\underbrace{e(\rho_1), \dots, e(\rho_1)}_{n_1 \text{ times}}, \dots, \underbrace{e(\rho_\ell), \dots, e(\rho_\ell)}_{n_\ell \text{ times}}, \underbrace{0, \dots, 0}_{n-m \text{ times}})$$

If π is square integrable, then

$$(e_*(\sigma), \beta_{n_1}) > 0, (e_*(\sigma), \beta_{n_1+n_2}) > 0, \cdots, (e_*(\sigma), \beta_{n_1+n_2+\cdots+n_\ell}) > 0.$$

The converse also holds: if the above inequalities hold for any α and σ as above, then π is square integrable (if instead of strict inequalities > 0 hold the weaker conditions ≥ 0 in all the above relations, then this is a criterion for π to be tempered).

17. Cuspidal reducibilities

Fix irreducible unitarizable cuspidal representations ρ and σ of $GL(p, F)$ and S_q respectively (irreducible cuspidal σ is automatically unitarizable). Now section 11. implies that if for some $\alpha \in \mathbb{R}$, $\nu^\alpha \rho \rtimes \sigma$ reduces then $\rho \cong \tilde{\rho}$ (representations satisfying this condition will be called **selfdual**). Conversely, considering complementary series, one gets that if $\rho \cong \tilde{\rho}$, then $\nu^\alpha \rho \rtimes \sigma$ reduces for some $\alpha \geq 0$. Induction from associate parabolic subgroups (from section 7.) implies that $\nu^\alpha \rho \rtimes \sigma$ reduces if and only if $\nu^{-\alpha} \rho \rtimes \sigma$ reduces. Therefore, regarding reducibility, it is enough to study only the non-negative exponents.

Determining the reducibility points is a very important and very hard problem. **Fundamental results** in that direction are the following:

(1) For selfdual ρ there is exactly one $\alpha \geq 0$ for which $\nu^\alpha \rho \rtimes \sigma$ reduces ([33]). This point will be denoted by

$$\alpha_{\rho,\sigma}.$$

(2) If ρ is selfdual and σ is **generic**, then

$$\alpha_{\rho,\sigma} \in \{0, 1/2, 1\}$$

([31], [32]).

In a particular case, when σ is the trivial representation of the trivial group (i.e. we are inducing from the Siegel parabolic subgroup), $\alpha_{\rho,1}$ is equal to 1 if and only if we work with symplectic groups and if $\rho = 1_{F^\times}$ (i.e. the induction is in $Sp(2, F) = SL(2, F)$) ([32], [22]).

(3) For $p > 1$ and ρ a selfdual representation of $GL(p, F)$ (it is enough that $p \geq 1$ and $\rho \not\cong 1_{F^\times}$), $\rho \rtimes 1$ reduces in $Sp(2p, F)$ if and only if it is irreducible in $SO(2p+1, F)$ ([32]).

(4) F. Shahidi has proved that for generic σ, we have

$$\alpha_{\rho,\sigma} - \alpha_{\rho,1} \in \mathbb{Z}.$$

In general, some important conjectures from the theory of automorphic forms would imply that for **any** σ the following (conjecture) holds

$$\text{(BA)} \qquad \alpha_{\rho,\sigma} - \alpha_{\rho,1} \in \mathbb{Z},$$

which will be called the **basic assumption** (see [20] for more details, together with section 12. of [24]). This conjecture would obviously imply that

$$\alpha_{\rho,\sigma} \in (1/2)\mathbb{Z}$$

for any σ.

Further, for fixed σ, there should exist only finitely many ρ with $\alpha_{\rho,\sigma} \notin \{0, 1/2\}$. There is a natural and precise conjecture of C. Mœglin regarding such ρ's (see [22] and [24], and also remarks in section 34.). This is the reason why reducibilities $\alpha_{\rho,\sigma} \notin \{0, 1/2\}$ can be called exceptional.

Some examples. *Let χ_0 be a character of order 2 of $F^\times$.*

(1) *As is well known, for $Sp(2, F) = SL(2, F)$ we have $\alpha_{\chi_0,1} = 0$ and $\alpha_{1_{F^\times},1} = 1$.*
(2) *For $SO(3, F)$, we have $\alpha_{\chi_0,1} = \alpha_{1_{F^\times},1} = 1/2$.*
(3) *One can get in a relatively simple way (using Clifford theory) the following fact: Suppose that ρ is selfdual and that the central character ω_ρ of ρ satisfies $\omega_\rho \not\equiv 1$. Then the representation $\rho \rtimes 1$ of $Sp(2p, F)$ reduces, and $\nu^\alpha \rho \rtimes 1$ is irreducible for $\alpha > 0$.*
(4) *F. Shahidi has proved the following: for $Sp(4, F)$ and ρ selfdual irreducible cuspidal representation of $GL(2, F)$ one has $\alpha_{\rho,1} = 0$ if $\omega_\rho \not\equiv 1$, and $\alpha_{\rho,1} = 1/2$ if $\omega_\rho \equiv 1$ ([31]). For $SO(5, F)$, the situation is reversed ([32]).*

18. Regular induced representations

Let σ be an irreducible cuspidal representation of a Levi subgroup M of a parabolic subgroup P of G. We shall say that σ is regular (in G), or that $\mathrm{Ind}_P^G(\sigma)$ is regular, if all (non-zero) Jacquet modules are multiplicity one representations. For this, it is equivalent to require one minimal non-zero Jacquet module to be a multiplicity one representation. Equivalently, this is the same as asking that all the representations $w\sigma$, when w runs over all representatives of $W_M \backslash W_G / W_M$ which normalize M (W_G and W_M denote the Weyl groups of G and M respectively) are nonequivalent.

This technical condition enables one to study composition series of such induced representations easily. Namely, let $\mathrm{Ind}_P^G(\sigma)$ be a regular representation as above, and $P' = M'N'$ a parabolic subgroup such that $r_{M'}^G(\mathrm{Ind}_P^G(\sigma))$ is non-zero. Then

$$\phi : \pi \mapsto \text{composition series of } r_{M'}^G(\pi)$$

can be interpreted as an injection from the partitive set of composition series of $\operatorname{Ind}_P^G(\sigma)$ into the partitive set of composition series of $r_{M'}^G(\operatorname{Ind}_P^G(\sigma))$. Further, $\phi(\pi_1 \cap \pi_2) = \phi(\pi_1) \cap \phi(\pi_2)$, and if $\phi(\pi)$ is irreducible, then π is irreducible.

This can be used to construct some interesting representations in a rather easy way, as we shall see soon.

19. Square integrable representations of Steinberg type

The square integrable representations, which we shall introduce now, are natural generalization of usual Steinberg representation. Their characteristic is, among others, that they have a very simple Jacquet modules, and therefore it is pretty easy to control representations parabolically induced from them.

Below we shall denote Jacquet module

$$r_{M_{(k)}}^{S_n}(\pi)$$

simply by

$$s_{(k)}(\pi).$$

Let ρ and σ be irreducible unitarizable cuspidal representations of groups $GL(p, F)$ and S_q respectively, such that $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha > 0$. Now the fact that

$$\text{s.s.}(s_{(p)}(\nu^\alpha\rho \rtimes \sigma)) = \nu^\alpha\rho \otimes \sigma + \nu^{-\alpha}\rho \otimes \sigma,$$

implies that $\nu^\alpha\rho \rtimes \sigma$ has a unique irreducible subrepresentation. We denote this subrepresentation by $\delta(\nu^\alpha\rho; \sigma)$. We get directly

$$\mu^*(\delta(\nu^\alpha\rho; \sigma)) = 1 \otimes \delta(\nu^\alpha\rho; \sigma) + \nu^\alpha\rho \otimes \sigma.$$

The square integrability criterion implies that $\delta(\nu^\alpha\rho; \sigma)$ is square integrable.

We can continue in the following way. Look at the subquotients

$$\nu^{\alpha+1}\rho \rtimes \delta(\nu^\alpha\rho; \sigma) \text{ and } \delta([\nu^\alpha\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma \text{ in } \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma.$$

We get the semi simplifications of GL-type Jacquet modules easily using formula (14.2). They are

$$\nu^{\alpha+1}\rho \times \nu^\alpha\rho \otimes \sigma + \nu^{-(\alpha+1)}\rho \times \nu^\alpha\rho \otimes \sigma$$

and

$$\delta([\nu^\alpha\rho, \nu^{\alpha+1}\rho]) \otimes \sigma + \nu^{\alpha+1}\rho \times \nu^{-\alpha}\rho \otimes \sigma + \delta([\nu^{-(\alpha+1)}\rho, \nu^{-\alpha}\rho]) \otimes \sigma$$

respectively (we use also (14.3)). From this we get that there exists a unique irreducible subquotient which has $\delta([\nu^\alpha\rho, \nu^{\alpha+1}\rho]) \otimes \sigma$ in its Jacquet module. Denote this subquotient by $\delta([\nu^\alpha\rho, \nu^{\alpha+1}\rho]; \sigma)$. It is easy to see that we can define this representation as the unique irreducible subrepresentation of $\delta([\nu^\alpha\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma$, and that we have

$$\begin{aligned}&\mu^*(\delta([\nu^\alpha\rho, \nu^{\alpha+1}\rho]; \sigma)\\ &\quad = 1 \otimes \delta([\nu^\alpha\rho, \nu^{\alpha+1}\rho]; \sigma) + \nu^{\alpha+1}\rho \otimes \delta(\nu^\alpha\rho; \sigma) + \delta([\nu^\alpha\rho, \nu^{\alpha+1}\rho]) \otimes \sigma.\end{aligned}$$

Continuing this procedure we get that the representation

$$\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha+1}\rho \times \nu^{\alpha}\rho \rtimes \sigma$$

has a unique irreducible subrepresentation, which we denote by

$$\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]; \sigma) \quad (n \geq 0),$$

and that

$$\mu^* \left(\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]; \sigma)\right) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+k}\rho]; \sigma)$$

(we take $\delta(\emptyset; \sigma)$ in the above formula to be just σ). The representation $\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]; \sigma)$ is square integrable and we have

$$\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]; \sigma)\tilde{} \cong \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]; \tilde{\sigma}).$$

It follows directly that $\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]; \sigma)$ is a subquotient of $\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho]) \rtimes \sigma$ (actually, a subrepresentation; moreover, the unique irreducible one).

20. A reducibility criterion

In a significant number of cases, reducibility of parabolically induced representations can be determined in the following way. Let $P_0 = M_0 N_0$ be a parabolic subgroup of G and σ an irreducible representation of M_0. Suppose that π and Π are representations of G of finite length, and that there exists a parabolic subgroup $P = MN$ such that the following three conditions hold

(R1) $\mathrm{Ind}_{P_0}^{G}(\sigma) \leq \Pi$, $\pi \leq \Pi$;

(R2) $r_M^G(\mathrm{Ind}_{P_0}^{G}(\sigma)) + r_M^G(\pi) \not\leq r_M^G(\Pi)$;

(R3) $r_M^G(\mathrm{Ind}_{P_0}^{G}(\sigma)) \not\leq r_M^G(\pi)$.

Then it follows directly that $\mathrm{Ind}_{P_0}^{G}(\sigma)$ is reducible (since the Jacquet module functor is exact). In applications of this criterion, one usually chooses π and Π to be parabolically induced representations (from other parabolic subgroups). Moreover, it is easy to get upper and lower bounds on the Jacquet modules of common irreducible subquotients of $\mathrm{Ind}_{P_0}^{G}(\sigma)$ and π (in a simpler cases they can give exact Jacquet modules).

In what follows we shall use the following convention for $\alpha \in \mathbb{R}$: the representation

$$\nu^{\alpha}\, 1_{F^\times}$$

of $F^\times$, will be simply denoted by

$$\nu^{\alpha}$$

(this will considerably simplify some formulas, making them more transparent).

Example 1. *We shall now prove that*

$$\pi_0 := \nu^{1/2} \rtimes St_{SO(3)} = \nu^{1/2} \rtimes \delta(\nu^{1/2}; 1)$$

reduces. Set

$$\pi = \delta([\nu^{-1/2}, \nu^{1/2}]) \rtimes 1,$$
$$\Pi = \nu^{1/2} \times \nu^{1/2} \rtimes 1.$$

Obviously, (R1) holds for $\mathrm{Ind}_{P_0}^G(\sigma) = \pi_0$. *We easily compute from (14.2)*

$$s_{GL}(\pi_0) = \nu^{1/2} \times \nu^{1/2} \otimes 1 \quad + \quad \nu^{-1/2} \times \nu^{1/2} \otimes 1, \tag{20.1}$$

$$s_{GL}(\pi) = \nu^{1/2} \times \nu^{1/2} \otimes 1 \quad + \quad 2\delta([\nu^{-1/2}, \nu^{1/2}]) \otimes 1, \tag{20.2}$$

$$s_{GL}(\Pi) =$$
$$\nu^{1/2} \times \nu^{1/2} \otimes 1 \ + \ 2\nu^{-1/2} \times \nu^{1/2} \otimes 1 \ + \ \nu^{-1/2} \times \nu^{-1/2} \otimes 1.$$

The multiplicity of $\nu^{1/2} \times \nu^{1/2} \otimes 1$ *in all three above Jacquet modules is 1. Therefore, (R2) holds. Observe that* $L(\nu^{-1/2}, \nu^{1/2}) \otimes 1$ *is a subquotient of (20.1), but not of (20.2). Therefore, (R3) also holds. Thus, we have proved that* π_0 *is reducible.* □

Remark. *The above considerations also easily give the reducibility of*

$$\pi = \delta([\nu^{-1/2}, \nu^{1/2}]) \rtimes 1.$$

Namely, the multiplicity of $\delta([\nu^{-1/2}, \nu^{1/2}]) \otimes 1$ *in (20.1) is 1, and in (20.2) is 2, which implies* $s_{GL}(\pi) \not\leq s_{GL}(\pi_0)$. *Therefore* π *is also reducible.*

Since π *is unitarizable, Frobenius reciprocity implies that each irreducible subquotient of* π *has* $\delta([\nu^{-1/2}, \nu^{1/2}]) \otimes 1$ *in its Jacquet module, which implies that the length of* π *is 2. Denote by* τ_+ *the unique irreducible subquotient of* π *which has* $\nu^{1/2} \times \nu^{1/2} \otimes 1$ *in its Jacquet module, and the other one by* τ_-. *Clearly* $\pi = \tau_+ \oplus \tau_-$. *Then the above discussion implies*

$$s_{GL}(\tau_+) = \delta([\nu^{-1/2}, \nu^{1/2}]) \otimes 1 + \nu^{1/2} \times \nu^{1/2} \otimes 1 \tag{20.3}$$

and

$$s_{GL}(\tau_-) = \delta([\nu^{-1/2}, \nu^{1/2}]) \otimes 1.$$

Observe that $\tau_+ \leq \nu^{1/2} \rtimes \delta(\nu^{1/2}; 1)$ *since* $\nu^{1/2} \times \nu^{1/2} \otimes 1$ *has multiplicity one in both (20.1) and (20.2) (use that the Jacquet functor is exact). It also has multiplicity one in* $s_{GL}(\Pi)$. *Thus*

$$\tau_+ \leq \nu^{-1/2} \rtimes \delta(\nu^{1/2}; 1) \text{ and } \tau_+ \leq \delta([\nu^{-1/2}, \nu^{1/2}]) \rtimes 1. \tag{20.4}$$

The above representation $\pi = \delta([\nu^{-1/2}, \nu^{1/2}]) \rtimes 1$ is a particular case covered by the following very simple but pretty general criterion (Theorem 13.2 in [39]; it also covers non-unitary situations):

Theorem. *Let* $\rho \in \mathcal{C}$ *be unitarizable, and let* σ *be an irreducible cuspidal representation of* S_q. *If* ρ *is selfdual, suppose that* $\nu^\alpha \rho \rtimes \sigma$ *is irreducible for all* $\alpha \in \mathbb{R} \setminus (1/2)\mathbb{Z}$. *Assume* $\Delta \in \mathcal{S}$ *such that* $\Delta \subset \{\nu^\alpha \rho; \alpha \in \mathbb{R}\}$. *Then*

$$\delta(\Delta) \rtimes \sigma \text{ reduces } \iff \rho' \rtimes \sigma \text{ reduces for some } \rho' \in \Delta.$$

Let us illustrate the power of the above result with an example. Let $\Delta = [\chi, \nu^n\chi]$, where $\chi \in (F^\times)\tilde{}$ and $n \geq 0$. Consider odd orthogonal groups in this example. Then $\delta(\Delta) \rtimes 1$ reduces if and only if

$$[\chi, \nu^n\chi] \in \{[\nu^{-n-1/2}\chi_0, \nu^{-1/2}\chi_0], [\nu^{-n+1/2}\chi_0, \nu^{1/2}\chi_0], \dots, [\nu^{1/2}\chi_0, \nu^{n+1/2}\chi_0]\}$$

for some $\chi_0 \in (F^\times)\hat{}$ satisfying $\chi_0^2 \equiv 1$. So, we have a complete answer for the reducibility points of the (not necessarily unitary) degenerate principal series of $SO(2n+3, F)$ which are induced from the Siegel parabolic subgroup.

Example 2. *Let $m \in \mathbb{Z}$, $m \geq 2$. Now we shall prove the reducibility of*

$$\pi_0 := St_{GL(2m)} \rtimes St_{SO(3)} = \delta([\nu^{-(m-1/2)}, \nu^{m-1/2}]) \rtimes \delta(\nu^{1/2}; 1).$$

Set

$$\pi = \delta([\nu^{-(m-1/2)}, \nu^{1/2}]) \rtimes \delta([\nu^{1/2}, \nu^{m-1/2}]; 1),$$
$$\Pi = \delta([\nu^{-(m-1/2)}, \nu^{1/2}]) \times \delta([\nu^{1/2}, \nu^{m-1/2}]) \rtimes 1.$$

Now one sees easily that condition (R1) above is satisfied with $\mathrm{Ind}_{P_0}^G(\sigma) = \pi_0$. *From (14.2) one gets directly that the multiplicity of $\delta([\nu^{1/2}, \nu^{m-1/2}])^2 \times \nu^{1/2} \otimes 1$ in the Jacquet modules of all three above representations is 1. This implies that condition (R2) holds.*

From (14.2) (and (14.3)) we get

$$s_{GL}(\pi) = \left(\sum_{i=-m-1/2}^{1/2} \delta([\nu^{-i}, \nu^{m-1/2}]) \times \delta([\nu^{i+1}, \nu^{1/2}]) \right) \times \delta([\nu^{1/2}, \nu^{m-1/2}]) \otimes 1,$$

$$s_{GL}(\pi_0) = \left(\sum_{i=m-1/2}^{-m-1/2} \delta([\nu^{-i}, \nu^{m-1/2}]) \times \delta([\nu^{i+1}, \nu^{m-1/2}]) \right) \times \nu^{1/2} \otimes 1.$$

The multiplicity of $\delta([\nu^{-(m-1/2)}, \nu^{m-1/2}]) \times \nu^{1/2} \otimes 1$ in the Jacquet modules of π and π_0 is 1 and 2 respectively (since for π we can get this subquotient only for $i = -m - 1/2$, and then the multiplicity is 1, and for π_0 we get it for $i = -m - 1/2$ and $m - 1/2$, and the multiplicity is one in both cases). This shows (R3). Therefore, the reducibility of π_0 is proved. $\square$

An interesting question is what happens for $m = 1$. The above strategy does not work in this case. We shall address this question later.

Remarks. (1) *The representation $\pi_0 = St_{GL(2m)} \rtimes 1$ of $SO(2m+1, F)$ is reducible (for $m \geq 1$). One proves this in a similar way as in the Example 2. One takes $\pi = \delta([\nu^{-(m-1/2)}, \nu^{-1/2}]) \rtimes \delta([\nu^{1/2}, \nu^{m-1/2}]; 1)$, $\Pi = \delta([\nu^{-(m-1/2)}, \nu^{-1/2}]) \times \delta([\nu^{1/2}, \nu^{m-1/2}]) \rtimes 1$ and considers the multiplicities of $\delta([\nu^{1/2}, \nu^{m-1/2}])^2 \otimes 1$ and $\delta([\nu^{-(m-1/2)}, \nu^{m-1/2}]) \otimes 1$ in the Jacquet modules.*

The above reducibility also follows from the last theorem.

(2) *Let* $\rho \not\cong 1_{F^\times}$ *be an irreducible unitarizable cuspidal representation of* $GL(p,F)$ *and let the representation* $\delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \rtimes 1$ *of* $SO(mp+1,F)$ *be reducible. Then*

$$\pi_0 := \delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \rtimes \delta(\nu^{1/2};1) \quad \textit{reduces.}$$

This can be proved in the following way. We get directly

$$\delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \rtimes 1 = \tau_1 \oplus \tau_2,$$

where the τ_i *are irreducible and the multiplicity of*

$$\delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \otimes 1$$

in the Jacquet module of τ_i *is 1. Now take* $\pi = \nu^{1/2} \rtimes \tau_i$ *and* $\Pi = \delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \times \nu^{1/2} \rtimes 1$*. The multiplicity of*

$$\delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \times \nu^{1/2} \otimes 1$$

in the Jacquet modules of π_0, π *and* Π *is 2, 1 and 2 respectively. This implies the reducibility of* π_0.

21. Proving irreducibility

The reducibility of a parabolically induced representation has strong implications for its Jacquet modules (we shall list them below). A way to show the irreducibility of such a representation is to show that these implications can not occur. We discuss this in more detail below.

Let σ be an irreducible representation of a Levi factor M_0 of a parabolic subgroup $P_0 = M_0N_0$. Suppose that $\mathrm{Ind}_{P_0}^G(\sigma)$ reduces. Then in the Grothendieck group we can write $\mathrm{Ind}_{P_0}^G(\sigma) = \pi_1 + \pi_2$, with both $\pi_1 > 0$, $\pi_2 > 0$. For any standard parabolic subgroup P with standard Levi decomposition $P = MN$, let

$$T_{i,P} = r_M^G(\pi_i), \quad i = 1,2.$$

We can consider $T_{1,P}$ and $T_{2,P}$ as elements of $\mathcal{R}(M)$. Then, the following must hold:

($\mathfrak{R}1$) $T_{i,P} \geq 0$ and $T_{1,P} \neq 0$ if and only if $T_{2,P} \neq 0$;

($\mathfrak{R}2$) $T_{1,P} + T_{2,P} = r_M^G(\mathrm{Ind}_{P_0}^G(\sigma))$ in $\mathcal{R}(M)$;

($\mathfrak{R}3$) $r_{M_2}^{M_1}(T_{i,P_1}) = T_{i,P_2}$ when $P_1 \supset P_2$.

The first property follows from section 10. (see (3) there).

Therefore, if one can show that there is no system of $T_{i,P} \in \mathcal{R}(G)$, $i = 1,2$, when P is running over a subset of standard parabolic subgroups, which satisfy ($\mathfrak{R}1$) - ($\mathfrak{R}3$), then $\mathrm{Ind}_{P_0}^G(\sigma)$ is irreducible. Actually, in showing irreducibility it is often possible to show the non-existence of such a system for three proper standard parabolic subgroups P, P_1, P_2 satisfying $P \subset P_1, P_2$.

Note that conditions ($\mathfrak{R}1$) - ($\mathfrak{R}3$) are necessary for reducibility. An existence of such a system does not prove reducibility.

In the case that σ is unitarizable, one can have a simpler way of proving irreducibility of $\mathrm{Ind}_{P_0}^G(\sigma)$. For example, suppose that

$$\mathrm{Ind}_{P_0}^G(\sigma) \hookrightarrow \mathrm{Ind}_{P'}^G(\sigma')$$

for some irreducible representation σ' of the standard Levi subgroup M' of a standard parabolic subgroup P' of G. Now, if the multiplicity of σ' in $r_{M'}^G(\mathrm{Ind}_{P_0}^G(\sigma))$ is one (by Frobenius reciprocity it must be ≥ 1), then $\mathrm{Ind}_{P_0}^G(\sigma)$ is irreducible (see Example 1 in the following section, where irreducibility is proved in this way). This follows directly from the exactness of the Jacquet functors. In the other examples in the following section, one can find more subtle ways of proving irreducibility when σ is unitarizable.

22. Some half-integral examples of irreducibility

The examples that we study in this section illustrate methods for proving irreducibility. These examples are interesting for understanding phenomena which show up for general square integrable representations. This is the reason that we consider (unitary) tempered induced representations. We start with a very simple example and proceed with more complicated ones (one can find simple examples of different types in the following section).

Example 1 - Irreducibility of $St_{GL(1)} \rtimes St_{SO(3)}$. *Let*

$$\pi_0 = St_{GL(1)} \rtimes St_{SO(3)} = \nu^0 \rtimes \delta(\nu^{1/2}; 1).$$

Applying (14.2) write

$$\begin{aligned}
\mu^*(\pi_0) &= M^*(\nu^0) \rtimes \mu^*(\delta(\nu^{1/2}; 1)) \\
&= (1 \otimes \nu^0 + 2\nu^0 \otimes 1) \rtimes (1 \otimes \delta(\nu^{1/2}; 1) + \nu^{1/2} \otimes 1) \\
&= 1 \otimes \pi_0 + \left[2\nu^0 \otimes \delta(\nu^{1/2}; 1) + \nu^{1/2} \otimes \nu^0 \rtimes 1\right] + 2\nu^0 \times \nu^{1/2} \otimes 1.
\end{aligned}$$

Let π be any irreducible subquotient of π_0. Then π is a subrepresentation since π_0 is unitarizable. Now we have

$$\pi \hookrightarrow \nu^0 \rtimes \delta(\nu^{1/2}; 1) \hookrightarrow \nu^0 \times \nu^{1/2} \rtimes 1 \cong \nu^{1/2} \times \nu^0 \rtimes 1.$$

Frobenius reciprocity implies that (irreducible representation) $\nu^{1/2} \otimes \nu^0 \rtimes 1$ is in the Jacquet module of π. Since the multiplicity of $\nu^{1/2} \otimes \nu^0 \rtimes 1$ in the Jacquet module of π_0 is 1, we conclude that the length of π_0 is 1 (since the Jacquet functor is exact), i.e. that π_0 is irreducible. □

Example 2 - Irreducibility of $St_{GL(2m+1)} \rtimes St_{SO(3)}$ for $m \in \mathbb{Z}_{\geq 1}$. *Let*

$$\pi_0 = St_{GL(2m+1)} \rtimes St_{SO(3)} = \delta([\nu^{-m}, \nu^m]) \rtimes \delta(\nu^{1/2}; 1).$$

We shall sketch the proof (it uses a similar idea as the proof of Proposition 4.2 in [39]; see there for more details). Applying (14.2) and (14.3) we get

$$\mu^*(\pi_0) = M^*(\delta([\nu^{-m}, \nu^m])) \rtimes \mu^*(\delta(\nu^{1/2}; 1))$$

$$= \Big(\sum_{i=-m-1}^{m} \sum_{j=i}^{m} \delta([\nu^{-i},\nu^m]) \times \delta([\nu^{j+1},\nu^m]) \otimes \delta([\nu^{i+1},\nu^j]) \Big)$$
$$\rtimes (1 \otimes \delta(\nu^{1/2};1) + \nu^{1/2} \otimes 1).$$

Let π be any irreducible subquotient of π_0. Then it is a subrepresentation since π_0 is unitarizable. Using simple facts about irreducibility of principal series in $GL(2,F)$ and $SO(3,F)$, and properties of $\rtimes$ and $\times$ (see sections 12. and 14.) we get

$$\begin{aligned}\pi \hookrightarrow \pi_0 &\hookrightarrow \nu^m \times \nu^{m-1} \times \cdots \times \nu^{-m+1} \times \nu^{-m} \times \nu^{1/2} \rtimes 1\\ &\cong \nu^m \times \nu^{m-1} \times \cdots \times \nu^{-m+1} \times \nu^{1/2} \times \nu^{-m} \rtimes 1\\ &\cong \nu^m \times \nu^{m-1} \times \cdots \times \nu^{-m+1} \times \nu^{1/2} \times \nu^{m} \rtimes 1\\ &\cong \nu^m \times \nu^{m-1} \times \cdots \times \nu^{-m+1} \times \nu^{m} \times \nu^{1/2} \rtimes 1\\ &\cdots\cdots\cdots\cdots\\ &\cong \nu^m \times \nu^{m-1} \times \cdots \times \nu^{0} \times \nu^{m} \times \nu^{m-1} \times \cdots \times \nu \times \nu^{1/2} \rtimes 1.\end{aligned}$$

Frobenius reciprocity implies that

$$\sigma_1 := \nu^m \otimes \nu^{m-1} \otimes \cdots \otimes \nu^0 \otimes \nu^m \otimes \nu^{m-1} \otimes \cdots \otimes \nu \otimes \nu^{1/2} \otimes 1$$

appears in the Jacquet module of π (as a quotient).

Now we want to see from which terms in $s_{(2m)}(\pi_0)$ we can get σ_1 as a subquotient (by the transitivity of Jacquet modules, σ_1 must be a subquotient of some term of $s_{(2m)}(\pi_0)$ since $\pi \leq \pi_0$). From the above formula for $\mu^*(\pi_0)$ we get

$$s.s.(s_{(2m)}(\pi_0)) = \Big(\sum_{i=-m-1}^{m} \delta([\nu^{-i},\nu^m]) \times \delta([\nu^{i+1},\nu^m]) \otimes \delta(\nu^{1/2};1) \Big)$$
$$+ \Big(\sum_{i=-m-1}^{m-1} \delta([\nu^{-i},\nu^m]) \times \delta([\nu^{i+2},\nu^m]) \times \nu^{1/2} \otimes \nu^{i+1} \rtimes 1 \Big).$$

We can get σ_1 as a subquotient of a term in $s.s.(s_{(2m)}(\pi_0))$ only from the first sum showing up in the formula for $s.s.(s_{(2m)}(\pi_0))$ (since irreducible subquotients of the Jacquet module with respect to the standard minimal parabolic subgroup of terms in the second sum have the form

$$\cdots\cdots\cdots\cdots \otimes \nu^{\pm(i+1)} \otimes 1,$$

with $\pm(i+1) \in \mathbb{Z}$, which cannot give σ_1 since we always have $\pm(i+1) \neq 1/2$). Further, since all exponents of the tensor factors of σ_1 are ≥ 0, we can get σ_1 for a subquotient of Jacquet module of π_0 only if $-i \geq 0$ and $i+1 \geq 0$ (use the remark in section 12. and the definition of $\delta(\Delta)$ in section 13.). Thus $i = -1$ or 0. Both choices give the same representation (up to an equivalence), the irreducible representation $\delta([\nu,\nu^m]) \times \delta([\nu^0,\nu^m]) \otimes \delta(\nu^{1/2};1)$. This representation also must be a subquotient of the Jacquet module of π.

Since

$$\delta([\nu,\nu^m])\times\delta([\nu^0,\nu^m])\otimes\delta(\nu^{1/2};1)$$
$$\hookrightarrow \nu^m\times\nu^{m-1}\times\cdots\times\nu^2\times\nu\times\nu^m\times\nu^{m-1}\times\cdots\times\nu\times\nu^0\otimes\nu^{1/2}\rtimes 1,$$

Frobenius reciprocity implies that

$$\sigma_2 := \nu^m\otimes\nu^{m-1}\otimes\cdots\otimes\nu^2\otimes\nu\otimes\nu^m\otimes\nu^{m-1}\otimes\cdots\otimes\nu\otimes\nu^0\otimes\nu^{1/2}\otimes 1$$

is in the Jacquet module of $\delta([\nu,\nu^m])\times\delta([\nu^0,\nu^m])\otimes\delta(\nu^{1/2};1)$*, and further also in the Jacquet module of* π*.*

Now we shall analyze from which subquotients of $s_{(2m-1)}(\pi_0)$ *we can get* σ_2 *in the Jacquet module. The formula for* $\mu^*(\pi_0)$ *implies*

$$s.s.(s_{(2m-1)}(\pi_0)) = \Big(\sum_{i=-m-1}^{m-1}\delta([\nu^{-i},\nu^m])\times\delta([\nu^{i+2},\nu^m])\otimes\nu^{i+1}\rtimes\delta(\nu^{1/2};1)\Big)$$
$$+\Big(\sum_{i=-m-1}^{m-2}\delta([\nu^{-i},\nu^m])\times\delta([\nu^{i+3},\nu^m])\times\nu^{1/2}\otimes\delta([\nu^{i+1},\nu^{i+2}])\rtimes 1\Big).$$

Again, in a similar way as before, we get that σ_2 *can come only from the first sum. Since the tensor product of* σ_2 *ends with* $\nu^0\otimes\nu^{1/2}\otimes 1$*, we conclude that* σ_2 *can come only from the term in the first sum corresponding to* $i+1=0$*, i.e.* $i=-1$*.*

Therefore, the only term of $s.s.(s_{(2m-1)}(\pi_0))$ *which can give* σ_2 *is*

$$\delta([\nu,\nu^m])\times\delta([\nu,\nu^m])\otimes\nu^0\rtimes\delta(\nu^{1/2};1).$$

This representation is irreducible by the previous example (and section 13.), and we have seen that it has multiplicity one in $s_{(2m-1)}(\pi_0)$*. Since* σ_2 *is in the Jacquet module of* π*, this representation is also in the Jacquet module of* π*. Now the multiplicity one property of this representation in* $s_{(2m-1)}(\pi_0)$ *implies that* $\pi=\pi_0$*, i.e. that* π_0 *is irreducible.* □

The following example of irreducibility is not quite as simple to prove, but it is very important since this irreducibility is in a sense singular. Namely, recall that we have proved in section 20. that $St_{GL(2m)}\rtimes St_{SO(3)}$ is reducible for all $m\geq 2$. The case of $m=1$ which we consider below is therefore a kind of surprise:

Example 3 - Irreducibility of $St_{GL(2)}\rtimes St_{SO(3)}$**.** *Our aim will now be to prove the irreducibility of*

$$\pi_0 = St_{GL(2)}\rtimes St_{SO(3)} = \delta([\nu^{-1/2},\nu^{1/2}])\rtimes\delta(\nu^{1/2};1).$$

Suppose that π_0 *reduces. First we get from (14.2) and (14.3))*

$$\mu^*(\pi_0) = 1\otimes\pi_0 + \tag{22.1}$$
$$\big[2\nu^{1/2}\otimes\nu^{-1/2}\rtimes\delta(\nu^{1/2};1)+\nu^{1/2}\otimes\delta([\nu^{-1/2},\nu^{1/2}])\rtimes 1\big]+$$
$$\big[2\boxed{\delta([\nu^{-1/2},\nu^{1/2}])\otimes\delta(\nu^{1/2};1)}+3\,\nu^{1/2}\times\nu^{1/2}\otimes\delta(\nu^{1/2};1)+2\nu^{1/2}\times\nu^{1/2}\otimes 1_{SO(3)}\big]$$

$$+\big[2\delta([\nu^{-1/2},\nu^{1/2}])\times\nu^{1/2}\otimes 1+\nu^{1/2}\times\nu^{1/2}\times\nu^{1/2}\otimes 1\big].$$

Since π_0 is unitarizable, each irreducible subquotient of π_0 is a subrepresentation. Now Frobenius reciprocity implies that $\delta([\nu^{-1/2},\nu^{1/2}])\otimes\delta(\nu^{1/2};1)$ is in the Jacquet module of each irreducible subquotient of π_0. Observe that the Jacquet module of $\delta([\nu^{-1/2},\nu^{1/2}])\otimes\delta(\nu^{1/2};1)$ with respect to the minimal parabolic subgroup is

$$\nu^{1/2}\otimes\nu^{-1/2}\otimes\nu^{1/2}\otimes 1. \tag{22.2}$$

Note that all the representations in the third and fourth lines of (22.1) are irreducible. Observe that the fourth line in (22.1) has length 3, and that $\nu^{1/2}\times\nu^{1/2}\times\nu^{1/2}\otimes 1$ has multiplicity one in that line. Denote by π_1 the (unique) irreducible subquotient of π_0 containing $\nu^{1/2}\times\nu^{1/2}\times\nu^{1/2}\otimes 1$ in its Jacquet module. The (semi simplification of the) Jacquet module of $\nu^{1/2}\times\nu^{1/2}\times\nu^{1/2}\otimes 1$ with respect to the minimal parabolic subgroup is

$$6\ \nu^{1/2}\otimes\nu^{1/2}\otimes\nu^{1/2}\otimes 1. \tag{22.3}$$

Recall that π_1 has (22.2) in its Jacquet module, which does not belong to (22.3). This fact and the reducibility of π_0 (together with the transitivity of Jacquet modules) imply

$$s_{(3)}(\pi_1)=\delta([\nu^{-1/2},\nu^{1/2}])\times\nu^{1/2}\otimes 1+\nu^{1/2}\times\nu^{1/2}\times\nu^{1/2}\otimes 1.$$

Now from the fourth line of (22.1), it follows that π_0 is a length two representation. Denote the other irreducible subquotient by π_2. Then

$$s_{(3)}(\pi_2)=\delta([\nu^{-1/2},\nu^{1/2}])\times\nu^{1/2}\otimes 1.$$

The above two formulas imply

$$s_{(1,1,1)}(\pi_1)= \tag{22.4}$$

$$\nu^{1/2}\otimes\nu^{-1/2}\otimes\nu^{1/2}\otimes 1+2\ \nu^{1/2}\otimes\nu^{1/2}\otimes\nu^{-1/2}\otimes 1+6\ \nu^{1/2}\otimes\nu^{1/2}\otimes\nu^{1/2}\otimes 1,$$

$$s_{(1,1,1)}(\pi_2)=\nu^{1/2}\otimes\nu^{-1/2}\otimes\nu^{1/2}\otimes 1+2\ \nu^{1/2}\otimes\nu^{1/2}\otimes\nu^{-1/2}\otimes 1. \tag{22.5}$$

From the other side, we see from the second line of (22.1) and from (20.4) that

$$3\nu^{1/2}\otimes\tau_+\leq s_{(1)}(\pi_0).$$

Observe that $\nu^{1/2}\otimes\tau_+\not\leq s_{(1)}(\pi_2)$ since

$$\nu^{1/2}\otimes s_{(1,1)}(\tau_+)=2\ \nu^{1/2}\otimes\nu^{1/2}\otimes\nu^{1/2}\otimes 1+\nu^{1/2}\otimes\nu^{1/2}\otimes\nu^{-1/2}\otimes 1$$

and $\nu^{1/2}\otimes\nu^{1/2}\otimes\nu^{1/2}\otimes 1$ does not show up in (22.5). Now $\pi_0=\pi_1\oplus\pi_2$ implies $3\nu^{1/2}\otimes\tau_+\leq s_{(1)}(\pi_1)$, which implies $3\ \nu^{1/2}\otimes s_{(1,1)}(\tau_+)\leq s_{(1,1,1)}(\pi_1)$. This further implies $3\ \nu^{1/2}\otimes\nu^{1/2}\otimes\nu^{-1/2}\otimes 1\leq s_{(1,1,1)}(\pi_1)$. This clearly contradicts (22.4). This contradiction completes the proof of the irreducibility of π_0. $\square$

Remarks. (1) *Let us sum up what we have proved up to now for*

$$St_{GL(m)} \rtimes St_{SO(3)} = \delta(1_{F^\times}, m) \rtimes \delta(\nu^{1/2}; 1).$$

This representation is

- **irreducible** *for all* **odd** m,
- **reducible** *for all* **even** m **except** $m = 2$.

(2) *One can prove that the representation $\delta([\nu^{-m}, \nu^m]) \rtimes 1$, $m \in \mathbb{Z}_{\geq 0}$ of $SO(2m+1, F)$ is irreducible (this follows from Proposition 4.2 in [39]; it can be proved in a similar way as the second example of this section).*

(3) *Suppose that ρ is an irreducible unitarizable cuspidal representation of some general linear group such that $\delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \rtimes 1$ is irreducible. In a similar way as in the second example of this section one can prove that*

$$\begin{aligned}\pi_0 := \delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \rtimes \delta(\nu^{1/2}; 1) \\ = \delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \rtimes St_{SO(3)}\end{aligned}$$

is irreducible.

A very nice fact is that the above considerations hold much more generally. They hold whenever we have irreducible unitarizable cuspidal representations ρ and σ of $GL(p, F)$ and S_q respectively, such that $\nu^{1/2}\rho \rtimes \sigma$ reduces, with completely analogous proofs. It is also not important which series of classical groups is in question, and whether ρ and σ are just characters or infinite dimensional representations, etc. Everything is done in $R(S)$ (our case was $p = 1$, $S_q = SO(1) = \{1\}$, $\rho = 1_{F^\times}$ and $\sigma = 1$). For example, in this case

$$\delta([\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]) \rtimes \delta(\nu^{1/2}\rho; \sigma)$$

is irreducible for each odd positive integer, and is reducible for each even m with one exception. This exception is 2.

Remark. *Now we take a close look at the following special case (which follows from previous considerations). We shall consider only representations of $SO(2n+1, F)$ in this remark. Let ρ be a selfdual irreducible cuspidal representation of $GL(p, F)$ such that the representation $\nu^{1/2}\rho \rtimes 1$ (of $SO(2p+1, F)$) reduces. Then for any selfdual irreducible cuspidal representation ρ' of $GL(p', F)$ the following holds: for exactly one parity of positive integers we have that*

(1) *$\delta(\rho', m) \rtimes \delta(\nu^{1/2}\rho; 1)$ reduces for all m from that parity, with possible finitely many exceptions;*

(2) *$\delta(\rho', m) \rtimes \delta(\nu^{1/2}\rho; 1)$ is irreducible for all m from the other parity.*

Note that the only exception in (1) shows up for $\rho' \cong \rho$ and $m = 2$. Therefore, we can characterize $\delta(\nu^{1/2}\rho; 1)$ with this exception.

The parity in (1) is determined by ρ': it is odd if $\nu^{1/2}\rho' \rtimes 1$ is irreducible, and even otherwise.

At this point the reader who is more interested in square integrability than in the reducibility questions can skip directly to section 28.

23. Some integral examples of irreducibility (and reducibility)

We shall illustrate how one gets reducibility points in the case of

$$\chi \rtimes St_{Sp(2)} = \chi \rtimes \delta(\nu; 1), \quad \chi \in (F^\times)\tilde{},$$

in a little bit different way than above.

Using (14.2), we write semi simplifications of Jacquet modules of $\chi \rtimes St_{Sp(2)}$ with respect to the maximal parabolic subgroups in the following way:

$$\begin{array}{ccccc}
\text{s.s.}(s_{(1)}(\chi \rtimes St_{Sp(2)})) & \longleftarrow & \chi \rtimes St_{Sp(2)} & \longrightarrow & \text{s.s.}(s_{(2)}(\chi \rtimes St_{Sp(2)})) \\
\Big\| & & & & \Big\| \\
\begin{array}{c}\chi \otimes St_{Sp(2)} + \chi^{-1} \otimes St_{Sp(2)} \\ + \ \underline{\nu \otimes \chi \rtimes 1}\end{array} & & & & \underline{\chi \times \nu \otimes 1} \ + \underline{\chi^{-1} \times \nu \otimes 1}.
\end{array}$$

Assume

$$\chi^2 \neq 1_{F^\times} \text{ and } \chi \neq \nu^{\pm 1}, \nu^{\pm 2}.$$

Then all five representations in the above two sums are irreducible. In particular,

$$\nu \otimes \chi \rtimes 1, \ \chi \times \nu \otimes 1 \text{ and } \chi^{-1} \times \nu \otimes 1$$

are irreducible. Now we shall concentrate our attention on the Jacquet modules of these three representations:

$$\begin{array}{cccc}
\text{---} + \text{---} + \nu \otimes \chi \rtimes 1 & \chi \times \nu \otimes 1 & + & \chi^{-1} \times \nu \otimes 1 \\
\downarrow & \downarrow & & \downarrow \\
\text{---} + \text{---} + \nu \otimes \chi \otimes 1 + \nu \otimes \chi^{-1} \otimes 1 & \begin{array}{c}\chi \otimes \nu \otimes 1 + \\ \underline{\nu \otimes \chi \otimes 1}\end{array} & + & \begin{array}{c}\chi^{-1} \otimes \nu \otimes 1 + \\ \underline{\nu \otimes \chi^{-1} \otimes 1}\end{array}
\end{array}$$

Let π be an irreducible subquotient of $\chi \rtimes St_{Sp(2)}$ which has

$$\nu \otimes \chi \rtimes 1$$

as a subquotient of the Jacquet module with respect to $P_{(1)}$ (clearly, such a π exists). Then π must have

$$\nu \otimes \chi \otimes 1 \text{ and } \nu \otimes \chi^{-1} \otimes 1$$

as subquotients of its Jacquet module with respect to the minimal parabolic subgroup (see the diagram). These characters are the underlined members on the right hand side of the above diagram. Since these underlined members are coming from different subquotients of the Jacquet module with respect to $P_{(2)}$, we see that π must have

$$\chi \times \nu \otimes 1 \text{ and } \chi^{-1} \times \nu \otimes 1$$

in its Jacquet module. Since these two representations form the whole corresponding Jacquet module of $\chi \rtimes St_{Sp(2)}$, the exactness of the Jacquet modules implies $\pi = \chi \rtimes St_{Sp(2)}$. Thus, we have proved that $\chi \rtimes St_{Sp(2)}$ is irreducible in this case.

Now we go to reducibility. The representation $\chi \rtimes St_{Sp(2)}$ is reducible if $\chi = \nu^{\pm 2}$ or $\chi^2 = \nu^0$. We shall explain how one proves the reducibility in the case $\chi = \nu^0$ (the case $\chi = \nu^{\pm 2}$ is easy, since then one immediately gets $St_{Sp(4)}$ as a subquotient). Obviously

$$\nu^0 \rtimes St_{Sp(2)} \leq \nu^0 \times \nu \rtimes 1, \tag{23.1}$$

$$\nu^{1/2} St_{GL(2)} \rtimes 1 = \delta([\nu^0, \nu]) \rtimes 1 \leq \nu^0 \times \nu \rtimes 1, \tag{23.2}$$

and further from (14.2) and the representation theory of $GL(n)$ (actually, of $GL(2)$), it follows that

$$\begin{gathered} s_{(2)}(\nu^0 \rtimes St_{Sp(2)}) = 2(\nu^0 \times \nu \otimes 1) \not\leq \\ s_{(2)}(\nu^{1/2} St_{GL(2)} \rtimes 1) = \delta([\nu^0, \nu]) \otimes 1 + \nu^0 \times \nu \otimes 1 + \delta([\nu^{-1}, \nu^0]) \otimes 1, \end{gathered} \tag{23.3}$$

$$\begin{gathered} s_{(2)}(\nu^0 \rtimes St_{Sp(2)})) + s_{(2)}(\nu^{1/2} St_{GL(2)} \rtimes 1) \not\leq \\ s_{(2)}(\nu^0 \times \nu \rtimes 1) = 2(\nu^0 \times \nu \otimes 1) + 2(\nu^0 \times \nu^{-1} \otimes 1). \end{gathered} \tag{23.4}$$

The relations (23.1) – (23.4) and the exactness of the Jacquet modules imply the reducibility of $\nu^0 \rtimes St_{Sp(2)}$. In a similar way, one shows reducibility if $\chi^2 = 1_{F^\times}$ but $\chi \not\equiv 1_{F^\times}$ (here one considers the subquotients $\chi \rtimes St_{Sp(2)}$ and $\nu \rtimes \tau$ of $\chi \times \nu \rtimes 1$, where τ is an irreducible subrepresentation of $\chi \rtimes 1$; here one also needs to consider Jacquet modules with respect to $P_{(1)}$).

In the case of $\chi = \nu^{\pm 1}$ we again get irreducibility, but the proof of this fact does not rely on Jacquet modules of $\nu^{\pm 1} \rtimes St_{Sp(2)}$ only (see the following section).

24. A delicate case

In this section, we consider the remaining case of irreducibility which we have mentioned above (this is a special case of Proposition 5.1 of [39]). In this case we can neither prove reducibility nor irreducibility by the above methods. This is one of a few cases where we have a system as in section 21., but we cannot prove reducibility using the strategy of section 20. The idea is to go to larger groups, and get a contradiction there (with Frobenius reciprocity and/or the theory of R-groups). Namely, if we suppose that some parabolically induced representation of $Sp(4, F)$ reduces, it does not affect only the representation theory of $Sp(4, F)$, but also of all $Sp(6, F)$, $Sp(8, F)$, etc. (for example, if the inducing representation is unitarizable, it affects all the unitary duals).

Such cases, which we call delicate cases, are infrequent in the following sense. For example, in the case of symplectic and odd-orthogonal groups, for solving the question of reducibility of any degenerate principal series induced from any maximal parabolic subgroup (i.e. representation induced by one-dimensional representation), which was settled by C. Jantzen in [14], there are only two such cases: irreducibility of the representations $\nu^{1/2}\psi_0 1_{GL(2)} \rtimes 1$ of $SO(5)$ and of $\nu \rtimes 1_{Sp(2)}$ of $Sp(4)$, where ψ_0 is a character of $GL(2,F)$ whose square is trivial (i.e., a character of order one or two). These cases (actually, much more general cases) were settled in [39] (Propositions 6.3 and 5.1). We shall give below the proof of the second irreducibility in the "dual" situation, since later on our primary interest will be square integrable representations. For the moment, let us observe that if we suppose reducibility of the first representation, a very simple case of the theory of R-groups leads to a contradiction in the representation theory of $SO(9)$, while for the second representation we get contradiction in the representation theory of $Sp(6)$ (this contradiction is related to Frobenius reciprocity).

Let us note that this (sometimes very effective) method of solving a problem by going to larger groups was applied systematically in our work on unitarizability of general linear groups ([34]).

We shall now handle a more general case which implies the irreducibility of the representation $\nu \rtimes St_{Sp(2)}$ of $Sp(4)$:

Proposition. *Let ρ be an irreducible unitarizable cuspidal representation of $GL(p,F)$ and σ an irreducible cuspidal representation of S_q. Assume that $\nu\rho \rtimes \sigma$ reduces (which implies that $\rho \rtimes \sigma$ does not reduce). Then $\nu\rho \rtimes \delta(\nu\rho;\sigma)$ is irreducible.*

Proof. Let

$$\pi_0 = \nu\rho \rtimes \delta(\nu\rho;\sigma).$$

We have

$$\begin{aligned}\mu^*(\pi_0) = 1 \otimes \pi_0 \\ &+[\nu\rho \otimes \nu\rho \rtimes \sigma + \nu\rho \otimes \delta(\nu\rho;\sigma) + \nu^{-1}\rho \otimes \delta(\nu\rho;\sigma)] \\ &+[\nu\rho \times \nu\rho \otimes \sigma + \nu^{-1}\rho \times \nu\rho \otimes \sigma].\end{aligned}$$

From this formula we see that there exists an irreducible subrepresentation π of π_0 satisfying $\nu\rho \times \nu\rho \otimes \sigma \leq s_{(2p)}(\pi)$. Suppose that $\nu\rho \rtimes \delta(\nu\rho;\sigma)$ reduces. Then since $s_{GL}(\pi_0)$ has length 2, we conclude

$$s_{(2p)}(\pi) = \nu\rho \times \nu\rho \otimes \sigma.$$

Now consider

$$\Pi = \rho \times \nu\rho \times \nu^{-1}\rho \rtimes \sigma.$$

Note that

$$\rho \rtimes \pi \leq \Pi, \quad \delta([\nu^{-1}\rho, \nu\rho]) \rtimes \sigma \leq \Pi \tag{24.1}$$

in the Grothendieck group. We have directly

$$\begin{aligned}&\text{s.s.}(s_{(3p)}(\Pi)) = \\ &\quad 4\rho \times \nu\rho \times \nu^{-1}\rho \otimes \sigma + 2\rho \times \nu^{-1}\rho \times \nu^{-1}\rho \otimes \sigma + 2\rho \times \nu\rho \times \nu\rho \otimes \sigma, \\ &\text{s.s.}(s_{(3p)}(\rho \rtimes \pi)) = 2\rho \times \nu\rho \times \nu\rho \otimes \sigma\end{aligned}$$

and

$$\text{s.s.}(s_{(3p)}(\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma)) = 2\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma + 2\delta([\rho,\nu\rho])\times\nu\rho\otimes\sigma.$$

Note that the multiplicity of

$$\nu\rho\times\delta([\rho,\nu\rho])\otimes\sigma$$

is two in the above three Jacquet modules. Now (24.1) implies that there exists an irreducible representation φ such that

$$\varphi \leq \delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma$$

and

$$\varphi \leq \rho\rtimes\pi.$$

Since

$$\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma$$

is unitarizable, each irreducible subquotient of it is actually a subrepresentation. Frobenius reciprocity implies the existence of a non-trivial intertwining

$$s_{(3p)}(\varphi) \twoheadrightarrow \delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma,$$

which implies

$$\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma \leq s_{(3p)}(\varphi).$$

Note that $\varphi \leq \rho\rtimes\pi$ implies

$$\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma \leq s_{(3p)}(\rho\rtimes\pi).$$

This contradicts

$$\text{s.s.}(s_{(3p)}(\rho\rtimes\pi)) = 2\rho\times\nu\rho\times\nu\rho\otimes\sigma.$$

This contradiction finishes the proof. □

Now we shall present another (more complicated) delicate case, where in the proof we use a consequence of the theory of R-groups (later on we shall recall this result). This is the reason why we restrict to the characteristic zero case. We shall only state the result here, which is Proposition 6.3 of [39]:

Proposition. *Let ρ and σ be irreducible cuspidal representations of groups $GL(p,F)$ and S_q respectively, such that ρ is selfdual (i.e. $\rho\cong\tilde{\rho}$). Suppose that* char $F=0$ *and that both $\rho\rtimes\sigma$ and $\nu\rho\rtimes\sigma$ are irreducible. Then $\delta([\rho,\nu\rho])\rtimes\sigma$ is also irreducible.*

25. Langlands parameters of irreducible subquotients

As we have already mentioned, one can solve the reducibility question in surprising generality using the above simple methods. Moreover, one can get in many cases also lengths and Langlands parameters of irreducible subquotients at the reducibility points (this is done in [14] for all degenerate principal series representations of groups S_n induced from a maximal parabolic subgroup). We shall not go much into this direction. Rather we shall show with a simple example the very basic idea of the method (C. Jantzen and G. Muić have improved the basic method introduced in [39] very significantly).

First we shall recall the Langlands classification in the case of the groups S_n. Denote by T the set of all equivalence classes of irreducible tempered representations of all S_n, $n \geq 0$. Set

$$D_+ = \{\delta \in D; e(\delta) > 0\}.$$

Let

$$t = (\delta_1, \dots, \delta_k; \tau) \in M(D_+) \times T.$$

Take a permutation p of $\{1, \dots, k\}$ such that

$$e(\delta_{p(1)}) \geq \cdots \geq e(\delta_{p(k)}).$$

Then the representation

$$\delta_{p(1)} \times \cdots \times \delta_{p(k)} \rtimes \tau \tag{25.1}$$

has a unique irreducible quotient, which we denote by $L(t)$. The mapping $t \mapsto L(t)$ is a bijection of $M(D_+) \times T$ onto $\cup_{n=0}^{\infty} \widetilde{S}_n$. This is the Langlands classification for the groups S_n.

The multiplicity of $L(t)$ in (25.1) is 1. Further, the representation

$$\tilde{\delta}_{p(1)} \otimes \cdots \otimes \tilde{\delta}_{p(k)} \otimes \tau$$

is a subquotient in the Jacquet module of $L(t)$ (later on, we shall use this fact).

Example. *Consider the following representation*

$$\pi_0 := \nu^{n+1} \rtimes L(\nu^n, \nu^{n-1}, \dots, \nu; 1).$$

Obviously we have an epimorphism

$$\phi : \nu^{n+1} \times \nu^n \times \cdots \times \nu \rtimes 1 \to \pi_0.$$

Therefore

$$L(\nu^{n+1}, \nu^n, \dots, \nu; 1)$$

is a (sub)quotient of π_0. *The embedding* $\delta([\nu^n, \nu^{n+1}]) \hookrightarrow \nu^{n+1} \times \nu^n$ *lifts to an embedding*

$$i : \delta([\nu^n, \nu^{n+1}]) \times \nu^{n-1} \times \cdots \times \nu \rtimes 1 \hookrightarrow \nu^{n+1} \times \nu^n \times \cdots \times \nu \rtimes 1.$$

We want to show

$$\phi \circ i \neq 0.$$

This would imply that

$$L(\delta([\nu^n,\nu^{n+1}]),\nu^{n-1},\dots,\nu;1)$$

is a subquotient of π_0 *(note that this would further imply that* π_0 *is reducible).*

Suppose $\phi\circ i=0$*. Then we would have an epimorphism from the quotient of the domain of* ϕ *by the domain of* i *onto* π_0*. This quotient is isomorphic to*

$$L(\nu^n,\nu^{n+1})\times\nu^{n-1}\times\cdots\times\nu\rtimes 1.$$

This implies

$$s_{GL}(\pi_0)\le s_{GL}(L(\nu^n,\nu^{n+1})\times\nu^{n-1}\times\cdots\times\nu\rtimes 1).$$

Now (14.2) and Bernstein-Zelevinsky theory implies that this is not possible. Thus, $\phi\circ i\neq 0$*.*

A simple analysis of Jacquet modules of π_0 *would give that the length of* π_0 *is 2. Thus, in the Grothendieck group we have*

$$\begin{aligned}\pi_0&=\nu^{n+1}\rtimes L(\nu^n,\nu^{n-1},\dots,\nu;1)\\&=L(\nu^{n+1},\nu^n,\dots,\nu;1)+L(\delta([\nu^n,\nu^{n+1}]),\nu^{n-1},\dots,\nu;1).\end{aligned}$$

One can find in [10] and [11] examples of the study of the internal structure of induced representations with help of Jacquet modules (the semi simplicity of certain induced representations is studied there, from which is obtained non-unitarizability of some (non-induced) representations).

26. An interesting integral tempered irreducibility

Now we shall prove the following proposition (later on, we shall explain why this example is interesting for our considerations).

Proposition. *The representation*

$$St_{GL(3)}\rtimes St_{Sp(2)}=\delta([\nu^{-1},\nu])\rtimes\delta(\nu;1)$$

is irreducible.

Proof. Denote the representation in the proposition by π_0. From (14.3), it follows that

$$M^*(\delta([\nu^{-1},\nu]))=\sum_{k=-2}^{1}\sum_{l=k}^{1}\delta([\nu^{-k},\nu])\times\delta([\nu^{l+1},\nu])\otimes\delta([\nu^{k+1},\nu^l]).$$

Recall $\mu^*(\delta(\nu;1))=1\otimes\delta(\nu;1)+\nu\otimes 1$. Now applying (14.2), we get directly

$$\mu^*(\pi_0)=1\otimes\pi_0$$

$$+\left[2(\nu\otimes\delta([\nu^{-1},\nu^0])\rtimes\delta(\nu;1))+\nu\otimes\delta([\nu^{-1},\nu])\rtimes 1\right]\tag{26.1}$$

(26.2)

$$+\left[2(\boxed{\delta([\nu^0,\nu])\otimes\nu^{-1}\rtimes\delta(\nu;1)})+2(\nu\times\nu\otimes\delta([\nu^{-1},\nu^0])\rtimes 1)+\nu\times\nu\otimes\nu^0\rtimes\delta(\nu,1)\right]$$

$$+\left[2\left(\boxed{\delta([\nu^{-1},\nu])\otimes\delta(\nu;1)}\right)+2(\delta([\nu^0,\nu])\times\nu\otimes\nu^{-1}\rtimes 1)\right. \tag{26.3}$$

$$\left. +2(\nu\times\delta([\nu^0,\nu])\otimes\delta(\nu;1))+\boxed{\nu\times\nu\times\nu\otimes\nu^0\rtimes 1}\right] \tag{26.4}$$

$$\left[2(\delta([\nu^{-1},\nu])\times\nu\otimes 1)+2\left(\boxed{\delta([\nu^0,\nu])\times\nu\times\nu\otimes 1}\right)\right]. \tag{26.5}$$

Now we analyze the above formulas. First observe that all boxed representations are irreducible (recall that the boxed representation in line (26.2) is irreducible by the first proposition in section 24.). Further, if we have an irreducible subquotient π' of π_0, then unitarizability implies that it is a subrepresentation. Now Frobenius reciprocity implies that it must have the boxed term in line (26.3) in its Jacquet module.

The transitivity of Jacquet modules implies that $s_{(1,1,1,1)}(\pi')$ must have

$$\nu\otimes\nu^0\otimes\nu^{-1}\otimes\nu\otimes 1$$

as a subquotient. Note that this subquotient cannot be in the Jacquet modules of the two non-boxed representations in line (26.2) (all irreducible subquotients of these terms start with $\nu\otimes\nu\otimes\dots$). This implies

$$\delta([\nu^0,\nu])\otimes\nu^{-1}\rtimes\delta(\nu;1)\le s_{(2)}(\pi'),$$

which implies

$$\nu\otimes\nu^0\otimes\nu\otimes\nu\otimes 1\le s_{(1,1,1,1)}(\pi').$$

Now $\nu\otimes\nu^0\otimes\nu\otimes\nu\otimes 1$ is not in the Jacquet module of the first term in line (26.5) (since there is always tensor factor ν^{-1}). This implies that

$$\delta([\nu^0,\nu])\times\nu\times\nu\otimes 1\le s_{(4)}(\pi') \tag{26.6}$$

holds for any irreducible subquotient π' of π_0.

Let π be the unique irreducible subquotient of π_0 which has the boxed term from line (26.4) in its Jacquet module. If we show that $s_{(4)}(\pi)$ contains the boxed term from line (26.5) with multiplicity two, then (26.6) implies $\pi=\pi_0$, i.e., π_0 is irreducible. Suppose that the multiplicity is one. Since

$$\nu\otimes\nu\otimes\nu\otimes\nu^0\otimes 1$$

is not in the Jacquet module of the first representation in line (26.5), the (semi simplification of the) Jacquet module of $\nu\times\nu\times\nu\otimes\nu^0\rtimes 1$ with respect to the minimal parabolic subgroup, which is equal to 12 $\nu\otimes\nu\otimes\nu\otimes\nu^0\otimes 1$, must be less then or equal to the Jacquet module of $\delta([\nu^0,\nu])\times\nu\times\nu\otimes 1$ with respect to the minimal parabolic subgroup, which is impossible (since the last Jacquet module is also a representation of length 12, but has $\nu\otimes\nu^0\otimes\nu\otimes\nu\otimes 1$ as a subquotient). This contradiction ends the proof of irreducibility. $\square$

Remark. *The representation*

$$St_{GL(m)} \rtimes St_{Sp(2)} = \delta(1_{F^\times}, m) \rtimes \delta(\nu; 1)$$

is

- **irreducible** *for all* **even** m,
- **reducible** *for all* **odd** m **except** $m = 3$.

For $m = 3$ this is proved in the last proposition. One proves the above reducibility claim in a similar way as we proved reducibility in the second example of section 20. The remaining irreducibility claim is proved in a similar way as we proved irreducibility in the second example of section 21.

27. On R-groups

We shall very briefly describe the R-groups that show up for the groups that we consider. Let G be such a group (some S_n), and let σ be an irreducible square integrable representation of the standard Levi subgroup M of a standard parabolic subgroup P. One considers the stabilizer $W(\sigma)$ of σ in the Weyl group of G. Let $W(\sigma)'$ be the subgroup of those $w \in W(\sigma)$ for which the corresponding normalized standard intertwining operator acts on $\mathrm{Ind}_P^G(\sigma)$ as a scalar. Then $W(\sigma)/W(\sigma)'$ is the R-group of σ. Actually, it is isomorphic to a subgroup of W (see [17]). Moreover, for our G one has $r^2 = 1$ for each element r of the R-group ([7]). So the R-group is commutative. Further, the intertwining algebra of $\mathrm{Ind}_P^G(\sigma)$ is isomorphic to the group algebra $\mathbb{C}[R]$ of R. Therefore, $\mathrm{Ind}_P^G(\sigma)$ is a multiplicity one representation. Further, characters of R naturally parameterize irreducible pieces of $\mathrm{Ind}_P^G(\sigma)$ (see the second section of [17]). Since the R-group is a free $\mathbb{Z}/2\mathbb{Z}$-module, characters of the R-group are in bijection with functions from (some fixed) basis of the R-group into $\{\pm 1\}$. Therefore, functions from the basis of the R-group into $\{\pm 1\}$ parameterize irreducible pieces of $\mathrm{Ind}_P^G(\sigma)$.

Now we shall present one consequence, quite interesting in our considerations (which also tells what the R-group is in this case):

Theorem. *Let $\delta_1, \dots, \delta_n \in D_u$, and let τ be an irreducible square integrable representation of S_q. Denote by l the number of nonequivalent representations δ among them having $\delta \rtimes \tau$ reducible. Then $\delta_1 \times \cdots \times \delta_n \rtimes \tau$ is a multiplicity one representation of length 2^l.*

This result was proved by Goldberg in [7] in characteristic 0.

On irreducible square integrable representations and their parameters

28. Introductionary remarks on invariants of square integrable representations

(1) Recall that by the Langlands classification of the non-unitary dual, we need to know irreducible tempered representations of Levi subgroups, which are tensor products of such representations of groups $GL(p, F)$ and of S_q. The Bernstein-Zelevinsky theory classifies the irreducible tempered representations of general linear groups modulo cuspidal representations. The previous section tells us that we have reduction of the irreducible tempered representations of groups S_q to the square integrable representations of the same groups and the knowledge when $\delta \rtimes \pi$ reduces, where δ and π are irreducible square integrable representations of $GL(p, F)$ and S_q respectively. Recall that $\delta = \delta(\rho, m)$ for some unitarizable irreducible cuspidal representation ρ of a general linear group and a positive integer m. If ρ is not selfdual, then it follows easily that $\delta(\rho, m) \rtimes \pi$ is irreducible.

It remains to consider the case of selfdual ρ. The following general phenomenon (which we have already observed in some examples) holds for such ρ:

for exactly one parity of positive integers the following holds:

- $\delta(\rho, m) \rtimes \pi$ reduces for all m from that parity, with possibly finitely many exceptions;
- $\delta(\rho, m) \rtimes \pi$ is irreducible for all m from the other parity.

We have seen in sections 20. and 22. that this holds for the representation $\pi = \delta(\nu^{1/2}\rho; 1)$ of $SO(2p+1, F)$, and that the only exception is $\delta = \delta(\rho, 2)$, and that this exception characterizes $\delta(\nu^{1/2}\rho; 1)$ (recall that we have chosen ρ such that $\nu^{1/2}\rho \rtimes 1$ reduces as a representation of $SO(2p+1, F)$).

Therefore, for understanding the non-unitary dual, it is crucial to know the parity which satisfies the above properties for the given selfdual ρ, and to know the exceptions. The set of all such exceptions, when ρ runs over all the selfdual irreducible cuspidal representations ρ of general linear groups will be denoted by

$$\mathrm{Jord}(\pi).$$

Actually, to abbreviate notation, we shall simply denote the elements of $\mathrm{Jord}(\pi)$ by (ρ, m) instead of $\delta(\rho, m)$ (recall that (ρ, m) completely determines the representation $\delta(\rho, m)$). The sets $\mathrm{Jord}(\pi)$ are finite (see [22]). The set $\mathrm{Jord}(\pi)$ is called the set of **Jordan blocks** of π. As we have already noted, this invariant is very important from the point of view of representation theory (for the tempered representations and the non-unitary dual). But it is also important for the Galois side, as we shall discuss later.

In the case that $\nu^{1/2}\rho \rtimes 1$ reduces, we have seen that $(\rho, 2) \in \mathrm{Jord}(\delta(\nu^{1/2}\rho; 1))$. Using the estimate (34.1) of C. Mœglin we get.

$$\mathrm{Jord}(\delta(\nu^{1/2}\rho; 1)) = \{(\rho, 2)\}.$$

In a special case, we have

$$\mathrm{Jord}(St_{SO(3)}) = \{(1_{F^\times}, 2)\}.$$

For an irreducible square integrable representation π of S_q and a selfdual irreducible cuspidal representation ρ of a general linear group let

$$\mathrm{Jord}_\rho(\pi) := \{m; (\rho, m) \in \mathrm{Jord}(\pi)\}.$$

This set is called the **Jordan blocks of π along ρ**.

Consider a more general representation

$$\delta([\nu^{1/2}\rho, \nu^{k-1/2}\rho]; \sigma), \quad k \in \mathbb{Z}_{\geq 1},$$

when $\nu^{1/2}\rho \rtimes \sigma$ reduces (σ is an irreducible cuspidal representation of some S_q). In this case, we would get the set of "exceptions"

$$\mathrm{Jord}_\rho(\delta([\nu^{1/2}\rho, \nu^{k-1/2}\rho]; \sigma)) = \{2k\}$$

(this is again a single element set; we shall see examples of bigger exception sets later).

(2) From our previous considerations and estimate (34.1), one sees that for the trivial representation of S_0 in the case of odd orthogonal groups we have

$$\mathrm{Jord}(1_{SO(1))}) = \emptyset,$$

and in the symplectic case we have

$$\mathrm{Jord}(1_{Sp(0)}) = \{(1_{F^\times}, 1)\}$$

(this interesting difference is related to the fact that the dual complex L-group of $SO(1, F)$ is $Sp(0, \mathbb{C})$, and of $Sp(0, F)$ is $SO(1, \mathbb{C})$).

(3) Now we shall show how Jordan blocks help distinguish among representations. In general we have

$$\mathrm{Jord}(St_{SO(2n+1)}) = \{(1_{F^\times}, 2n)\}.$$

In particular,

$$\mathrm{Jord}(St_{SO(5)}) = \{(1_{F^\times}, 4)\}.$$

Take any two different characters χ_1, χ_2 of $F^\times$ satisfying $\chi_1^2 = \chi_2^2 \equiv 1$. Then the following inequalities hold

$$\begin{aligned}
\nu^{1/2}\chi_1 \rtimes \delta(\nu^{1/2}\chi_2; 1) &\leq \nu^{1/2}\chi_1 \times \nu^{1/2}\chi_2 \rtimes 1, \\
\nu^{1/2}\chi_2 \rtimes \delta(\nu^{1/2}\chi_1; 1) &\leq \nu^{1/2}\chi_1 \times \nu^{1/2}\chi_2 \rtimes 1.
\end{aligned}$$

It is very easy to show that the two representations on the left hand side have exactly one irreducible subquotient in common, which we shall denote by

$$\delta(\nu^{1/2}\chi_1, \nu^{1/2}\chi_2; 1),$$

and that

$$\nu^{1/2}\chi_1 \otimes \nu^{1/2}\chi_2 \otimes 1 + \nu^{1/2}\chi_2 \otimes \nu^{1/2}\chi_1 \otimes 1$$

is its Jacquet module with respect to the minimal parabolic subgroup. Now Casselman's square integrability criterion implies that the representation $\delta(\nu^{1/2}\chi_1, \nu^{1/2}\chi_2; 1)$ is square integrable. Similar calculations as before give

$$\mathrm{Jord}(\delta(\nu^{1/2}\chi_1, \nu^{1/2}\chi_2; 1)) = \{(\chi_1, 2), (\chi_2, 2)\}.$$

This is an example of Jordan blocks with more then one element (more interesting examples of this type will come later). So we see that in this case Jordan blocks distinguish $\delta(\nu^{1/2}\chi_1, \nu^{1/2}\chi_2; 1)$ and $St_{SO(5)}$.

Later on, we shall have examples where Jordan blocks are not enough to distinguishing square integrable representations that we consider. For the moment, let us mention one simple example of this type. One gets such representations when one has nonequivalent irreducible subrepresentations of $\rho|SL(2)$, where ρ is an irreducible cuspidal representation of $GL(2) = GSp(2)$ (R. Langlands and J.-P. Labesse called these representations L-indistinguishable). More interesting examples will be given later.

(4) From the point of view of the local Langlands conjectures for irreducible square integrable representations (discussed in section 4.), we know that the first parameter attached to an irreducible square integrable representation should be a (discrete) admissible homomorphism of the Weil-Deligne group into the complex dual group. Further, we know that the second parameter of an irreducible square integrable representation should be an irreducible representation of the component group, which is one-dimensional in our case, i.e., a character.

Let us take a closer look, from the point of view of the local Langlands correspondences, at the exceptions that we have obtained and discussed in the last few sections. Look first at $St_{SO(2n+1,F)}$. Recall that by the local Langlands correspondence for odd-orthogonal groups, $St_{SO(2n+1,F)}$ should correspond to an admissible homomorphism

$$W_F \times SL(2, \mathbb{C}) \to Sp(2n, \mathbb{C}).$$

Recall that

$$\mathrm{Jord}(St_{SO(2n+1,F)}) = \{(1_{F^\times}, 2n)\},$$

which means that the only exception is the square integrable representation $\delta(1_{F^\times}, 2n) = St_{GL(2n,F)}$. Since this exception is a representation of $GL(2n, F)$, we can apply the local Langlands correspondence Φ for general linear groups (discussed in section 13) to it. We get

$$\Phi(\delta(1_{F^\times}, 2n)) = 1_{W_F} \otimes E_{2n}.$$

A well known fact states that E_m is:

- symplectic (i.e., respects a non-degenerate symplectic form on $\mathbb{C}^m$) if m is even;
- orthogonal (i.e., respects a non-degenerate orthogonal form on $\mathbb{C}^m$) if m is odd.

This implies that in fact

$$\Phi(\delta(1_{F^\times}, 2n)) : W_F \times SL(2, \mathbb{C}) \to Sp(2n, \mathbb{C}).$$

Therefore, $\Phi(\delta(1_{F^\times}, 2n))$ goes just where the admissible homomorphism corresponding to $St_{SO(2n+1,F)}$ should go.

If we consider $St_{Sp(2n,F)}$, where

$$\mathrm{Jord}(St_{Sp(2n,F)}) = \{(1_{F^\times}, 2n + 1)\},$$

then

$$\Phi(\delta(1_{F^\times}, 2n + 1)) : W_F \times SL(2, \mathbb{C}) \to SO(2n + 1, \mathbb{C}),$$

and we again get that $\Phi(\delta(1_{F^\times}, 2n + 1))$ goes just where the admissible homomorphism corresponding to $St_{Sp(2n,F)}$ should go.

Consider the square integrable representation $\delta(\nu^{1/2}\chi_1, \nu^{1/2}\chi_2; 1)$ of $SO(5, F)$ considered in (3). Recall

$$\mathrm{Jord}(\delta(\nu^{1/2}\chi_1, \nu^{1/2}\chi_2; 1)) = \{(\chi_1, 2), (\chi_2, 2)\}.$$

Now

$$\bigoplus_{(\rho,k)\in \mathrm{Jord}(\delta(\nu^{1/2}\chi_1,\nu^{1/2}\chi_2;1))} \Phi(\delta(\rho, k)) =$$

$$\Phi(\chi_1) \otimes E_2 \oplus \Phi(\chi_2) \otimes E_2 : W_F \times SL(2, \mathbb{C}) \to Sp(4, \mathbb{C})$$

($\Phi(\chi_i)$ is here simply obtained by the Artin reciprocity of local class field theory). So, this representation goes precisely where the admissible homomorphism corresponding to $\delta(\nu^{1/2}\chi_1, \nu^{1/2}\chi_2; 1)$ should go. Actually, G. Muić conjectured in section 3. of [26] that in this way one should get natural candidates for the admissible homomorphisms corresponding to the generic square integrable representations of classical groups (see 28.1 below).

Further, considering the cases of Langlands functoriality between classical and general linear groups which are expected to hold, C.

Mœglin realized that it is natural to expect that the (discrete) admissible homomorphism corresponding to a general irreducible square integrable representation π of S_n should be

$$\bigoplus_{(\rho,k)\in\mathrm{Jord}(\pi)} \Phi(\delta(\rho,k)) = \bigoplus_{(\rho,k)\in\mathrm{Jord}(\pi)} \Phi(\rho)\otimes E_k \tag{28.1}$$

(see remark (7) below for a few more details regarding this).

Suppose for a moment that (28.1) holds.

(5) Consider the case $S_n = SO(2n+1,F)$. Denote by $e_{(\rho,k)}$ the linear operator on $\mathbb{C}^{2n}$ which acts as multiplication by -1 on $\Phi(\rho)\otimes E_k$, and as identity on all other summands in (28.1). Then the component group is

$$\left\{ \prod_{(\rho,k)\in\mathrm{Jord}(\pi)} e_{(\rho,k)}^{\mu_{(\rho,k)}}; \mu_{(\rho,k)} \in \mathbb{Z}/2\mathbb{Z}\right\} \Big/ \{\pm \mathrm{Id}_{\mathbb{C}^{2n}}\}.$$

Now characters (i.e., irreducible representations) of this group are in a natural bijection with each of the following sets:

- functions $\mathrm{Jord}(\pi) \to \{\pm 1\}$ which have a prescribed value on one fixed $(\rho,k) \in \mathrm{Jord}(\pi)$;
- functions $\mathrm{Jord}(\pi) \to \{\pm 1\}$ which take the value 1 on even number of times in the case $\mathrm{card}(\mathrm{Jord}(\pi)) \in 2\mathbb{Z}$;
- functions $\{e_{(\rho_i,k_i)}e^{-1}_{(\rho_{i+1},k_{i+1})}; i = 1,\dots,m-1\} \to \{\pm 1\}$, if we enumerate $\mathrm{Jord}(\pi) = \{(\rho_i,k_i); i = 1\dots,m\}$.

Note that in all three above cases we can forget the Galois side. We can work simply with functions related to $\mathrm{Jord}(\pi)$ (in the last case we can work formally with $(\rho_i,k_i)(\rho_{i+1},k_{i+1})^{-1}$'s). So, if we expect that (28.1) is the admissible homomorphism corresponding the irreducible square integrable representation π, for the purpose of classifying square integrable representations, we may try to parameterize them by functions related to $\mathrm{Jord}(\pi)$.

Therefore, regarding the second parameter of an irreducible square integrable representation π (i.e. an irreducible representation of the component group, which is a character in our case), we shall not work on the Galois side (except giving Galois side interpretations at some places). We shall relate such a character to a function related to $\mathrm{Jord}(\pi)$, and consider it as an invariant of π. This function, denoted by

$$\epsilon_\pi,$$

will be defined only partially on $\mathrm{Jord}(\pi)$ (for reasons explained below), and will be called the **partially defined function** attached to π. All the non-cuspidal square integrable representations considered up to now can be already distinguished by $\mathrm{Jord}(\pi)$. In the cases of more complicated square integrable representations, we shall need this function for further distinction of that representations.

Let us mention that this function has some relation with the function that parameterizes irreducible tempered pieces in section 27. We shall not go into more detail here, but this may become clear if we know that the construction of square integrable representations is closely related to the question of (reducibility of) tempered induction (see [41] for examples of this very direct connection). In the present paper, we shall see the relation of tempered induction to square integrable representations only in the simplest case.

Note that in the case of general linear groups, tempered induction is irreducible, i.e., parabolic induction carries irreducible tempered representations to irreducible (tempered) ones, and the component groups are trivial (so its irreducible representations can be only trivial).

(6) Since the classification that we present here is modulo cuspidal data (similarly as it was in the case of Bernstein and Zelevinsky for general linear groups; there the reducibility point is always 1), we shall have a third invariant, which is an irreducible cuspidal representation of the classical group that shows up in the construction of a square integrable representation. This invariant is called the **partial cuspidal support**. In our constructions up to now it was the trivial representation 1 of S_0 in the case of Steinberg representation, and σ in the case of the generalized Steinberg representation. In general, the partial cuspidal support of a general irreducible square integrable representation π of S_q is an irreducible cuspidal representation

$$\pi_{\text{cusp}}$$

of some $S_{q'}, q' \leq q$, such that there exists a representation ψ of $GL(q-q', F)$, such that

$$\pi \hookrightarrow \psi \rtimes \pi_{\text{cusp}}.$$

Note that π_{cusp} should have its own character $\epsilon_{\pi_{\text{cusp}}}$ which parameterizes it. Therefore, roughly the (complete) character corresponding to π should be determined by both ϵ_π and $\epsilon_{\pi_{\text{cusp}}}$. This partially explains why ϵ_π is only partially defined. The other reason is that $\text{Jord}(\pi)$ does not correspond exactly to the basis of the component group: in the orthogonal case the dual group has non-trivial center, while in the symplectic case one needs to take care of the determinant one condition. Our discussion above is not very precise (for a precise discussion, one should consult Mœglin's papers).

The invariants attached to π will form the triple

$$(\text{Jord}(\pi), \epsilon_\pi, \pi_{\text{cusp}}).$$

Recall that in the local Langlands conjecture we should have only two parameters. We do not have (the analogue of) the partial cuspidal support. Actually, the pair consisting of Jord(π) and the (complete) character corresponding to π should determine π_{cusp}. Therefore, in our triple Jord(π) and ϵ_π need to have "control" of π_{cusp} in an appropriate way (to start with fixed Jord(π) we should have at most finitely many possibilities for π_{cusp}). The condition among Jord(π), ϵ_π and π_{cusp} is a technical condition which we shall discuss later. Triples which satisfy this condition will be called **admissible triples**. C. Mœglin has attached to an irreducible square integrable representation π such an admissible triple, and has shown that the process of attaching triples is injective (assuming a natural technical conjecture (BA), which shows up only in proofs).

(7) The Langlands program started in the 1960's with Langlands' conjectures for $GL(n)$-groups. These conjectures generalize the Artin reciprocity law from class field theory (which corresponds to the case $GL(1)$). R.P. Langlands soon realized that these conjectures should be a very special case of a general principle called functoriality. We shall recall here very briefly only the case of functoriality that interests us. For simplicity, we shall consider only the case of odd orthogonal groups $SO(2n+1,F)$ (a completely analogous situation holds with symplectic groups). Recall ${}^LSO(2n+1,F)^0 = Sp(2n,\mathbb{C})$ and ${}^LGL(2n,F)^0 = GL(2n,\mathbb{C})$. The inclusion

$$i : Sp(2n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{C})$$

yields the inclusion $\varphi \mapsto i\circ\varphi$ of (discrete) admissible homomorphisms for $SO(2n+1,F)$ into admissible homomorphisms for $GL(2n,F)$:

$$\begin{array}{ccc} Sp(2n,\mathbb{C}) & \xrightarrow{\quad i \quad} & GL(2n,\mathbb{C}) \\ {\scriptstyle\varphi}\uparrow & & {\scriptstyle i\circ\varphi}\uparrow \\ W_F \times SL(2,\mathbb{C}) & = & W_F \times SL(2,\mathbb{C}). \end{array} \tag{28.2}$$

The existences of the local Langlands correspondences for $SO(2n+1,F)$ and $GL(2n,F)$ (see section 4.) would directly give the lift of the mapping $\varphi \mapsto i\circ\varphi$ to the level of representations and L-packets (this would be an instance of a local Langlands functoriality, as well as a local Langlands correspondence is an instance of it). Very roughly (and little bit oversimplified), the functoriality predicts analogous "natural" lifting on the level of adelic groups over global fields ("natural" means that some basic invariants like L-functions should be respected; see [5] for details). Local and global lifts should be compatible, and on unramified factors the lift should have naturally expected form.

Recall that the local Langlands conjecture for general linear groups is proved. Observe now that the knowledge of the local lift of irreducible representations of $SO(2n+1,F)$ to irreducible representations of $GL(2n,F)$ would imply the knowledge of the local Langlands correspondence for $SO(2n+1,F)$. Further, the knowledge of the global lift from $SO(2n+1)$ to $GL(2n)$ and the requirement that the local and global lifts should be compatible, would imply the knowledge of the action of the local Langlands correspondence for $SO(2n+1)$ on local factors of automorphic representations.

The lifts of generic cuspidal automorphic representations of adelic classical groups over number fields to general linear groups were obtained in [5] using a converse theorem for global representations (regarding being automorphic; this theorem generalizes the classical converse theorems of Hecke and Weil). Therefore, [5] gives what will be admissible homomorphisms corresponding to local factors of generic cuspidal automorphic representations of the adelic group $SO(2n+1)$ (they are also generic). The formula obtained in the case of generic square integrable representations of $SO(2n+1,F)$ is the formula that we mentioned in (4) (and which was conjectured by G. Muić in [26]).

Functoriality is not established in the non-generic case, but some expectations of the modern theory of automorphic forms would imply the formula also holds in this case (see [20]). The above discussion also gives support to the basic assumption (see also [32]).

29. An important simple example of construction of square integrable representations

Take an irreducible unitarizable cuspidal representation ρ of $GL(p,F)$ and an irreducible cuspidal representation σ of S_q such that $\rho \rtimes \sigma$ reduces (then $\rho \cong \tilde{\rho}$ and $\nu^\alpha \rho \rtimes \sigma$ does not reduce for all $\alpha \in \mathbb{R}^\times$). From the Jacquet module $s_{(p)}(\rho \rtimes \sigma)$, one gets that $\rho \rtimes \sigma$ is a sum of two irreducible representations. Further, Frobenius reciprocity implies that $\rho \rtimes \sigma$ is a multiplicity one representation. Write

$$\rho \rtimes \sigma = \tau_1 \oplus \tau_2,$$

where τ_1 and τ_2 are irreducible ($\tau_1 \not\cong \tau_2$). Then it follows directly that

$$\mu^*(\tau_i) = 1 \otimes \tau_i + \rho \otimes \sigma.$$

We shall work below with symplectic groups and take ρ to be a character ψ of order 2 of $F^\times$ and $\sigma = 1_{Sp(0)}$ (=1), but the entire construction holds for general ρ and σ as above (i.e., with reducibility at 0). We shall comment a more general case later.

We consider

$$\delta([\psi, \nu\psi]) \rtimes 1 \;\hookrightarrow\; \nu\psi \times \psi \rtimes 1 \;=\; \nu\psi \rtimes \tau_1 \;\oplus\; \nu\psi \rtimes \tau_2.$$

Observe that

$$\begin{aligned}(29.1)\qquad \mu^*(\nu\psi \rtimes \tau_i) = 1 \otimes \nu\psi \rtimes \tau_i + \\ [\nu\psi \otimes \tau_i + \nu^{-1}\psi \otimes \tau_i + \psi \otimes \nu\psi \rtimes 1] + \\ [\nu\psi \times \psi \otimes 1 + \nu^{-1}\psi \times \psi \otimes 1].\end{aligned}$$

From this, using Frobenius reciprocity, we see that $\nu\psi \rtimes \tau_i$ has a unique irreducible subrepresentation (since the multiplicity of $\nu\psi \otimes \tau_i$ in the above Jacquet module is 1). We denote this subrepresentation by

$$\delta([\psi, \nu\psi]_{\tau_i}; 1).$$

Observe

$$\begin{aligned}(29.2)\qquad \mu^*(\delta([\psi, \nu\psi]) \rtimes 1) = 1 \otimes \delta([\psi, \nu\psi]) \rtimes 1 + \\ [\nu\psi \otimes \tau_1 + \nu\psi \otimes \tau_2 + \psi \otimes \nu\psi \rtimes 1] + \\ \left[\delta([\psi, \nu\psi]) \otimes 1 + \psi \times \nu\psi \otimes 1 + \delta([\nu^{-1}\psi, \psi]) \otimes 1\right].\end{aligned}$$

This implies that both representations $\delta([\psi, \nu\psi]_{\tau_i}; 1)$ are subrepresentations of the representation $\delta([\psi, \nu\psi]) \rtimes 1$ (because of imbedding (29.1) and the fact that the multiplicity of $\nu\psi \otimes \tau_i$ in the above Jacquet module is also 1). From the fact that $\delta([\psi, \nu\psi]_{\tau_i}; 1)$ is in both representations, we conclude

$$s_{(2)}(\delta([\psi, \nu\psi]_{\tau_i}; 1)) \leq \nu\psi \times \psi \otimes 1 + \delta([\nu^{-1}\psi, \psi]) \otimes 1.$$

The second summand on the right hand side, i.e. $\delta([\nu^{-1}\psi, \psi]) \otimes 1$, is in the Jacquet module of the Langlands quotient of $\delta([\psi, \nu\psi]) \rtimes 1$ (see section 25). This implies

$$s_{(2)}(\delta([\psi, \nu\psi]_{\tau_i}; 1)) \leq \nu\psi \times \psi \otimes 1,$$

which yields $s_{(1,1)}(\delta([\psi, \nu\psi]_{\tau_i}; 1)) \leq \nu\psi \otimes \psi \otimes 1 + \psi \otimes \nu\psi \otimes 1$.

From the Casselman criterion we conclude that $\delta([\psi, \nu\psi]_{\tau_i}; 1)$ is tempered. Now we claim that $\nu\psi \times \psi \rtimes 1$ does not have tempered subquotients which are not square integrable. Suppose that it has. Then there exists a proper standard parabolic subgroup $P = MN$ and an irreducible square integrable modulo center representation δ of M such that $\nu\psi \times \psi \rtimes 1$ and $\mathrm{Ind}_P^{Sp(4,F)}(\delta)$ have an irreducible subquotient in common. If P is the minimal parabolic subgroup, then $\mathrm{Ind}_P^{Sp(4,F)}(\delta) = \chi_1 \times \chi_2 \rtimes 1$ for some $\chi_1, \chi_2 \in (F^\times)\hat{}$. Now, section 10. implies that the above two representations cannot have an irreducible subquotient in common. Suppose that P is the Siegel parabolic subgroup. Then from section 10., it follows that δ cannot be cuspidal. This implies $\mathrm{Ind}_P^{Sp(4,F)}(\delta) = (\chi \circ \det) St_{GL(2,F)} \rtimes 1 \leq \nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes 1$ for some $\chi \in (F^\times)\hat{}$. Now again section 10. implies that the representations $\nu\psi \times \psi \rtimes 1$ and $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes 1$ can not have an irreducible subquotient in common. We handle the remaining proper standard parabolic subgroup similarly.

In this way we have proved that $\delta([\psi, \nu\psi]_{\tau_i}; 1)$ is square integrable. From this it follows that

$$s_{(2)}(\delta([\psi, \nu\psi]_{\tau_i}; 1)) = \delta([\psi, \nu\psi]) \otimes 1,$$

which further implies

$$\mu^*\left(\delta([\psi,\nu\psi]_{\tau_i};1)\right) = 1\otimes\delta([\psi,\nu\psi]_{\tau_i};1)+\nu\psi\otimes\tau_i+\delta([\psi,\nu\psi])\otimes 1.$$

Remark. *The above simple example is important since this is how, in systematic way, we shall get two square integrable representations of bigger groups (in this case of $Sp(4,F)$) from a square integrable representation of a smaller group (in this case from the trivial representation of $Sp(0,F)$).*

The above proof of square integrability is an illustration of the simplest example of a much more general proof of square integrability in [24] (in sections 9. - 11. of that paper; that proof is a crucial part of [24] and it completes the classification of square integrable representations of classical groups modulo cuspidal data).

Now we can analyze representations induced by the above two square integrable representations. We obtain

$$\mathrm{Jord}(\delta([\psi,\nu\psi]_{\tau_i};1)) = \{(\psi,1),(\psi,3),(1_{F^\times},1)\}.$$

So the Jordan blocks do not distinguish these non-isomorphic square integrable representations.

In general, suppose that $\rho\rtimes\sigma$ reduces (ρ and σ are as usual). Then we can write

$$\rho\rtimes\sigma=\tau_1\oplus\tau_2,$$

for nonequivalent irreducible τ_1 and τ_2. We can continue the above construction: we define

$$\delta([\rho,\nu^k\rho]_{\tau_i};\sigma)$$

as the unique irreducible subrepresentation of

$$\delta([\nu\rho,\nu^k\rho])\rtimes\tau_i.$$

Similarly one proves that $\delta([\rho,\nu^k\rho]_{\tau_i};\sigma)$ are square integrable (one can also determine $\mu^*(\delta([\rho,\nu^k\rho]_{\tau_i};\sigma))$ - it is a simple formula, similar to that for the Steinberg type representations).

We shall now briefly illustrate

30. A little bit more complicated example of construction of square integrable representations

In this section we consider odd orthogonal groups. From section 20. we know that

$$\delta([\nu^{-1/2},\nu^{1/2}])\rtimes 1=\tau_+\oplus\tau_-.$$

Now

$$\delta([\nu^{-1/2},\nu^{3/2}])\rtimes 1\hookrightarrow\nu^{3/2}\times\delta([\nu^{-1/2},\nu^{1/2}])\rtimes 1=\nu^{3/2}\rtimes\tau_+\oplus\nu^{3/2}\rtimes\tau_-.$$

The multiplicity of $\nu^{3/2}\otimes\tau_\pm$ in $\mu^*(\nu^{3/2}\rtimes\tau_\pm)$ (and $\delta([\nu^{-1/2},\nu^{3/2}])\rtimes 1$) is 1. This implies that $\nu^{3/2}\otimes\tau_\pm$ has a unique irreducible subrepresentation, which will be denoted by

$$\delta([\nu^{-1/2},\nu^{3/2}]_\pm;1).$$

In calculations below we shall write

$$\pi_\pm = \delta([\nu^{-1/2}, \nu^{3/2}]_\pm; 1).$$

Since the multiplicity of $\nu^{3/2} \otimes \tau_\pm$ in $\mu^*(\delta([\nu^{-1/2}, \nu^{3/2}]) \rtimes 1))$ is also 1, both $\pi_\pm$ must be subrepresentations of $\delta([\nu^{-1/2}, \nu^{3/2}]) \rtimes 1$. In this way we get two upper bounds for the Jacquet module of $\delta([\nu^{-1/2}, \nu^{3/2}]_\pm; 1)$. From them, similarly as in the previous section, we first conclude that $\nu^{3/2} \otimes \tau_\pm$ is tempered, and then again similarly as in the previous section, that it must be square integrable. It is not hard to show

$$s_{GL}(\pi_+) = \delta([\nu^{-1/2}, \nu^{3/2}]) \otimes 1 + \delta([\nu^{1/2}, \nu^{3/2}]) \times \nu^{1/2} \otimes 1,$$

$$s_{GL}(\pi_-) = \delta([\nu^{-1/2}, \nu^{3/2}]) \otimes 1.$$

Now we shall recall of a general principle regarding Jordan blocks. From it, we shall easily see what are the Jordan blocks of $\pi_\pm$. Below we shall assume that basic (technical) assumption (BA) from the section 17. holds.

Proposition 2.1 from [24] tells:

Proposition. *Let π' be an irreducible square integrable representation of S_q and let $x, y \in (1/2)\mathbb{Z}$ such that $x - y \in \mathbb{Z}_{\geq 0}$. Let ρ be an irreducible selfdual cuspidal representation of $GL(p)$. We assume that x, $y \in \mathbb{Z}$ if and only if $\nu^t\rho \rtimes 1$ reduces for some $t \in \mathbb{Z}$. Further, suppose that there is an irreducible square integrable representation π embedded in the induced representation*

$$\pi \hookrightarrow \nu^x\rho \times \cdots \times \nu^{x-i+1}\rho \times \cdots \times \nu^y\rho \rtimes \pi'.$$

Then:
(i) If $y > 0$, then $2y - 1 \in \mathrm{Jord}_\rho(\pi')$ and

$$\mathrm{Jord}_\rho(\pi) = \big(\mathrm{Jord}_\rho(\pi') \setminus \{(\rho, 2y-1)\}\big) \cup \{(\rho, 2x+1)\}.$$

(ii) If $y \leq 0$, then

$$\mathrm{Jord}_\rho(\pi) = \mathrm{Jord}_\rho(\pi') \cup \{(\rho, 2x+1), (\rho, -2y+1)\}.$$

From this directly follows

$$\mathrm{Jord}(\delta([\nu^{-1/2}, \nu^{3/2}]_\pm; 1)) = \{(1_{F^\times}, 2), (1_{F^\times}, 4)\}.$$

Further, all the other Jordan blocks that we have computed up to now can be obtained from the above proposition.

We can also often see elements of the Jordan blocks from Jacquet modules:

Lemma. *Suppose that π is an irreducible square integrable representation of S_q. Let $\nu^x\rho\otimes\tau$ be an irreducible subquotient of a standard Jacquet module of π, where ρ is an irreducible selfdual cuspidal representation of $GL(p, F)$, $x \in \mathbb{R}$, and τ is an irreducible representation of $S_{q'}$. Then*

$$(\rho, 2x+1) \in \mathrm{Jord}(\pi).$$

31. Partially defined function

We shall now explain how one can distinguish the above two representations. This will be done by partially defined function on $\mathrm{Jord}(\pi)$, where π is a square integrable representation. Partially defined function will essentially be a character defined on a subgroup of the free $\mathbb{Z}/2\mathbb{Z}$-module with basis $\mathrm{Jord}(\pi)$ (a character of such a subgroup must take values in $\{\pm 1\}$). Therefore, if this character is defined on the whole $\mathrm{Jord}(\pi)$, it is enough to define the function on $\mathrm{Jord}(\pi)$ (this is the case for reducibilities in $\frac{1}{2}+\mathbb{Z}$). Otherwise, we shall again define it on generators and again talk about a function (with values in $\{\pm 1\}$), rather then about a character.

Now we shall explain how to make a difference between representations π_+ and π_- from the previous section, without going into the internal structure of these representations (recall that we know certain Jacquet modules which happend to be different). Recall

$$s_{GL}(\pi_+) = \delta([\nu^{-1/2}, \nu^{3/2}]) \otimes 1 + \delta([\nu^{1/2}, \nu^{3/2}]) \times \nu^{1/2} \otimes 1.$$

Both representations on the right hand side are irreducible, and have different central characters. Therefore, we have a direct sum. Now the Frobenius reciprocity implies

$$\pi_+ \hookrightarrow \nu^{1/2} \times \delta([\nu^{1/2}, \nu^{3/2}]) \rtimes 1,$$

which easily implies that

$$\pi_+ \hookrightarrow \nu^{1/2} \rtimes \pi' = \nu^{(2-1)/2} \rtimes \pi'$$

for some irreducible representation π' (use the elementary Lemma 3.2 of [24]). From the other side,

$$s_{GL}(\pi_-) = \delta([\nu^{-1/2}, \nu^{3/2}]) \otimes 1$$

implies that always

$$\pi_- \not\hookrightarrow \nu^{1/2} \rtimes \pi' = \nu^{(2-1)/2} \rtimes \pi'.$$

This is the reason that for $\pi \in \{\pi_\pm\}$ can define

$$\epsilon_\pi((1_{F^\times}, 2)) = 1 \iff \pi \hookrightarrow \nu^{(2-1)/2} \rtimes \pi'$$

for some irreducible representation π' (of $SO(5, F)$). Now obviously

$$\epsilon_{\pi_\pm}((1_{F^\times}, 2)) = \pm 1 \tag{31.1}$$

(i.e. $\epsilon_{\pi_+}((1_{F^\times}, 2)) = 1$ and $\epsilon_{\pi_-}((1_{F^\times}, 2)) = -1$). Next question is what is $\epsilon_{\pi_\pm}((1_{F^\times}, 4))$. One can get answer to that question from a general principle covering the case when square integrable representations are obtained in the above way. Before explaining that principle, let us observe that

$$\pi_\pm \hookrightarrow \delta([\nu^{-1/2}, \nu^{3/2}]) \rtimes 1 \hookrightarrow \nu^{3/2} \times \nu^{1/2} \times \nu^{-1/2} \rtimes 1,$$

which implies (again using Lemma 3.2 of [24]) that for some irreducible π' we have

$$\pi_\pm \hookrightarrow \nu^{3/2} \rtimes \pi' = \delta([\nu^{(2+1)/2}, \nu^{(4-1)/2}]) \rtimes \pi'.$$

Now we present definition of the partially defined function attached to a general square integrable representation.

Definition. *Let π be an irreducible square integrable representation of S_q.*

(1) *Suppose that $(\rho, a), (\rho, c) \in \mathrm{Jord}(\pi)$ are such that $c < a$, $\mathrm{Jord}_\rho(\pi) \cap \{x; c < x < a\} = \emptyset$. Then we shall denote c by*
$$a_-.$$

(2) *For $(\rho, b) \in \mathrm{Jord}(\pi)$ the partially defined function may or may not be defined on $(\rho, b) \in \mathrm{Jord}(\pi)$. If it is defined, it takes values in $\{\pm 1\}$. For a, a_- as in (1), $\epsilon_\pi((\rho, a)(\rho, a_-)^{-1})$ is always defined, and it must be equal to*
$$\epsilon_\pi((\rho, a))\epsilon_\pi((\rho, a_-))^{-1}$$
if $\epsilon_\pi((\rho, a))$ and $\epsilon_\pi((\rho, a_-))$ are defined. If $\epsilon_\pi((\rho, a))$ and $\epsilon_\pi((\rho, a_-))$ are not defined, we shall nevertheless write $\epsilon_\pi((\rho, a)(\rho, a_-)^{-1})$ formally as $\epsilon_\pi((\rho, a))\epsilon_\pi((\rho, a_-))^{-1}$. This is an element of $\{\pm 1\}$, and it is 1 if and only if
$$\pi \hookrightarrow \delta([\nu^{(a_-+1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'.$$
for some irreducible representation π' of some $S_{q'}, q' < q$.

(3) *For $(\rho, a) \in \mathrm{Jord}(\pi)$, $\epsilon_\pi((\rho, a))$ is always defined if a is even. Using (2), we can compute $\epsilon_\pi((\rho, a))$ if we know $\epsilon_\pi((\rho, a_{\min}))$, where $a_{\min} = min(\mathrm{Jord}_\rho(\pi))$. One defines*
$$\epsilon_\pi((\rho, a_{\min})) \quad \text{to be} \quad 1$$
if and only if
$$\pi \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a_{\min}-1)/2}\rho]) \rtimes \pi'$$
for some irreducible representation π' of some $S_{q'}, q' < q$.

(4) *For $(\rho, a) \in \mathrm{Jord}(\pi)$, $\epsilon_\pi((\rho, a))$ is not defined if a is odd and if*
$$\mathrm{Jord}_\rho(\pi_{\mathrm{cusp}}) \neq \emptyset.$$

(5) *For $(\rho, a) \in \mathrm{Jord}(\pi)$, $\epsilon_\pi((\rho, a))$ is defined if a is odd and*
$$\mathrm{Jord}_\rho(\pi_{\mathrm{cusp}}) = \emptyset.$$
Note that in that case $\rho \rtimes \pi_{\mathrm{cusp}}$ reduces into two non-equivalent irreducible representations. One makes the choice of the irreducible component of $\rho \rtimes \pi_{\mathrm{cusp}}$ to which 1 is attached (-1 is attached to the other one), i.e.
$$\rho \rtimes \pi_{\mathrm{cusp}} = \tau_1 \oplus \tau_{-1}.$$
Then C. Mœglin used normalized intertwining operators to define $\epsilon_\pi((\rho, a))$. We can define partially defined function in a different way, which we now briefly explain. Using (2), we are able to compute $\epsilon_\pi((\rho, a))$ if we know $\epsilon_\pi((\rho, a_{\max}))$, where $a_{\max} = max(\mathrm{Jord}_\rho(\pi))$. We define
$$\epsilon_\pi((\rho, a_{\max})) \quad \text{to be} \quad s \in \{\pm 1\}$$

if and only if

$$\pi \hookrightarrow \pi' \rtimes \delta([\nu\rho, \nu^{(a_{\max}-1)/2}\rho]) \rtimes \tau_s$$

for some irreducible representation π' of a general linear group (see [43] for more details).

Remarks. (1) *In our case (for square integrable representations defined in section 30.) we have*

$$\epsilon_{\pi_\pm}((1_{F^\times}, 2)) = \epsilon_{\pi_\pm}((1_{F^\times}, 4)) = \pm 1$$

(see (31.1)).

(2) *(Add (2) of the previous definition) At this point, it would be interesting to have example when $\epsilon_\pi((\rho, a))\epsilon_\pi((\rho, a_-))^{-1} = -1$. The first example (in the even parity) not involving cuspidal cases, occurs for $SO(13)$. One considers*

$$\delta([\nu^{-1/2}, \nu^{3/2}]) \rtimes \delta([\nu^{1/2}, \nu^{5/2}]; 1) = \delta([\nu^{-1/2}, \nu^{3/2}]) \rtimes St_{SO(7)}.$$

This representation has two irreducible subrepresentations. Both are square integrable. Their Jordan blocks are

$$\{(1_{F^\times}, 2), (1_{F^\times}, 4), (1_{F^\times}, 6)\}.$$

One of these subrepresentations contains

$$\delta([\nu^{1/2}, \nu^{5/2}]) \times \delta([\nu^{1/2}, \nu^{3/2}]) \times \nu^{1/2} \otimes 1$$

in its Jacquet module. This subrepresentation has partially defined function identically equal to 1. The partially defined function of the other subrepresentation is 1 at $(1_{F^\times}, 6)$, and otherwise -1. So 4 and 6 are neighbors in the Jordan blocks along $1_{F^\times}$, and partially defined function takes different values on them.

(3) *(Add (3) of the previous definition) We have seen examples (in the even parity) of different $\epsilon_\pi((\rho, a_{\min}))$'s (the representations $\pi_\pm$ from section 30.).*

(4) *Look now at the symplectic groups. We have the following simplest example regarding (4) of the above definition.*

The representation $\delta([\nu^{-1}, \nu^2]) \rtimes 1$ has two irreducible subrepresentations. Denote them by $\pi_\pm$. Both are square integrable. We get from the proposition of the section 30. that their Jordan blocks are

$$\{(1_{F^\times}, 1), (1_{F^\times}, 3), (1_{F^\times}, 5)\}.$$

Their partially defined functions are (up to an order)

$$\epsilon_{\pi_+}((1_{F^\times}, 3))\epsilon_{\pi_+}((1_{F^\times}, 1))^{-1} = 1, \epsilon_{\pi_+}((1_{F^\times}, 5))\epsilon_{\pi_+}((1_{F^\times}, 3))^{-1} = 1;$$

$$\epsilon_{\pi_-}((1_{F^\times}, 3))\epsilon_{\pi_-}((1_{F^\times}, 1))^{-1} = -1, \epsilon_{\pi_-}((1_{F^\times}, 5))\epsilon_{\pi_-}((1_{F^\times}, 3))^{-1} = 1.$$

Here, in the full principal series, we have one more irreducible square integrable subquotient (the third one). Now we shall describe it.

The representation $\delta([\nu^0,\nu]) \rtimes \delta([\nu,\nu^2];1)$ *has two irreducible subrepresentations. Both are square integrable and one of them is* π_+. *Denote the remaining one by* π_3. *Its Jordan blocks are again*

$$\{(1_{F^\times},1),(1_{F^\times},3),(1_{F^\times},5)\}.$$

Its partially defined function is

$$\epsilon_{\pi_3}((1_{F^\times},3))\epsilon_{\pi_+}((1_{F^\times},1))^{-1}=1, \epsilon_{\pi_+}((1_{F^\times},5))\epsilon_{\pi_+}((1_{F^\times},3))^{-1}=-1.$$

Up to now, we have constructed 3 representations with the same Jordan blocks. Actually, the classification tells that the representation which would have both values -1, must be strongly positive (see below for the definition of this type of representations).

In the following section, we shall discuss square integrable representations which are the first step in construction of the general square integrable representations from the cuspidal ones.

32. Some examples of strongly positive representations

The criterion of W. Casselman for square integrability tells us that certain sums of exponents must be positive, but all the exponents $e(\rho_i)$ considered in the criterion do not need to be positive. For example, representations in sections 29. and 30. are square integrable, but all their exponents are not positive. From the other side, for the Steinberg representations, not only the sums are positive, but each term of each sum is positive. Such representations, for which all $e(\rho_i)$ in the criterion for square integrability are strictly positive, will be called **strongly positive representations**. They are obviously square integrable, and they are starting building blocks of general square integrable representations.

Examples of such representations are, besides generalized Steinberg representations, irreducible cuspidal representations (which can be viewed as a trivial case of generalized Steinberg representations). Combining several generalized Steinberg representations, one can get a new (just a little bit) more general examples of strongly positive representations (examples of such representations are representations $\delta(\nu^{1/2}\chi_1,\nu^{1/2}\chi_2;1)$ from section 28.; see [40] for more such regular examples).

In general, more characteristic examples of these type of representations have shown up in representation theory of p-adic groups pretty late. To get an idea about these (little bit unusual) representations, it is good to see some characteristic examples.

Before we give the examples, let us explain how one can characterize these representations in terms of admissible triples. To an irreducible square integrable representation π, we have up to now attached $\mathrm{Jord}(\pi)$, π_{cusp} and partially defined function ϵ_π, which may not be completely defined, but the differences $\epsilon_\pi((\rho,a))\epsilon_\pi((\rho,a_-))^{-1}$ are always defined. This is what we need.

We shall say that a triple $(\mathrm{Jord}(\pi), \epsilon_\pi, \pi_{\mathrm{cusp}})$ is **alternated** if

$$\epsilon_\pi((\rho, a))\epsilon_\pi((\rho, a_-))^{-1} = -1$$

for all exiting (neighbor) pairs $(\rho, a), (\rho, a_-)$ in $\mathrm{Jord}(\pi)$.

Mœglin has proved in [21] that an irreducible square integrable representation is strongly positive if and only if the attached triple is alternated.

Observe that irreducible cuspidal representations obviously have alternated triples, since they can not be subquotients in representations parabolically induced from any proper parabolic subgroup. Also, generalized Steinberg representations (and their "combinations") have alternated triples for obvious reason: their $\mathrm{Jord}_\rho(\pi)$ has at most one element (so the condition being alternated is automatically satisfied).

Really interesting examples here are the cases where condition of being alternated is not satisfied for such obvious reasons. For this, we need to have a reducibility bigger then 1. Therefore, the involved representation can not be generic. For a long time, we did not know such examples (we knew examples of non-generic representations, but we did not know reducibility points). This explains why such representations has shown relatively lately in the representation theory.

C. Mœglin has constructed an example of (non-generic) irreducible cuspidal representation σ of $Sp(8, F)$ such that $\nu^3 \rtimes \sigma$ reduces. Now the theorem from section 20. implies that $\mathrm{Jord}_{1_{F^\times}}(\sigma)$ is $\{1, 3, 5\}$.

Remarks. (1) *C. Mœglin has explained us how she got the example, using the Howe correspondence (Waldspurger published in 1980'es an example [44] of such use of Howe correspondence for determining reducibility points for* $GSp(4)$*). Later on, we shall also describe examples of reducibilities* > 1 *obtained by M. Reeder.*

(2) *As we noted already, Jordan blocks of* σ *(as above) along the trivial character of* $F^\times$ *consist exactly of 1, 3 and 5. Since the sum is 9, we conclude from the estimate (34.1) (which we shall introduce later), that these are all the Jordan blocks of* σ*. Therefore, this is the representation missing in (4) of Remarks in section 31.*

Now we can explain how should look like all the other cuspidal reducibilities of an irreducible selfdual cuspidal representation ρ *of a general linear group, with* σ*. The above fact, describing Jordan blocks of* σ*, tells that each other (integral and half-integral) reducibility is either 0 or 1/2. Moreover, basic assumption states that it should hold*

$$\alpha_{\rho,\sigma} - \alpha_{\rho,1} \in \mathbb{Z}.$$

Therefore, for other ρ*'s,* $\alpha_{\rho,\sigma}$ *is completely determined by* $\alpha_{\rho,1}$ *(so it depends only on* ρ*, not on* σ*).*

(3) *Suppose that the cuspidal reducibility* $\alpha_{\rho,1}$ *is*

$$1/2.$$

Then for any irreducible cuspidal representation σ' the reducibility should be some exponent in $1/2+\mathbb{Z}_{\geq 0}$. Therefore, the Jordan blocks along ρ should be even, i.e. of the form

$$(\rho, 2k).$$

Recall that even dimensional algebraic representations of $SL(2,\mathbb{C})$ are symplectic (while odd-dimensional are orthogonal). In the local Langlands correspondence for $Sp(2n,F)$ we need to get orthogonal representation, which should be of the form

$$\varphi(\rho)\otimes E_{2k},$$

where E_l denotes the l-dimensional irreducible algebraic representation of $SL(2,\mathbb{C})$. Therefore, in the local Langlands correspondence for general linear groups $\varphi(\rho)$ should be symplectic representation.

Because of that, if a representation $\nu^{1/2}\rho \rtimes 1$ of the symplectic group reduces, we say that ρ is a symplectic representation.

For a similar reason, if a representation $\nu^{1/2}\rho\rtimes 1$ of the symplectic group is irreducible, we say that ρ is an orthogonal representation.

F. Shahidi has shown that for selfdual irreducible cuspidal representation ρ of a general linear group, nonequivalent to $1_{F^\times}$, the reducibility in symplectic and orthogonal case of $\rho \rtimes 1$ is always in $\{0, 1/2\}$, and that it is irreducible in one case if and only if it is reducible in the other one (recall $\rho \not\cong 1_{F^\times}$). So orthogonality excludes being symplectic and conversely.

(4) *The basic assumption and the theorem from section 20 imply that if (ρ, k) is a Jordan block of an irreducible cuspidal representation of a classical group, where $k \geq 3$, then $(\rho, k-2)$ is always a Jordan block of the same (cuspidal) representation.*

Let us return to Mœglin's example, where $\nu^3 \rtimes \sigma$ reduces. We obtain a series of generalized Steinberg representations

$$\delta([\nu^3,\nu^k];\sigma)$$

for $k \geq 3$ (we can formally say that $k=2$ produces σ itself). All their Jacquet modules are essentially square integrable. For example, Siegel-like Jacquet module is

$$\delta([\nu^3,\nu^k])\otimes\sigma.$$

Here $\mathrm{Jord}_{1_{F^\times}}$ is $\{1,3,2k+1\}$.

In particular, $\mathrm{Jord}_{1_{F^\times}}(\delta(\nu^3;\sigma)) = \{1,3,7\}$.

Looking now at the regular representation $\nu^2\times\nu^3\rtimes\sigma$ and its subrepresentations

$$\nu^2\rtimes\delta(\nu^3;\sigma) \quad \text{and} \quad L(\nu^2,\nu^3)\rtimes\sigma,$$

we directly get that there exists a unique common subrepresentation π. This subrepresentation satisfies

$$\mu^*(\pi) = 1\otimes\pi + \nu^2\otimes\delta(\nu^3;\sigma) + L(\nu^2,\nu^3)\otimes\sigma.$$

From this follows that π is square integrable. Further analysis, using the proposition from section 30., gives

$$\mathrm{Jord}_{1_{F^\times}}(\pi) = \{1, 5, 7\},$$

and from the Jacquet modules follows that ϵ_π on the (difference of the) pair 5, 7 takes value -1 (otherwise, we would have a representation of the form $\nu^3 \otimes \tau$ in that Jacquet module, which is impossible). This representation will be denoted by

$$\delta([\nu^2, \nu^3]; \sigma).$$

Analogously, one can define representation

$$\delta([\nu, \nu^3]; \sigma)$$

which has

$$L(\nu, \nu^2, \nu^3) \otimes \sigma$$

in its Jacquet module, and for which the Jordan blocks along $1_{F^\times}$ are $\{3, 5, 7\}$.

"Combining" $\nu^4 \rtimes \delta([\nu, \nu^3]; \sigma)$ and $L([\nu, \nu^2]) \rtimes \delta([\nu^3, \nu^4]; \sigma)$ we get an irreducible (square integrable) subrepresentation, denoted by

$$\delta([\nu, \nu^4]; \sigma),$$

which has

$$L(\nu, \nu^2, \delta([\nu^3, \nu^4])) \otimes \sigma$$

in its Jacquet module, and for which Jordan blocks along $1_{F^\times}$ are $\{3, 5, 9\}$.

A more complicated task is to show that $\nu^3 \rtimes \delta([\nu, \nu^4]; \sigma)$ has an irreducible square integrable subquotient. Its Jordan blocks along $1_{F^\times}$ are $\{3, 7, 9\}$.

All above representations are strongly positive.

At this point, we have enough different examples to be able to get the idea of basic phenomena which show up in classification modulo cuspidal data of general square integrable representations. Instead of giving all details in the general case, which is pretty complicated, we shall describe the classification here on some interesting examples.

First we shall give a reduction to the

33. Cuspidal lines

Let ρ be an irreducible unitarizable cuspidal representation of a general linear group, and let

$$\mathrm{Irr}^{GL}_{\nu^{\mathbb{R}}\rho}$$

be the set of all irreducible subquotients of representations of the form

$$\nu^{e_1}\rho \times \cdots \times \nu^{e_k}\rho, \quad e_i \in \mathbb{R},\ k \geq 1.$$

Then each irreducible representation π of a general linear group can be decomposed

$$\pi \cong \mathop{\times}_{i=1}^{l} \pi(\rho_i)$$

where $\pi(\rho_i) \in \mathrm{Irr}^{GL}_{\nu^{\mathbb{R}}\rho_i}$, for some finite set of non-equivalent irreducible unitarizable cuspidal representations ρ_i of general linear groups. The above decomposition is unique up to a permutation. This decomposition, established by A.V. Zelevinsky, enables one to work in cuspidal lines $\mathrm{Irr}^{GL}_{\nu^{\mathbb{R}}\rho}$. Work with irreducible representations in particular cuspidal line is essentially independent of the corresponding cuspidal representation ρ.

We can make similar decomposition for classical groups S_n, but one needs to involve not only parabolic induction, but also Jacquet modules. We shall not go into details here, which appear to be rather elementary in the setting related to the square integrability (see [42]). This decomposition was done in full generality by C. Jantzen ([15]).

The main work which one needs to do in construction of irreducible square integrable representations for groups S_n is in the cuspidal lines. Therefore, we shall concentrate to a cuspidal line

$$\mathrm{Irr}^{S}_{\nu^{\mathbb{R}}\rho,\sigma},$$

which is the set of all irreducible subquotients of representations of the form

$$\nu^{e_1}\rho \times \cdots \times \nu^{e_k}\rho \rtimes \sigma, \quad e_i \in \mathbb{R},\ k \geq 0$$

(ρ will be assumed to be cuspidal and selfdual; this is enough to get all the square integrable representations). Square integrable classes in $\mathrm{Irr}^{S}_{\nu^{\mathbb{R}}\rho,\sigma}$ will be denoted by

$$\mathcal{D}(\rho;\sigma).$$

Similarly as in the case of general linear groups (where the situation in cuspidal lines was independent of ρ), here the construction depends on the number

$$\alpha_{\rho,\sigma},$$

but essentially not on ρ and σ.

Now we shall go back to

34. General strongly positive representations

Instead of giving a complete description of irreducible strongly positive representations , we shall follow the example

$$\alpha_{\rho,\sigma} = 3.$$

Note that in all examples of strongly positive square integrable representations in section 32., Jordan blocks had 3 elements. Actually, this holds in general. To any set $\{2k_1+1, 2k_2+1, 2k_3+1\}$ of 3 odd positive integers (we assume $0 \leq k_1 < k_2 < k_3$), one attaches a representation: the unique irreducible subrepresentation of

$$\Pi := \delta([\nu\rho, \nu^{k_1}\rho]) \times \delta([\nu^2\rho, \nu^{k_2}\rho]) \times \delta([\nu^3\rho, \nu^{k_3}\rho]) \rtimes \sigma$$

(observe that if we chose $k_i = i-1$, then factors corresponding to $j \leq i$ do not show up effectively). This subrepresentation is strongly positive, and the Jordan blocks along ρ are $\{2k_1+1, 2k_2+1, 2k_3+1\}$. In this way one gets all strongly positive representations in $\mathcal{D}(\rho;\sigma)$. Now we can easily get previous examples specifying particular choices of k_i's.

The description for any other positive integral reducibility $\alpha_{\rho,\sigma}$ is analogous to the above one (in the product Π we have $\alpha_{\rho,\sigma}$ GL-factors, instead of 3). Note that for $\alpha_{\rho,\sigma}=1$, each strongly positive representation in $\mathcal{D}(\rho;\sigma)$ is a generalized Steinberg representations.

For $\alpha_{\rho,\sigma}=0$, σ is the only strongly positive representation in $\mathcal{D}(\rho;\sigma)$.

Now we shall explain situation in the case of half-integral reducibility.

Similar as above, the generalized Steinberg representations are the only strongly positive representations in $\mathcal{D}(\rho;\sigma)$ for $\alpha_{\rho,\sigma}=1/2$.

Consider now the case $\alpha_{\rho,\sigma}=5/2$ (this reducibility shows up in an example of M. Reeder, in the case of irreducible degenerate unipotent cuspidal representations; see the remarks below). The theorem in section 20. implies that the Jordan blocks of σ along ρ are

$$\mathrm{Jord}_\rho=\{2,4\}.$$

One has generalized Steinberg representations $\delta([\nu^{5/2}\rho,\nu^{5/2+k}];\sigma)$, $k\geq -1$ (which are strongly positive), for which Jordan blocks along ρ consist of $\{2,2k+6\}$. In particular, for $\delta(\nu^{5/2}\rho;\sigma)$ Jordan blocks along ρ consists of $\{2,6\}$. From the definition of the partially defined function, it is clear that its value is -1 on $(\rho,2)$.

Similarly as above, we have a unique irreducible subrepresentation of

$$\nu^{3/2}\rho\rtimes\delta(\nu^{5/2};\sigma),$$

denoted by

$$\delta([\nu^{3/2}\rho,\nu^{5/2}];\sigma).$$

Here Jordan blocks along ρ consists of $\{4,6\}$. From the definition of the partially defined function, it is clear that its value is -1 on $(\rho,4)$.

Continuing with $\nu^{1/2}\rho\rtimes\delta([\nu^{3/2}\rho,\nu^{5/2}];\sigma)$, and its unique irreducible subrepresentation denoted by $\delta([\nu^{1/2}\rho,\nu^{5/2}];\sigma)$ (which is square integrable), we would get Jordan blocks along ρ are $\{2,4,6\}$. From the definition of the partially defined function, it is clear that its value is 1 on $(\rho,2)$.

Now we can say how strongly positive representations in $\mathcal{D}(\rho;\sigma)$ look like for this reducibility (i.e. at 5/2). Here Jordan blocks along ρ can have 2 or 3 elements.

Consider first the case when the Jordan blocks along ρ have two elements $\{2k_1,2k_2\}$, $0<k_1<k_2$. Then corresponding strongly positive representation is the unique irreducible subrepresentation of

$$\delta([\nu^{3/2}\rho,\nu^{(2k_1-1)/2}])\times\delta([\nu^{5/2}\rho,\nu^{(2k_2-1)/2}])\rtimes\sigma.$$

In this case, the partially defined function takes the value -1 on $(\rho,2k_1)$.

Consider now the case when the Jordan blocks along ρ have three elements $\{2k_1,2k_2,2k_3\}$, $0<k_1<k_2<k_3$. Then corresponding strongly positive representation is unique irreducible subrepresentation of

$$\delta([\nu^{1/2}\rho,\nu^{(2k_1-1)/2}])\times\delta([\nu^{3/2}\rho,\nu^{(2k_2-1)/2}])\times\delta([\nu^{5/2}\rho,\nu^{(2k_3-1)/2}])\rtimes\sigma.$$

In this case partially defined function is 1 on $(\rho, 2k_1)$.

This is general way how one constructs strongly positive representations in the half-integral case. In general, for a half-integral reducibility $\alpha_{\rho,\sigma}$, the Jordan blocks will have $\alpha_{\rho,\sigma} \pm 1/2$ elements. The corresponding partially defined function is 1 on the minimal element if and only if the Jordan blocks have $\alpha_{\rho,\sigma} + 1/2$ elements.

"Control" of π (or $\mathrm{Jord}(\pi)$ and ϵ_π) over π_{cusp} roughly corresponds to the condition related to the Jordan blocks of a strongly positive representation coming with π, and from the Jordan blocks of π_{cusp} along each ρ (roughly, in the case of integral reducibility they should have cardinality equal to the value of the reducibility point, while in the half-integral case the cardinality needs to be equal to the value of the reducibility point $\pm 1/2$; in the half-integral case there is additional condition on partially defined function depending on sign of $1/2$). We shall not go here in more details (which one can find in [21], or [24], or [42]).

Remarks. (1) *M. Reeder proved that there exists the following reducibility: For $k \geq 1$, denote $n = k^2 + k + 1$. He has shown, using computation of Plancherel measures, that for certain irreducible unipotent degenerate cuspidal representations $\theta(k)$, the induced representation*

$$\nu^{k+1/2} \rtimes \theta(k)$$

reduces. Further, $\theta(1)$ is a representation of $SO(5, F)$, $\theta(2)$ is a representation of $SO(13, F)$, $\theta(3)$ is a representation of $SO(26, F)$.

(2) *C. Mœglin has proved in [22] that*

$$\sum_{(\rho,k)\in Jord(\pi)} kd_\rho \leq 2n \ (resp. \ \leq 2n+1) \tag{34.1}$$

holds if π is a square integrable representation of $SO(2n + 1)$ (resp. $Sp(2n)$), where

$$d_\rho$$

is defined with requirement that ρ is a representation of $GL(d_\rho, F)$. It is expected that in (34.1) holds the equality.

(3) *For a generic square integrable representation π, Theorem 8.1 of [5] implies the equality in (34.1). Note that equality in (34.1) is equivalent to the fact that the expected admissible homomorphism (28.1) attached to π goes to the dual group of expected rank.*

(4) *Suppose that σ is an irreducible cuspidal representation of $SO(2n + 1, F)$ (resp. $Sp(2n, F)$). Let X be the set of all non-equivalent self-dual irreducible cuspidal representations ρ of general linear groups $GL(d_\rho, F)$ such that*

$$\alpha_{\rho,\sigma} \geq 1$$

(and $\alpha_{\rho,\sigma} \in (1/2)\mathbb{Z}$). Then the theorem in section 20. and (34.1) imply

$$\sum_{\rho\in X,\, \alpha_{\rho,\sigma}\in\mathbb{Z}} \alpha^2_{\rho,\sigma} d_\rho + \sum_{\rho\in X,\, \alpha_{\rho,\sigma}\in(1/2)\mathbb{Z}} (\alpha^2_{\rho,\sigma} - 1/4) d_\rho \leq 2n \ (resp.\ 2n+1).$$

For generic (and cuspidal) σ, we have

$$\sum_{\rho\in X} d_\rho = 2n \ (resp.\ 2n+1).$$

35. The general step

We shall explain a general principle which is used to construct all the square integrable representations, starting from the strongly positive ones.

Theorem. *Let π be an irreducible square integrable representation of S_q, ρ an irreducible cuspidal selfdual representation of $GL(p,F)$, and $a, a_- \in \mathbb{Z}_{\geq 1}$ such that $a - a_- \in 2\mathbb{Z}_{\geq 1}$. We shall assume that (ρ, a) satisfies the parity condition: a is even $\iff$ $\nu^{1/2}\rho \rtimes 1_{S_0}$ reduces (in the series of groups that we consider). Further we assume*

$$\mathrm{Jord}_\rho(\pi) \cap \{x; a_- \leq x \leq a\} = \emptyset.$$

Then the representation

$$\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi \tag{35.1}$$

contains exactly two irreducible subrepresentations. They are not equivalent.

Further, we can decompose

$$\delta([\nu^{-(a_--1)/2}\rho, \nu^{(a_--1)/2}\rho]) \rtimes \pi = T_1 \oplus T_{-1}$$

as a sum of two nonequivalent irreducible (tempered) representations. Then

$$\delta([\nu^{(a_--1)/2+1}\rho, \nu^{(a-1)/2}\rho]) \rtimes T_\eta$$

contains a unique irreducible subrepresentation, which we denote by π_η. These irreducible representations are exactly the irreducible subrepresentations of (35.1).

Representations π_η are square integrable,

$$\mathrm{Jord}(\pi_\eta) = \mathrm{Jord}(\pi) \cup \{(\rho, a), (\rho, a_-)\},$$

and

$$\epsilon_{\pi_\eta}(\rho, a) = \epsilon_{\pi_\eta}(\rho, a_-).$$

Further, ϵ_{π_η} restricted to $\mathrm{Jord}(\pi)$ is just ϵ_π.

Proofs of square integrability in sections 29. and 30. illustrate on simple examples the proof of above theorem (proof in general case is much more complicated).

Theorem. *Supposing that the basic assumption holds, one gets each irreducible square integrable representation in finitely many steps as described in the previous theorem, starting from an irreducible strongly positive square integrable representation. Admissible triples give one to one parameterization of irreducible square integrable representations.*

For a more precise statement and more details, one should consult [24].

We have already noted that this classification is modulo irreducible cuspidal representations and cuspidal reducibilities (i.e. it reduces irreducible square integrable representations to the irreducible cuspidal representations ρ and σ and cuspidal reducibilities $\alpha_{\rho,\sigma}$). As we have already noted, we can also expect that it will give a reduction in establishing local Langlands correspondence for classical groups to the cuspidal case: the admissible homomorphism corresponding to an irreducible square integrable representation π should be

$$\bigoplus_{(\rho,k)\in \mathrm{Jord}(\pi)} \Phi(\rho)\otimes E_k,$$

where Φ in the above formula denotes the local Langlands correspondence for general linear groups.

From the other side, if we know which Jordan blocks are attached to an irreducible cuspidal representation σ of a classical group (which would determine the admissible homomorphism of the Weil-Deligne group corresponding to σ), then the reducibility point would be:

- $\alpha_{\rho,\sigma} = \frac{\max(\mathrm{Jord}_\rho(\sigma))+1}{2}$ if $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$;
- $\alpha_{\rho,\sigma} = \alpha_{\rho,1}$ if $\mathrm{Jord}_\rho(\sigma) = \emptyset$

(the second equality follows from the basic assumption).

To know the reducibility $\alpha_{\rho,1}$, it is enough to know if the local Langlands correspondence carries ρ into symplectic or orthogonal representation (see Remarks in section 32.; if an irreducible cuspidal representation goes to the symplectic group, then in the second case reducibility should be for symplectic series of groups at $1/2$ and for the orthogonal series of groups at 0; if it goes to the orthogonal group, we should have converse situation).

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