ON UNITARIZABILITY IN THE CASE OF CLASSICAL p-ADIC GROUPS

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ABSTRACT. In the introduction of this paper we discuss a possible approach to the unitarizability problem for classical p-adic groups. In this paper we give some very limited support that such approach is not without chance. In a forthcoming paper we shall give additional evidence in generalized cuspidal rank (up to) three.

1. INTRODUCTION

Some important classes of irreducible unitary representations of classical p-adic groups have been classified. Still, classification of the whole unitary dual of these groups does not seem to be in sight in the moment. Since the case of general linear groups is well-understood, we shall start with description of the unitarizability in the case of these groups, the history related to this and what this case could suggest us regarding the unitarizability for the classical p-adic groups.

Fix a local field F. Denote by $GL(n,F)\widehat{\ }$ the set of all equivalence classes of irreducible unitary representations of $GL(n,F)$. We shall use a well-known notation $\times$ of Bernstein and Zelevinsky for parabolic induction of two representations π_i of $GL(n_i,F)$:

$$\pi_1 \times \pi_2 = \mathrm{Ind}^{GL(n_1+n_2,F)}(\pi_1 \otimes \pi_2)$$

(the above representation is parabolically induced from a suitable parabolic subgroup containing upper triangular matrices whose Levi factor is naturally isomorphic to the direct product $GL(n_1,F) \times GL(n_2,F)$). Denote by ν the character $|\det|_F$ of a general linear group. Let $D_u = D_u(F)$ be the set of all the equivalence classes of the irreducible square integrable (modulo center) representations of all $GL(n,F)$, $n \geq 1$. For $\delta \in D_u$ and $m \geq 1$ denote by

$$u(\delta, m)$$

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the unique irreducible quotient of $\nu^{(m-1)/2}\delta\times\nu^{(m-1)/2-1}\delta\times\ldots\times\nu^{-(m-1)/2}\delta$. This irreducible quotient is called a Speh representation. Let B_{rigid} be the set of all Speh representations, and

$$B = B(F) = B_{rigid} \cup \{\nu^{\alpha}\sigma \times \nu^{-\alpha}\sigma; \sigma \in B_{rigid}, 0 < \alpha < 1/2\}.$$

Denote by $M(B)$ the set of all finite multisets in B. Then the following simple theorem solves the unitarizability problem for the archimedean and the non-archimedean general linear groups in a uniform way:

Theorem 1.1. ([56], [65]) *A mapping*

$$(\sigma_1,\ldots,\sigma_k) \mapsto \sigma_1 \times \ldots \times \sigma_k$$

defined on $M(B)$ goes into $\cup_{n\geq 0} GL(n,F)\widehat{\ }$, and it is a bijection.

The above theorem was first proved in the p-adic case in mid 1980-es (in [56]). Since the claim of the theorem makes sense also in the archimedean case, immediately became evident that the theorem extends also to the archimedean case, with the same strategy of the proof (the main ingredients of the proof were already present in that time, although one of them was announced by Kirillov, but the proof was not complete in that time). One can easily get an idea of the proof from [55] (there is considered the p-adic case, but exactly the same strategy holds in the archimedean case). Vogan's classification in the archimedean case (Theorem 6.18 of [70]) gives a very different description of the unitary dual (it is equivalent to Theorem 1.1, but it is not obvious to see that it is equivalent - see section 12 of [6]).

In the rest of this paper, we shall consider only the case of non-archimedean field F. Although the representation theory of reductive p-adic groups started with the F. Mautner paper [36] from 1958, the ideas that lead to the proof of the above theorem can be traced back to the paper of I. M. Gelfand and M. A. Naimark [15] from 1947, and together with the work on the unitarizability of general linear groups over division algebras, we may say that spans a period of seven decades.

The proof of Theorem 1.1 in [56] is based on a very subtle Bernstein-Zelevinsky theory based on the derivatives ([71]), and on the Bernstein's paper [10]. Among others , the Bernstein's paper [10] proves a fundamental fact about distributions on general linear groups. It is based on the geometry of these groups (a key idea of that paper can be traced back to the Kirillov's paper [28] from 1962, which is motivated by a result of the Gelfand-Naimark book [16]). One of the approaches to the unitarizability of the Speh representations is using the H. Jacquet's construction of the residual spectrum of the spaces of the square integrable automorphic forms in [23], which generalizes an earlier construction of B. Speh in [53].

We presented in [57] what we expected to be the answer to the unitarizability question for general linear groups over a local non-archimedean division algebra $\mathcal{A}$[1]. We have reduced in [57] a proof of the expected answer to two expected facts. They were proved by J. Badulescu and D. Renard ([5]), and V. Sécheree ([50]). As well as in the field case, these proofs (together with the theories that they require) are far from being simple (the Sécheree proof is particularly technically complicated since it requires knowledge of a complete theory of types for these groups).

As a kind of surprise came a recent work [31] of E. Lapid and A. Mínguez in which they gave another (surprisingly simple in comparison with the earlier) proof of the Sécheree result (relaying on the Jacquet module methods). Besides, J. Badulescu gave earlier in [3] another very simple (local) proof of his and Renard's result.

Thanks to this new development, we have a pretty simple approach to the unitarizability for general linear groups over non-archimedean division algebras, using only very standard non-unitary theory and knowledge of the reducibility point between two irreducible cuspidal representations of general linear groups, i.e. when $\rho \times \rho'$ reduces for ρ and ρ' irreducible cuspidal representations[2]. It is very important that we have such a relative simple approach to the irreducible unitary representations in this case, since these representations are basic ingredients of some very important unitary representations, like the representations in the spaces of the square integrable automorphic forms, and their knowledge can be quite useful (see [33], or [20] or [21])

Thanks to the work of J. Arthur, C. Mœglin and J.-L. Waldspurger, we have now classification of the irreducible cuspidal representations of classical p-adic groups[3] in the characteristic zero (Theorem 1.5.1 of [40] and Corollary 3.5 of [42]). Their parameters give directly the reducibility points with irreducible cuspidal representations of general linear groups (see for example (ii) in Remarks 4.5.2 of [44] among other papers). These reducibilities are any of $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$.

Therefore, a natural and important question is if we can have an approach to the unitarizability in the case of classical p-adic groups based only on the cuspidal reducibility points. We shall try to explain a possible strategy for such an approach based only on the reducibility points.

We fix a series of classical p-adic groups (see the section 2 for more details). First we shall introduce a notation for parabolic induction for the classical p-adic groups. The

[1]For $\delta \in D(\mathcal{A})_u$ denote by $\nu_\delta := \nu^{s_\delta}$, where s_δ is the smallest non-negative number such that $\nu^{s_\delta}\delta \times \delta$ reduces. Introduce $u(\delta, n)$ in the same way as above, except that we use ν_δ in the definition of $u(\delta, n)$ instead of ν. Then the expected answer is the same as in the Theorem 1.1, except that one replaces ν by ν_δ in the definition of $B(\mathcal{A})$.

[2]In the field case it reduces if and only if $\rho' \cong \nu^{\pm 1}\rho$.

[3]By classical groups we mean symplectic, orthogonal and unitary groups (see the following sections for more details). In this introduction and in the most of the paper we shall deal with symplectic and orthogonal groups. The case of unitary groups is discussed in the last section of the paper.

multiplication $\times$ between representations of general linear groups defined using parabolic induction has a natural generalization to a multiplication

$$\rtimes$$

between representations of general linear and classical groups defined again using the parabolic induction (see the second section of this paper).

Now we shall recall of the reduction of the unitarizability problem obtained in [66]. An irreducible representation π of a classical group is called weakly real if there exist self contragredient irreducible cuspidal representations ρ_i of general linear groups, an irreducible cuspidal representation σ of a classical group, and $x_i \in \mathbb{R}$, such that

$$\pi \hookrightarrow \nu^{x_1}\rho_1 \times \ldots \times \nu^{x_k}\rho_k \rtimes \sigma.$$

Then [66] reduces in a simple way the unitarizability problem for the classical p-adic groups to the case of the weakly real representations of that series of groups (see Theorems 2.1 and 2.2 of this paper). This is the reason that we shall consider only the weakly real representations in the sequel.

Let X be some set of irreducible cuspidal representations of the general linear groups. For an irreducible representation τ of a general linear group one says that it is supported by X if there exist $\rho_i \in X$ such that $\tau \hookrightarrow \rho_1 \times \ldots \times \rho_k$. Let additionally σ be an irreducible cuspidal representations of a classical p-adic group and assume $X = \tilde{X} := \{\tilde{\rho}; \rho \in X\}$ ($\tilde{\rho}$ denotes the contragredient representation of ρ). Then for an irreducible representation π of a classical p-adic group one says that it is supported by $X \cup \{\sigma\}$ if there exist $\rho_i \in X$ such that $\pi \hookrightarrow \rho_1 \times \ldots \times \rho_k \rtimes \sigma$. In that case we say that

$$\sigma$$

is a partial cuspidal support of π.

Let ρ be an irreducible self contragredient cuspidal representation of a general linear group. Denote $X_\rho := \{\nu^x\rho; x \in \mathbb{R}\}$. We call X_ρ a line of cuspidal representations. Further, denote by

$$Irr(X_\rho; \sigma)$$

the set of all equivalence classes of irreducible representations of classical groups supported by $X_\rho \cup \{\sigma\}$.

Let π be an irreducible (weakly real) representation of a classical p-adic group and denote its partial cuspidal support by σ. Then for ρ as above, there exists a unique irreducible representation $X_\rho(\pi)$ of a classical group supported in $X_\rho \cup \{\sigma\}$ and an irreducible representation $X_\rho^c(\pi)$ of a general linear group supported out of X_ρ such that

$$\pi \hookrightarrow X_\rho^c(\pi) \rtimes X_\rho(\pi). \tag{1.1}$$

One can chose a finite set $\rho_1, \ldots, \rho_k$ of irreducible selfcontragredient cuspidal representations of general linear groups such that for other selfcontragredient representations ρ of general linear groups we have always $X_\rho(\pi) = \sigma$.

Fix some set $\rho_1, \dots, \rho_k$ as above. Then C. Jantzen has shown in [24] that the correspondence

$$\pi \leftrightarrow (X_{\rho_1}(\pi), \dots, X_{\rho_k}(\pi)) \tag{1.2}$$

is a bijection from the set of all irreducible representations of classical groups supported by $X_{\rho_1}, \dots, X_{\rho_k} \cup \{\sigma\}$ onto the direct product $\prod_{i=1}^{k} Irr(X_{\rho_i}; \sigma)$. Moreover, C. Jantzen has shown that the above correspondence reduces some of the most basic data from the non-unitary theory about general parabolically induced representations (like for example the Kazhdan-Lusztig multiplicities) to the corresponding data for such representations supported by single cuspidal lines.

Regarding the unitarizability, it would be very important to know the answer to the following

Preservation of unitarizability question: Is π is unitarizable if and only if all $X_{\rho_i}(\pi)$ are unitarizable, $i = 1, \dots, k$.

If we would know that the answer to the above question is positive, then this would give a reduction of the unitarizability of a general irreducible representation to the unitarizability for the irreducible representations of classical p-adic groups supported in single cuspidal lines. Such a line $X_\rho \cup \{\sigma\}$ is determined by ρ and σ, for which there exists a unique $\alpha_{\rho,\sigma} \geq 0$ such that

$$\nu^{\alpha_{\rho,\sigma}} \rho \rtimes \sigma$$

reduces. In this paper we shall consider only the cases when

$$\alpha_{\rho,\sigma} \in (1/2)\mathbb{Z}. \tag{1.3}$$

Actually, from the recent work of J. Arthur, C. Mœglin and J.-L. Waldspurger, this assumption is known to hold if $\mathrm{char}(F) = 0$ (Theorem 3.1.1 of [41], and [42]).

Now suppose that we have additional pair ρ', σ' as ρ, σ. Assume that

$$\alpha_{\rho,\sigma} = \alpha_{\rho',\sigma'}.$$

If $\alpha_{\rho,\sigma} > 0$, then there exists a canonical bijection

$$E : Irr(X_\rho, \sigma) \to Irr(X_{\rho'}; \sigma')$$

(see the section 12 for the definition[4] of E). If $\alpha_{\rho,\sigma} = 0$, then we have two possibilities for such a bijection (see again the section 12). Chose any one of them and denote it again by E.

As we already mentioned, the parameter which determines the set $Irr(X_\rho, \sigma)$ (whose unitarizable representations we would like to determine) is the pair ρ, σ. This pair determines the cuspidal reducibility point $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}$, which is a very simple object in comparison with the pair ρ, σ. Therefore, a natural question would be to try to see if the unitarizability depends only on $\alpha_{\rho,\sigma}$, and not on ρ, σ itself. More precisely, we have the following

[4] It is there denoted by $E_{1,2}$.

Independence of unitarizability question: Let $\pi \in Irr(X_\rho, \sigma)$. Does it hold that π is unitarizable if and only if $E(\pi)$ is unitarizable.

In this paper we give some very limited evidence that one could expect positive answers to the above two questions. Using the classification of the generic unitary duals in [32], we get that the both above questions have positive answers in the case of irreducible generic representations (see the section 11). Also the classification of the unramified unitary duals in [45] implies that we have positive answer to the first question in the case of irreducible unramified representations.

Further, very limited evidence for positive answer to the second question give papers [18] and [19]. They imply that Independence of unitarizability question has positive answer for irreducible representations which have the same infinitesimal character as a generalized Steinberg representation[5]. The biggest part of this paper is related to Preservation of unitarizability question for representation whose one Jantzen component $X_\rho(\pi)$ has the same infinitesimal character like a generalized Steinberg representation. We are able to prove the following very special case related to Preservation of unitarizability question for such representations:

Theorem 1.2. *Suppose that π is an irreducible unitarizable representation of a classical group, and suppose that the infinitesimal character of some $X_\rho(\pi)$ is the same as the infinitesimal character of a generalized Steinberg representation corresponding to a reducibility point $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}$[6]. Then $X_\rho(\pi)$ is the generalized Steinberg representation, or its Aubert-Schneider-Stuhler dual. If char$(F) = 0$, then $X_\rho(\pi)$ is unitarizable.*

For the case when $\pi = X_\rho(\pi)$ (i.e. when π is supported by a single cuspidal line), in [18] and [19] is proved that π is a generalized Steinberg representation or its Aubert-Schneider-Stuhler dual (which are both unitarizable in characteristic zero). Our first idea to prove the above theorem (more precisely, to prove that $X_\rho(\pi)$ is a generalized Steinberg representation or its Aubert-Schneider-Stuhler dual) was to use the strategy of that two papers and the methods of [24]. While we were successful in extending [18], we were not for [19]. This was a reason for a search of a new (uniform) proof for [18] and [19], which is easy to extend to the proof of the above theorem (using [24]). This new proof is based on the following fact.

Proposition 1.3. *Fix irreducible cuspidal representations ρ and σ of a general linear and a classical group respectively, such that ρ is self contragredient. Suppose that $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha \in \frac{1}{2}\mathbb{Z}$, $\alpha > 0$. Let γ be an irreducible subquotient of*

$$\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma, \tag{1.4}$$

[5]Generalized Steinberg representations are defined and studied in [61]. See the section 3 of this paper for a definition

[6]As we already noted, this is know to hold if char$(F) = 0$.

different from the generalized Steinberg representation and its Aubert-Schneider-Stuhler involution[7]*. Then there exists an irreducible selfcontragredient unitarizable representation* τ *of a general linear group with support in* $\{\nu^k\rho; k \in \frac{1}{2}\mathbb{Z}\}$*, such that the length of*

$$\tau \rtimes \gamma$$

is at least 5, and that the multiplicity of $\tau \otimes \gamma$ *in the Jacquet module of* $\tau \rtimes \gamma$ *is at most 4.*

A bigger part of this paper is the proof of the above proposition. The proof is pretty technical and we shall say only a few remarks about it here. We often use Proposition 5.3 of [66][8] in proving that the length of $\tau \rtimes \gamma$ is at lest five. Two irreducible sub quotients we get directly applying this proposition. For remaining irreducible sub quotients we consider $\tau \rtimes \gamma$ as a part of a bigger representation Π which has the same semi simplification as $\tau \rtimes \gamma + \Pi'$, for some Π'. An advantage of Π in comparison to $\tau \rtimes \gamma$ is that we can easily write some irreducible sub quotients of it (using Proposition 5.3 of [66]). Next step is to show that these irreducible sub quotients are not sub quotients of Π'. For this is particularly useful the Geometric lemma, which is systematically applied through the structure of a twisted Hopf module which exists on the sum of the Grothendieck groups of the categories of the finite length representations of the classical groups. Further, the multiplicity of $\tau \otimes \gamma$ in the Jacquet module of $\tau \rtimes \gamma$ is estimated using the combinatorics which provides the above structure of the twisted Hopf module.

Now we shall recall a little bit of the history of the unitarizability of the irreducible representations which have the same infinitesimal character as a generalized Steinberg representation. The first case is the case of the Steinberg representations. The question of their unitarizability in this case came from the question of cohomologically non-trivial irreducible unitary representations. Their non-unitarizability or unitarizability was proved by W. Casselman ([13]). His proof of the non-unitarizability relays on the study of the Iwahori Hecke algebra. The importance of this non-unitarizability is very useful in considerations of the unitarizability in low ranks, since it implies also the non-existence of complementary series which would end by the trivial representation (it also reproves the classical result of Kazhdan from [27] in the p-adic case).

A. Borel and N. Wallach observed that the Casselman's non-unitarizability follows from the Howe-Moore theorem about asymptotics of the matrix coefficients of the irreducible unitary representations ([22]) and the Casselman's asymptotics of the matrix coefficients of the admissible representations of reductive p-adic groups ([12]). Neither of that two methods can be used for the case of the generalized Steinberg representation. This was a motivation to write papers [18] and [19]. The strategy of the proofs of that two papers was for a γ from Proposition 1.3 to find an irreducible unitarizable representation τ of a general

[7]The generalized Steinberg representation is a unique irreducible subrepresentation of (1.4), while its Aubert-Schneider-Stuhler involution is the unique irreducible quotient of (1.4).

[8]This is an extension to the case of classical groups of Proposition 8.4 of [71], which in the terms of the Langlands classification tells that $L(a+b) \leq L(a) \times L(b)$ (see the section 2 for notation).

linear group such that $\tau \rtimes \gamma$ is not semisimple. The semisimplicity of $\tau \rtimes \gamma$ (using the Frobenius reciprocity) would imply that $\tau \otimes \gamma$ is in the Jacquet module of each irreducible subquotient θ of $\tau \rtimes \gamma$. In [18] and [19], there were found τ and θ such that $\tau \otimes \gamma$ is not in the Jacquet module of θ. This implied the non-unitarizability of γ.

In this paper our strategy is to find τ such that the length of $\tau \rtimes \gamma$ is strictly bigger then the multiplicity of $\tau \otimes \sigma$ in the Jacquet module of $\tau \rtimes \sigma$ (the above proposition implies this).

We are particularly thankful to C. Jantzen for reading the section 8 of this paper, and giving suggestions about it (in that section are presented the main results of C. Jantzen from [24] in a slightly reformulated form). We are very thankful to C. Mœglin for her explanations regarding references related to some assumptions considered in this paper. We are also thankful to M. Hanzer, E. Lapid and A. Moy for useful discussions during the writing of this paper, and to the referee for useful suggestions.

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In a forthcoming paper we shall prove that the above two questions have positive answers for irreducible (weakly real) subquotients of representations

$$\rho_1 \times \ldots \times \rho_k \rtimes \sigma, \qquad k \leq 3,$$

where ρ_i are irreducible cuspidal representations of general linear groups and σ is an irreducible cuspidal representations of a classical p-adic group. Moreover, we shall classify such subquotients.

We shall now briefly review the contents of the paper. The second section brings the notation that we use in the paper, while the third one describes the irreducible representations that we shall consider. The fourth section recalls of Proposition 1.3 and explains what are the first two stages of its proof. In the fifth section is the first stage of the proof (when the essentially square integrable representation of a general linear group with the lowest exponent that enters the Langlands parameter of γ is non-cuspidal, and the tempered representation of the classical group which enters the Langlands parameter of γ is cuspidal). The following section considers the situation as in the previous section, except that the essentially square integrable representation of a general linear group with the lowest exponent that enters the Langlands parameter of γ is now cuspidal. Actually, we could handle these two cases as a single case. Nevertheless, we split it, since the first case is simpler, and it is convenient to consider it first. The seventh section handles the remaining case, when the tempered representation of a classical group which enters the Langlands parameter of γ is not cuspidal. This case is obtained from the previous two sections by a simple application of the Aubert-Schneider-Stuhler involution. At the end of this section we get the main results of [18] and [19] as a simple application of Proposition 1.3. In the following section we recall of the Jantzen decomposition of an irreducible representation of a classical p-adic group in a slightly modified version then in [24], while the ninth section discusses the decomposition into the cuspidal lines. In the tenth section we give a proof of Theorem 1.1, while in the following section we show that the unitarizability is preserved in the case of the irreducible generic representations of classical p-adic groups. In a similar way, using [45], one can see also that the unitarizability is preserved for the irreducible unramified representations of the classical groups considered in [45] (i.e. for the split classical p-adic groups). In the twelfth section we formulate a question if the unitarizability for the irreducible representations of classical groups supported by a single cuspidal line depends only on the reducibility point (i.e., not on the particular cuspidal representations which have that reducibility). The last section discusses the case of unitary groups.

2. Notation and Preliminaries

Now we shall briefly introduce the notation that we shall use in the paper. One can find more details in [60] and [43].

We fix a local non-archimedean field F of characteristic different from two. We denote by $|\ \ |_F$ the normalized absolute value on F.

For the group $\mathcal{G}$ of F-rational points of a connected reductive group over F, we denote by $\mathcal{R}(\mathcal{G})$ the Grothendieck group of the category $\mathrm{Alg}_{f.l.}(\mathcal{G})$ of all smooth representations of $\mathcal{G}$ of finite length. We denote by

$$\text{s.s.}$$

the semi simplification map $\mathrm{Alg}_{f.l.}(\mathcal{G}) \to \mathcal{R}(\mathcal{G})$. The irreducible representations of $\mathcal{G}$ are also considered as elements of $\mathcal{R}(\mathcal{G})$.

We have a natural ordering $\leq$ on $\mathcal{R}(\mathcal{G})$ determined by the cone $\text{s.s.}(\mathrm{Alg}_{f.l.}(\mathcal{G}))$.

If $\text{s.s.}(\pi_1) \leq \text{s.s.}(\pi_2)$ for $\pi_i \in \mathrm{Alg}_{f.l.}(\mathcal{G})$, then we write simply $\pi_1 \leq \pi_2$.

Now we go to the notation of the representation theory of general linear groups (over F), following the standard notation of the Bernstein-Zelevinsky theory ([71]). Denote

$$\nu : GL(n,F) \to \mathbb{R}^\times, \quad \nu(g) = |\det(g)|_F.$$

The set of equivalence classes of all irreducible essentially square integrable modulo center[9] representations of all $GL(n,F)$, $n \geq 1$, is denoted by

$$D.$$

For $\delta \in D$ there exists a unique $e(\delta) \in \mathbb{R}$ and a unique unitarizable representation δ^u (which is square integrable modulo center), such that

$$\delta \cong \nu^{e(\delta)}\delta^u.$$

[9]These are irreducible representations which become square integrable modulo center after twist by a (not necessarily unitary) character of the group.

The subset of cuspidal representations in D is denoted by

$$\mathcal{C}.$$

For smooth representations π_1 and π_2 of $GL(n_1,F)$ and $GL(n_2,F)$ respectively, $\pi_1\times\pi_2$ denotes the smooth representation of $GL(n_1+n_2,F)$ parabolically induced by $\pi_1\otimes\pi_2$ from the appropriate maximal standard parabolic subgroup (for us, the standard parabolic subgroups will be those parabolic subgroups which contain the subgroup of the upper triangular matrices). We use the normalized parabolic induction in the paper.

We consider

$$R=\underset{n\geq 0}{\oplus}\mathcal{R}(GL(n,F))$$

as a graded group. The parabolic induction $\times$ lifts naturally to a $\mathbb{Z}$-bilinear mapping $R\times R\to R$, which we denote again by $\times$. This $\mathbb{Z}$-bilinear mapping factors through the tensor product, and the factoring homomorphism is denoted by $m:R\otimes R\to R$.

Let π be an irreducible smooth representation of $GL(n,F)$. The sum of the semi simplifications of the Jacquet modules with respect to the standard parabolic subgroups which have Levi subgroups $GL(k,F)\times GL(n-k,F)$, $0\leq k\leq n$, defines an element of $R\otimes R$ (see [71] for more details). The Jacquet modules that we consider in this paper are normalized. We extend this mapping additively to the whole R, and denote the extension by

$$m^*:R\to R\otimes R.$$

In this way, R becomes a graded Hopf algebra.

For an irreducible representation π of $GL(n,F)$, there exist $\rho_1,\dots,\rho_k\in\mathcal{C}$ such that π is isomorphic to a subquotient of $\rho_1\times\cdots\times\rho_k$. The multiset of equivalence classes $(\rho_1,\dots,\rho_k)$ is called the cuspidal support of π.

Denote by $M(D)$ the set of all finite multisets in D. We add multi sets in a natural way:

$$(\delta_1,\delta_2,\dots,\delta_k)+(\delta_1',\delta_2',\dots,\delta_{k'}')=(\delta_1,\delta_2,\dots,\delta_k,\delta_1',\delta_2',\dots,\delta_{k'}').$$

For $d=(\delta_1,\delta_2,\dots,\delta_k)\in M(D)$ take a permutation p of $\{1,\dots,k\}$ such that

$$e(\delta_{p(1)})\geq e(\delta_{p(2)})\geq\cdots\geq e(\delta_{p(k)}).$$

Then the representation

$$\lambda(d):=\delta_{p(1)}\times\delta_{p(2)}\times\cdots\times\delta_{p(k)}$$

(called the standard module) has a unique irreducible quotient, which is denoted by

$$L(d).$$

The mapping $d\mapsto L(d)$ defines a bijection between $M(D)$ and the set of all equivalence classes of irreducible smooth representations of all the general linear groups over F. This is a formulation of the Langlands classification for general linear groups. We can describe $L(d)$ as a unique irreducible subrepresentation of

$$\delta_{p(k)} \times \delta_{p(k-1)} \times \cdots \times \delta_{p(1)}.$$

The formula for the contragredient is

$$L(\delta_1, \delta_2, \dots, \delta_k)\tilde{} \cong L(\tilde{\delta}_1, \tilde{\delta}_2, \dots, \tilde{\delta}_k).$$

A segment in $\mathcal{C}$ is a set of the form

$$[\rho, \nu^k\rho] = \{\rho, \nu\rho, \dots, \nu^k\rho\},$$

where $\rho \in \mathcal{C}, k \in \mathbb{Z}_{\geq 0}$. We shall denote a segment $[\nu^{k'}\rho, \nu^{k''}\rho]$ also by

$$[k', k'']^{(\rho)},$$

or simply by $[k', k'']$ when we fix ρ (or it is clear from the context which ρ is in question). We denote $[k, k]^{(\rho)}$ simply by $[k]^{(\rho)}$.

The set of all such segments is denoted by

$$\mathcal{S}.$$

For a segment $\Delta = [\rho, \nu^k\rho] = \{\rho, \nu\rho, \dots, \nu^k\rho\} \in \mathcal{S}$, the representation

$$\nu^k\rho \times \nu^{k-1}\rho \times \cdots \times \nu\rho \times \rho$$

contains a unique irreducible subrepresentation, which is denoted by

$$\delta(\Delta)$$

and a unique irreducible quotient, which is denoted by

$$\mathfrak{s}(\Delta).$$

The representation $\delta(\Delta)$ is an essentially square integrable representation modulo center. In this way we get a bijection between $\mathcal{S}$ and D. Further, $\mathfrak{s}(\Delta) = L(\rho, \nu\rho, \dots, \nu^k\rho)$ and

$$m^*(\delta([\rho, \nu^k\rho])) = \sum_{i=-1}^{k} \delta([\nu^{i+1}\rho, \nu^k\rho]) \otimes \delta([\rho, \nu^i\rho]), \tag{2.1}$$

$$m^*(\mathfrak{s}([\rho, \nu^k\rho])) = \sum_{i=-1}^{k} \mathfrak{s}([\rho, \nu^i\rho]) \otimes \mathfrak{s}([\nu^{i+1}\rho, \nu^k\rho]).$$

Using the above bijection between D and $\mathcal{S}$, we can express Langlands classification in terms of finite multisets $M(\mathcal{S})$ in $\mathcal{S}$:

$$L(\Delta_1, \dots, \Delta_k) := L(\delta(\Delta_1), \dots, \delta(\Delta_k)).$$

The Zelevinsky classification tells that

$$\mathfrak{s}(\Delta_{p(1)}) \times \mathfrak{s}(\Delta_{p(2)}) \times \cdots \times \mathfrak{s}(\Delta_{p(k)}),$$

has a unique irreducible subrepresentation, which is denoted by

$$Z(\Delta_1, \dots, \Delta_k)$$

(p is as above).

Since the ring R is a polynomial ring over D, the ring homomorphism $\pi \mapsto \pi^t$ on R determined by the requirement that $\delta(\Delta) \mapsto \mathfrak{s}(\Delta)$, $\Delta \in \mathcal{S}$, is uniquely determined by this condition. It is an involution, and is called the Zelevinsky involution. It is a special case of an involution which exists for any connected reductive group, called the Aubert-Schneider-Stuhler involution. This extension we shall also denote by $\pi \mapsto \pi^t$. A very important property of the Zelevinsky involution, as well as of the Aubert-Schneider-Stuhler involution, is that it carries irreducible representations to the irreducible ones ([2], Corollaire 3.9; also [49]).

The Zelevinsky involution ^{t} on the irreducible representations can be introduced by the requirement

$$L(a)^t = Z(a),$$

for any multisegment a. Then we define ^{t} on the multisegments by the requirement

$$Z(a)^t = Z(a^t).$$

For $\Delta = [\rho, \nu^k\rho] \in \mathcal{S}$, let

$$\Delta^- = [\rho, \nu^{k-1}\rho],$$

and for $d = (\Delta_1, \dots, \Delta_k) \in M(\mathcal{S})$ denote

$$d^- = (\Delta_1^-, \dots, \Delta_k^-).$$

Then the ring homomorphism $\mathcal{D} : R \to R$ determined by the requirement that $\mathfrak{s}(\Delta)$ goes to $\mathfrak{s}(\Delta) + \mathfrak{s}(\Delta^-)$ for all $\Delta \in \mathcal{S}$, is called the derivative. This is a positive mapping. Let $\pi \in R$ and $\mathcal{D}(\pi) = \sum \mathcal{D}(\pi)_n$, where $\mathcal{D}(\pi)_n$ is in the n-th grading group of R. If k is the lowest index such that $\mathcal{D}(\pi)_k \neq 0$, then $\mathcal{D}(\pi)_k$ is called the highest derivative of π, and denoted by h.d.(π). Obviously, the highest derivative is multiplicative (since R is an integral domain). Further

$$\text{h.d.}(Z(\Delta_1, \dots, \Delta_k)) = Z(\Delta_1^-, \dots, \Delta_k^-)$$

(see [71]).

We now very briefly recall basic notation for the classical p-adic groups. We follow [43]. Fix a Witt tower $V \in \mathcal{V}$ of symplectic of orthogonal vector spaces starting with an anisotropic space V_0 of the same type (see sections III.1 and III.2 of [30] for details). Consider the group of isometries of $V \in \mathcal{V}$, while in the case of odd-orthogonal groups one requires additionally that the determinants are 1. The group of split rank n will be denoted by S_n (for some other purposes a different indexing may be more convenient). For $0 \leq k \leq n$, one chooses a parabolic subgroup whose Levi factor is isomorphic to $GL(k,F) \times S_{n-k}$ (see [30], III.2)[10]. Then using parabolic induction one defines in a natural way multiplication

$$\pi \rtimes \sigma$$

of a representations π and σ of $GL(k,F)$ and S_{n-k} respectively.

We do not follow the case of split even orthogonal groups in this paper, although we expect that the results of this paper hold also in this case, with the same proofs (split even orthogonal groups are not connected, which requires some additional checkings).

Let F' be a quadratic extension of F, and denote by Θ the non-trivial element of the Galois group. In analogous way one defines the Witt tower of unitary spaces over F', starting with an anisotropic hermitian space V_0, and consider the isometry groups. One denotes by S_n the group of F-split rank n. Here multiplication $\rtimes$ is defined among representations of groups $GL(k,F')$ and S_{n-k}. Except in the last section, the classical groups that we consider in this paper are symplectic and orthogonal groups (introduced above), excluding split even orthogonal groups (what we have already mentioned). In the last section is commented the case of the unitary groups.

An irreducible representation of a classical group will be called weakly real if it is a subquotient of a representation of the form

$$\nu^{r_1}\rho_1 \times \dots \times \nu^{r_k}\rho_k \rtimes \sigma,$$

where $\rho_i \in \mathcal{C}$ are selfcontragredient, $r_i \in \mathbb{R}$ and σ is an irreducible cuspidal representation of a classical group.

The following theorems reduce the unitarizability problem for classical p-adic groups to the weakly real case (see [66]).

Theorem 2.1. *If π is an irreducible unitarizable representation of some S_q, then there exist an irreducible unitarizable representation θ of a general linear group and a weakly real irreducible unitarizable representation π' of some $S_{q'}$ such that*

$$\pi \cong \theta \rtimes \pi'.$$

Denote by $\mathcal{C}_u$ the set of all unitarizable classes in $\mathcal{C}$. For a set X of equivalence classes of irreducible representations, we denote by $\tilde{X} := \{\tilde{\tau}; \tau \in X\}$ (recall that $\tilde{\tau}$ denotes the contragredient of τ). Theorem 4.2 of [66] gives a more precise formulation of the above reduction:

[10]One can find in [60] matrix realizations of the symplectic and split odd-orthogonal groups. In a similar way one can make matrix realizations also for other orthogonal groups (and for unitary groups which are discussed a little bit later).

Theorem 2.2. *Let $\mathcal{C}'_u$ be a subset of $\mathcal{C}_u$ satisfying $\mathcal{C}'_u \cap \widetilde{\mathcal{C}'_u} = \emptyset$, such that $\mathcal{C}'_u \cup \widetilde{\mathcal{C}'_u}$ contains all $\rho \in \mathcal{C}_u$ which are not self contragredient. Denote*

$$\mathcal{C}' = \{\nu^\alpha \rho;\ \alpha \in \mathbb{R},\ \rho \in \mathcal{C}'_u\}.$$

Let π be an irreducible unitarizable representation of some S_q Then there exists an irreducible representation θ of a general linear group with support contained in $\mathcal{C}'$, and a weakly real irreducible representation π' of some $S_{n'}$ such that

$$\pi \cong \theta \rtimes \pi'.$$

Moreover, π determines such θ and π' up to an equivalence. Further, π is unitarizable (resp. Hermitian) if and only both θ and π' are unitarizable (resp. Hermitian).

The direct sum of Grothendieck groups $\mathcal{R}(S_n)$, $n \geq 0$, is denoted by $R(S)$. As in the case of general linear groups, one lifts $\rtimes$ to a mapping $R \times R(S) \to R(S)$ (again denoted by $\rtimes$). Factorization through $R \otimes R(S)$ is denoted by μ. In this way $R(S)$ becomes an R-module.

We denote by

$$s_{(k)}(\pi)$$

the Jacquet module of a representation π of S_n with respect to the parabolic subgroup $P_{(k)}$. If there exists $0 \leq k \leq n$ and an irreducible cuspidal representation σ of S_q, $q \leq n$, such that any irreducible sub quotient τ of $s_{(k)}(\pi)$ is isomorphic to $\theta_\tau \otimes \sigma$ for some representation θ_τ of a general linear group, then we shall denote $s_{(k)}(\pi)$ also by

$$s_{GL}(\pi).$$

Then σ is called a partial cuspidal support of π.

For an irreducible representation π of S_n, the sum of the semi simplifications of $s_{(k)}(\pi)$, $0 \leq k \leq n$, is denoted by

$$\mu^*(\pi) \in R \otimes R(S).$$

We extend μ^* additively to $\mu^* : R(S) \to R \otimes R(S)$. With this comultiplication, $R(S)$ becomes an R-comodule.

Further, $R \otimes R(S)$ is an $R \otimes R$-module in a natural way (the multiplication is denoted by $\rtimes$). Let $\sim : R \to R$ be the contragredient map and $\kappa : R \otimes R \to R \otimes R$, $\sum x_i \otimes y_i \mapsto \sum_i y_i \otimes x_i$. Denote

$$M^* = (m \otimes \mathrm{id}_R) \circ (\sim \otimes m^*) \circ \kappa \circ m^*.$$

Then ([60] and [43])

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma) \tag{2.2}$$

for $\pi \in R$ and $\sigma \in R(S)$ (or for admissible representations π and σ of $GL(n, F)$ and S_m respectively). A direct consequence of the formulas (2.2) and (2.1) is the following formula:

$$M^*(\delta([\nu^a\rho,\nu^c\rho])) \quad = \quad \sum_{s=a-1}^{c}\sum_{t=i}^{c}\delta([\nu^{-s}\tilde{\rho},\nu^{-a}\tilde{\rho}]) \;\times\; \delta([\nu^{t+1}\rho,\nu^{c}\rho]) \;\otimes\; \delta([\nu^{s+1}\rho,\nu^{t}\rho]).$$

Let π be a representation of some $GL(m,F)$. Then the sum of the irreducible subquotients of the form $*\otimes 1$ in $M^*(\pi)$ will be denoted by

$$M^*_{GL}(\pi)\otimes 1.$$

Let $m^*(\pi)=\sum x\otimes y$. Then easily follows that

$$M^*_{GL}(\pi)=\sum x\times\tilde{y}. \tag{2.3}$$

Let π be a sub quotient of $\rho_1\times\ldots\times\rho_l$ where ρ_i are irreducible cuspidal representations of general linear groups, and let σ be an irreducible cuspidal representations of S_q. Then the

$$\text{s.s.}(s_{GL}(\pi\rtimes\sigma))=M^*_{GL}(\pi)\otimes\sigma.$$

Further, the sum of the irreducible subquotients of the form $1\otimes *$ in $M^*(\tau)$ is

$$1\otimes\tau. \tag{2.4}$$

Now we shall recall of the Langlands classification for groups S_n ([52], [11], [29], [46] and [71]). Set

$$D_+=\{\delta\in D; e(\delta)>0\}.$$

Let T be the set of all equivalence classes of irreducible tempered representations of S_n, for all $n\geq 0$. For $t=((\delta_1,\delta_2,\ldots,\delta_k),\tau)\in M(D_+)\times T$ take a permutation p of $\{1,\ldots,k\}$ such that

$$\delta_{p(1)}\geq\delta_{p(2)}\geq\cdots\geq\delta_{p(k)}. \tag{2.5}$$

Then the representation

$$\lambda(t):=\delta_{p(1)}\times\delta_{p(2)}\times\cdots\times\delta_{p(k)}\rtimes\tau$$

has a unique irreducible quotient, which is denoted by

$$L(t).$$

The mapping

$$t\mapsto L(t)$$

defines a bijection from the set $M(D_+)\times T$ onto the set of all equivalence classes of the irreducible smooth representations of all S_n, $n\geq 0$. This is the Langlands classification for classical groups. The multiplicity of $L(t)$ in $\lambda(t)$ is one.

Let $t=((\delta_1,\delta_2,\ldots,\delta_k),\tau)\in M(D_+)\times T$ and suppose that a permutation p satisfies (2.5). Let $\delta_{p(i)}$ be a representation of $GL(n_i,F)$ and $L(t)$ a representation of S_n. Denote by

$$e_*(t)=(\underbrace{\delta_{p(1)},\ldots,\delta_{p(1)}}_{n_1\text{ times}},\ldots,\underbrace{\delta_{p(k)},\ldots,\delta_{p(k)}}_{n_k\text{ times}},\underbrace{0,\ldots,0}_{n'\text{ times}}),$$

where $n' = n - n_1 - \cdots - n_k$. Consider a partial ordering on $\mathbb{R}^n$ given by $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ if and only if

$$\textstyle\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_i, \quad 1 \leq j \leq n.$$

Suppose $t, t' \in M(D_+) \times T$ and $L(t')$ is a sub quotient of $\lambda(t)$. Then

$$\epsilon_*(t') \leq e_*(t), \text{ and the equality holds in the previous relation } \iff t' = t \tag{2.6}$$

(see section 6. of [59] for the symplectic groups - this holds in the same form for the other classical groups different from the split even orthogonal groups).

For $\Delta \in \mathcal{S}$ define $\mathfrak{c}(\Delta)$ to be $e(\delta(\Delta))$. Let

$$\mathcal{S}_+ = \{\Delta \in \mathcal{S}; \mathfrak{c}(\Delta) > 0\}.$$

In this way we can define in a natural way the Langlands classification $(a, \tau) \mapsto L(a; \tau)$ using $M(\mathcal{S}_+) \times T$ for the parameters.

Let τ and ω be irreducible representations of $GL(p, F)$ and S_q, respectively, and let π an admissible representation of S_{p+q}. Then a special case of the Frobenius reciprocity tells us

$$\mathrm{Hom}_{S_{p+q}}(\pi, \tau \rtimes \omega) \cong \mathrm{Hom}_{GL(p,F) \times S_q}(s_{(p)}(\pi), \tau \otimes \omega),$$

while the second second adjointness implies

$$\mathrm{Hom}_{S_{p+q}}(\tau \rtimes \omega, \pi) \cong \mathrm{Hom}_{GL(p,F) \times S_q}(\tilde{\tau} \otimes \omega, s_{(p)}(\pi)).$$

We could write down the above formulas for the parabolic subgroups which are not necessarily maximal.

3. On the irreducible sub quotients of $\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha+1}\rho \times \nu^{\alpha}\rho \rtimes \sigma$

Let ρ and σ be irreducible cuspidal representations of $GL(p, F)$ and S_q respectively, such that ρ is self contragredient. Then ρ is unitarizable (cuspidality implies that σ is unitarizable since the center of S_q is compact - more precisely, finite). Then

$$\nu^{\alpha_{\rho,\sigma}}\rho \rtimes \sigma$$

reduces for some $\alpha_{\rho,\sigma} \geq 0$. This reducibility point $\alpha_{\rho,\sigma}$ is unique by [52]. In this paper we shall assume that

$$\alpha_{\rho,\sigma} \in (1/2)\mathbb{Z}. \tag{3.1}$$

Actually, from the recent work of J. Arthur, C. Mœglin and J.-L. Waldspurger, this assumption is known to hold if $\mathrm{char}(F) = 0$ (Theorem 3.1.1 of [41] tells this for the quasi split case, while [42] extends it to the non-quasi split classical groups).

In the most of this paper we shall deal with the case

$$\alpha_{\rho,\sigma} > 0, \tag{3.2}$$

at least in the following several sections. We shall denote the reducibility point $\alpha_{\rho,\sigma}$ simply by

$$\alpha.$$

So $\alpha > 0$ and $\alpha \in \frac{1}{2}\mathbb{Z}$. We shall deal with irreducible sub quotients of

$$\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha+1}\rho \times \nu^{\alpha}\rho \rtimes \sigma.$$

The above representation has a unique irreducible subrepresentation, which is denoted by

$$\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho];\sigma) \ \ (n \geq 0).$$

This subrepresentation is square integrable and it is called a generalized Steinberg representation. We have

$$\mu^*\left(\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho];\sigma)\right) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+k}\rho];\sigma),$$

$$\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho];\sigma)\tilde{} \cong \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho];\tilde{\sigma}).$$

Further applying the Aubert-Schneider-Stuhler involution, we get

$$\mu^*\left(L(\nu^{\alpha+n}\rho.\ldots,\nu^{\alpha+1}\rho,\nu^{\alpha}\rho;\sigma)\right) = \\ \sum_{k=-1}^{n} L(\nu^{-(\alpha+n)}\rho,\ldots,\nu^{-(\alpha+k+2)}\rho,\nu^{-(\alpha+k+1)}\rho) \otimes L(\nu^{\alpha+k}\rho.\ldots,\nu^{\alpha+1}\rho,\nu^{\alpha}\rho;\sigma).$$

We say that a sequence of segments $\Delta_1,\ldots,\Delta_l$ is decreasing if $\mathfrak{c}(\Delta_1) \geq \cdots \geq \mathfrak{c}(\Delta_l)$.

Now we recall of Lemma 3.1 from [18] which we shall use several times in this paper:

Lemma 3.1. *Let $n \geq 1$. Fix an integer c' satisfying $0 \leq c' \leq n-1$. Let $\Delta_1,\ldots,\ \Delta_k$ be a sequence of decreasing mutually disjoint non-empty segments such that*

$$\Delta_1 \cup \ldots \cup\ \Delta_k = \{\nu^{\alpha+c'+1}\rho,\ldots,\nu^{\alpha+n-1}\rho,\nu^{\alpha+n}\rho\}.$$

Let $\Delta_{k+1},\ldots,\Delta_l$, $k<l$, be a sequence of decreasing mutually disjoint segments satisfying

$$\Delta_{k+1} \cup \cdots \cup\ \Delta_l = \{\nu^{\alpha}\rho,\nu^{\alpha+1}\rho,\ldots,\nu^{\alpha+c'}\rho\},$$

such that $\Delta_{k+1},\ldots,\Delta_{l-1}$ are non-empty. Let

$$a = (\Delta_1,\ldots,\Delta_{k-1}),$$

$$b = (\Delta_{k+2},\ldots,\Delta_{l-1}).$$

Then in $R(S)$ we have:

(1) *If $k+1<l$, then*

$$L(a+(\Delta_k)) \rtimes L((\Delta_{k+1})+b;\delta(\Delta_l;\sigma)) = \\ L(a+(\Delta_k,\Delta_{k+1})+b;\delta(\Delta_l;\sigma)) + L(a+(\Delta_k\cup\Delta_{k+1})+b;\delta(\Delta_l;\sigma)).$$

(2) *If $k+1=l$, then*

$$L(a+(\Delta_k))\rtimes\delta(\Delta_{k+1};\sigma)=L(a+(\Delta_k);\delta(\Delta_{k+1};\sigma))+L(a;\delta(\Delta_k\cup\Delta_{k+1};\sigma)).\quad\square$$

We assume

$$n\geq 1,$$

and consider irreducible subquotients of $\nu^{\alpha+n}\rho\times\nu^{\alpha+n-1}\rho\times\cdots\times\nu^{\alpha}\rho\rtimes\sigma$. Each irreducible subquotient can be written as

$$\gamma=L(\Delta_1,\dots,\Delta_k;\delta(\Delta_{k+1};\sigma))$$

for some $k\geq 0$, where $\Delta_1,\dots,\Delta_{k+1}$ is a sequence of decreasing mutually disjoint segments such that

$$\Delta_1\cup\ldots\cup\Delta_k\cup\Delta_{k+1}=\{\nu^{\alpha}\rho,\dots,\nu^{\alpha+n}\rho\},$$

and that $\Delta_1,\dots,\Delta_k$ are non-empty[11].

Remark 3.2. *Observe that*

$$\left(\nu^{\alpha+n}\rho\times\nu^{\alpha+n-1}\rho\times\ldots\times\nu^{\alpha+1}\rho\rtimes\delta(\nu^{\alpha}\rho;\sigma)\right)^t=\nu^{\alpha+n}\rho\times\nu^{\alpha+n-1}\rho\times\ldots\times\nu^{\alpha+1}\rho\rtimes L(\nu^{\alpha}\rho;\sigma).$$

Irreducible subquotients of $\nu^{\alpha+n}\rho\times\nu^{\alpha+n-1}\rho\times\ldots\times\nu^{\alpha+1}\rho\rtimes\delta(\nu^{\alpha}\rho;\sigma)$ satisfy $\Delta_{k+1}\neq\emptyset$, while irreducible sub quotients of $\nu^{\alpha+n}\rho\times\nu^{\alpha+n-1}\rho\times\ldots\times\nu^{\alpha+1}\rho\rtimes L(\nu^{\alpha}\rho;\sigma)$ satisfy $\Delta_{k+1}=\emptyset$. From this directly follows that the Aubert-Schneider-Stuhler involution is a bijection between the irreducible sub quotients for which $\Delta_{k+1}\neq\emptyset$ and the irreducible subquotients for which $\Delta_{k+1}=\emptyset$.

4. Key proposition

A bigger part of this paper we shall spend to prove the following

Proposition 4.1. *Let ρ and σ be irreducible cuspidal representations of $GL(p,F)$ and S_q respectively, such that ρ is self contragredient and that $\nu^{\alpha}\rho\rtimes\sigma$ reduces for some positive $\alpha\in\frac{1}{2}\mathbb{Z}$. Further, let γ be an irreducible subquotient of $\nu^{\alpha+n}\rho\times\nu^{\alpha+n-1}\rho\times\cdots\times\nu^{\alpha}\rho\rtimes\sigma$, different from*

$$L(\nu^{\alpha}\rho,\nu^{\alpha+1}\rho,\dots,\nu^{\alpha+n}\rho;\sigma)\ \text{and}\ \delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho];\sigma).$$

Then there exits an irreducible selfcontragredient unitarizable representation π of a general linear group with support in $[-\alpha-n,\alpha+n]^{(\rho)}$, such that the length of

$$\pi\rtimes\gamma$$

[11] It is easy to see that Langlands parameter of γ must be of above form. Namely, for the beginning, the tempered piece of the Langlands parameter must be square integrable (this follows from the fact that ρ is self contragredient and the fact that $\nu^{\alpha+n}\rho\times\cdots\times\nu^{\alpha}\rho\rtimes\sigma$ is a regular representations, i.e. all the Jacquet modules of it are multiplicity one representations). Further, one directly sees that this square integrable representation must be some $\delta(\Delta_{k+1};\sigma)$. Now considering the support, and using the fact that $\mathfrak{c}(\Delta_i)>0$, we get that the Langlands parameter of γ must be of the above form.

is at least 5, and that

$$5 \cdot \pi \otimes \gamma \not\leq \mu^*(\pi \rtimes \gamma).$$

We shall consider γ as in the proposition, and write $\gamma = L(\Delta_1, \dots, \Delta_k; \delta(\Delta_{k+1}; \sigma))$ as in the previous section (recall, $\Delta_1, \dots, \Delta_k$ are non-empty mutually disjoint decreasing segments, and additionally Δ_k, Δ_{k+1} are decreasing if $\Delta_{k+1} \neq \emptyset$). Since γ is different from $\delta([\nu^\alpha \rho, \nu^{\alpha+n}\rho]; \sigma)$, we have

$$k \geq 1,$$

and since γ is different from $L(\nu^\alpha \rho, \nu^{\alpha+1}\rho, \dots, \nu^{\alpha+n}\rho; \sigma)$ we have

$$\Delta_{k+1} \neq \emptyset \text{ or } \Delta_{k+1} = \emptyset \text{ and card } (\Delta_i) > 1 \text{ for some } 1 \leq i \leq k. \tag{4.1}$$

We shall first study γ for which $\Delta_{k+1} = \emptyset$. We split our proof of the case Δ_{k+1} of the above proposition into two stages. Each of them is one of the following two sections.

5. The case of card$(\Delta_k) > 1$ and $\Delta_{k+1} = \emptyset$

We continue with the notation introduced in the previous section. In this section we assume card$(\Delta_k) > 1$ and $\Delta_{k+1} = \emptyset$. Denote

$$\Delta_k = [\nu^\alpha \rho, \nu^c \rho],$$

$$\Delta_u = [\nu^{-\alpha} \rho, \nu^\alpha \rho],$$

$$\Delta = \Delta_k \cup \Delta_u = [\nu^{-\alpha} \rho, \nu^c \rho].$$

Then

$$\alpha < c.$$

Denote

$$a = (\Delta_1, \Delta_2, \dots, \Delta_{k-1}),$$

$$a_1 = (\Delta_1, \Delta_2, \dots, \Delta_{k-2}), \quad \text{if } a \neq \emptyset.$$

For $L(a, \Delta_k) \rtimes \sigma$ in the Grothendieck group we have

$$L(a, \Delta_k) \rtimes \sigma = L(a + (\Delta_k); \sigma) + L(a; \delta(\Delta_k; \sigma)). \tag{5.1}$$

We shall denote $L(a + (\Delta_k); \sigma)$ below simply by $L(a, \Delta_k; \sigma)$.

Our first goal in this sections is to prove:

Lemma 5.1. *The representation $\delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma)$ is of length at least 5 if card$(\Delta_k) > 1$.*

We know from Theorem 13.2 of [62] that $\delta(\Delta_u) \rtimes \sigma$ reduces. Frobenius reciprocity implies that it reduces into two non-equivalent tempered irreducible pieces. Denote them by $\tau((\Delta_u)_+;\sigma)$ and $\tau((\Delta_u)_-;\sigma)$. Now Proposition 5.3 of [66] implies

$$L(a,\Delta_k;\tau((\Delta_u)_+;\sigma)), \quad L(a,\Delta_k;\tau((\Delta_u)_-;\sigma)) \quad \leq \quad \delta(\Delta_u) \rtimes L(a,\Delta_k;\sigma). \tag{5.2}$$

Therefore, $\delta(\Delta_u) \times L(a,\Delta_k;\sigma)$ has length at least two.

Now we shall recall of a simple Lemma 4.2 from [18]:

Lemma 5.2. *If $|\Delta_k| > 1$, then we have*

$$L(a+(\Delta)) \times \nu^\alpha\rho \leq \delta(\Delta_u) \times L(a+(\Delta_k)),$$

and the representation on the left hand side is irreducible. □

The above lemma now implies

$$L(a,\Delta) \times \nu^\alpha\rho \rtimes \sigma \leq \delta(\Delta_u) \times L(a,\Delta_k) \rtimes \sigma =$$
$$\delta(\Delta_u) \rtimes L(a,\Delta_k;\sigma) + \delta(\Delta_u) \rtimes L(a;\delta(\Delta_k;\sigma)).$$

Since by Proposition 4.2 of [66],

$$L(a,\Delta,\nu^\alpha\rho;\sigma)$$

is a sub quotient of $L(a,\Delta) \times \nu^\alpha\rho \rtimes \sigma$, it is also a sub quotient of

$$\delta(\Delta_u) \rtimes L(a,\Delta_k;\sigma) + \delta(\Delta_u) \rtimes L(a;\delta(\Delta_k;\sigma)).$$

Suppose

$$L(a,\Delta,\nu^\alpha\rho;\sigma) \leq \delta(\Delta_u) \rtimes L(a;\delta(\Delta_k;\sigma)).$$

Then

$$L(a,\Delta,\nu^\alpha\rho;\sigma) \leq \lambda(a) \rtimes \tau = \lambda(a,\tau),$$

where τ is some irreducible subquotient of $\delta(\Delta_u) \rtimes \delta(\Delta_k;\sigma)$. Now (2.6) implies that this is not possible (since $\alpha > 0$). Therefore

$$L(a,\Delta,\nu^\alpha\rho;\sigma) \leq \delta(\Delta_u) \rtimes L(a,\Delta_k;\sigma). \tag{5.3}$$

Now we know that $\delta(\Delta_u) \times L(a,\Delta_k;\sigma)$ has length at least three, since obviously the representation $L(a,\Delta,\nu^\alpha\rho;\sigma)$ is not in $\{L(a,\Delta_k;\tau((\Delta_u)_+;\sigma)), L(a,\Delta_k;\tau((\Delta_u)_-;\sigma))\}$ (consider the tempered parts of the Langlands parameters).

From Theorem in the introduction of [63] (see also [35]) we know that $\delta(\Delta) \rtimes \sigma$ has two nonequivalent irreducible sub representations, and that they are square integrable. They are denoted there by $\delta(\Delta_+;\sigma)$ and $\delta(\Delta_-;\sigma)$. This and Proposition 5.3 of [66] imply

$$L(a,\nu^\alpha\rho;\delta(\Delta_\pm;\sigma)) \leq L(a,\nu^\alpha\rho) \rtimes \delta(\Delta_\pm;\sigma) \leq L(a) \times \nu^\alpha\rho \rtimes \delta(\Delta) \rtimes \sigma \leq$$
$$L(a) \times \delta(\Delta_k) \times \delta(\Delta_u) \rtimes \sigma.$$

Therefore

$$L(a,\nu^\alpha\rho;\delta(\Delta_\pm;\sigma)) \leq L(a) \times \delta(\Delta_k) \times \delta(\Delta_u) \rtimes \sigma.$$

If $a=\emptyset$, then formally

$$L(a,\nu^\alpha\rho;\delta(\Delta_\pm;\sigma))\leq L(a,\Delta_k)\times\delta(\Delta_u)\rtimes\sigma \tag{5.4}$$

since then $L(a)\times\delta(\Delta_k)=L(\Delta_k)=L(a,\Delta_k)$.

Now we shall show that (5.4) holds also if $a\neq\emptyset$. In this case we have

$$L(a,\nu^\alpha\rho;\delta(\Delta_\pm;\sigma))\leq L(a)\times\delta(\Delta_k)\times\delta(\Delta_u)\rtimes\sigma=$$

$$L(a,\Delta_k)\times\delta(\Delta_u)\rtimes\sigma+L(a_1,\Delta_{k-1}\cup\Delta_k)\times\delta(\Delta_u)\rtimes\sigma$$

(here we have used that $L(a)\times\delta(\Delta_k)=L(a,\Delta_k)+L(a_1,\Delta_{k-1}\cup\Delta_k)$, which is easy to prove). Suppose

$$L(a,\nu^\alpha\rho;\delta(\Delta_\pm;\sigma))\leq L(a_1,\Delta_{k-1}\cup\Delta_k)\times\delta(\Delta_u)\rtimes\sigma.$$

Observe that $L(a,\nu^\alpha\rho;\delta(\Delta_\pm;\sigma))\hookrightarrow L(\tilde a,\nu^{-\alpha}\rho)\rtimes\delta(\Delta_\pm;\sigma)\hookrightarrow L(\tilde a,\nu^{-\alpha}\rho)\times\delta(\Delta)\rtimes\sigma$. This implies that the Langlands quotient $L(a,\nu^\alpha\rho;\delta(\Delta_\pm;\sigma))$ has in the GL-type Jacquet module an irreducible sub quotient

$$\beta$$

which has exponent c in its Jacquet module, but has not $c+1$.

From the case of general linear groups we know $L(a_1,\Delta_{k-1}\cup\Delta_k)\leq L(a_1)\rtimes\delta(\Delta_{k-1}\cup\Delta_k)$ (see for example Proposition A.4 of [56]). Now application of tensoring, parabolic induction and Jacquet modules imply

$$s_{GL}(L(a_1,\Delta_{k-1}\cup\Delta_k)\times\delta(\Delta_u)\rtimes\sigma)\leq s_{GL}(L(a_1)\times\delta(\Delta_{k-1}\cup\Delta_k)\times\delta(\Delta_u)\rtimes\sigma).$$

Therefore, (on the level of semisimplifications) we have

$$s_{GL}(L(a_1,\Delta_{k-1}\cup\Delta_k)\times\delta(\Delta_u)\rtimes\sigma)\leq$$
$$M^*_{GL}((L(a_1))\times M^*_{GL}(\delta(\Delta_{k-1}\cup\Delta_k))\times M^*_{GL}(\delta(\Delta_u))\rtimes(1\otimes\sigma).$$

Now we shall examine how we can can get exponents c and $c+1$ in the support of the left hand side tensor factor of the last line. Since the left hand side is a product of three terms, we shall analyze each of them. Recall $\Delta_u=[\nu^{-\alpha}\rho,\nu^\alpha\rho]$. Since $0<\alpha<c$, now (2.3) and (2.1) imply that neither $\nu^c\rho$ nor $\nu^{c+1}\rho$ is in the support of $M^*_{GL}(\delta(\Delta_u))$.

Further, recall that $a=(\Delta_1,\dots,\Delta_{k-1})$ and $a_1=(\Delta_1,\dots,\Delta_{k-2})$. Since $\Delta_1\cup\cdots\cup\Delta_{k-1}=[\nu^{c+1}\rho,\nu^{\alpha+n}\rho]$, we have $\Delta_1\cup\cdots\cup\Delta_{k-2}=[\nu^t\rho,\nu^{\alpha+n}\rho]$ for some $t\geq c+2$ (clearly, $c+2>0$). This implies that no one of $\nu^{\pm c}\rho$ or $\nu^{\pm(c+1)}\rho$ is in the support of $L(a_1)$ or $L(a_1)\tilde{}$. This implies that neither $\nu^c\rho$ nor $\nu^{c+1}\rho$ is in the support of $M^*_{GL}(L(a_1))$ (use the fact that $L(a_1)\leq\prod_{i=1}^s\tau_i$ for some τ_i in the support of $L(a_i)$, and the formula that $M^*_{GL}(\tau_i)=\tau_i+\tilde\tau_i$ since τ_i are cuspidal).

Since the exponent c shows up in the support of β, it must show up in

$$M^*_{GL}(\delta(\Delta_{k-1}\cup\Delta_k))=M^*_{GL}(\delta([\alpha,d]))=\sum_{x=\alpha-1}^{d}\delta([-x,-\alpha])\times\delta([x+1,d]),$$

where $c+1 \leq d$. Now the above formula for $M^*_{GL}(\delta([\alpha, d]))$ implies that whenever we have in the support c, we must have it in a segment which ends with d, and therefore, we must have in the support also $c+1$. Therefore, β cannot be a sub quotient of $L(a_1, \Delta_{k-1} \cup \Delta_k) \times \delta(\Delta_u) \rtimes \sigma$. This contradiction implies

$$L(a, \nu^\alpha\rho; \delta(\Delta_\pm; \sigma)) \not\leq L(a_1, \Delta_{k-1} \cup \Delta_k) \times \delta(\Delta_u) \rtimes \sigma.$$

Now the following relation (which we have already observed above)

$$L(a, \nu^\alpha\rho; \delta(\Delta_\pm; \sigma)) \leq L(a, \Delta_k) \times \delta(\Delta_u) \rtimes \sigma + L(a_1, \Delta_{k-1} \cup \Delta_k) \times \delta(\Delta_u) \rtimes \sigma$$

implies

$$L(a, \nu^\alpha\rho; \delta(\Delta_\pm; \sigma)) \leq L(a, \Delta_k) \times \delta(\Delta_u) \rtimes \sigma.$$

Therefore (in both cases) we have

$$L(a, \nu^\alpha\rho; \delta(\Delta_\pm; \sigma)) \leq L(a, \Delta_k) \times \delta(\Delta_u) \rtimes \sigma =$$

$$\delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma) + \delta(\Delta_u) \rtimes L(a; \delta(\Delta_k; \sigma)).$$

Suppose

$$L(a, \nu^\alpha\rho; \delta(\Delta_\pm; \sigma)) \leq \delta(\Delta_u) \rtimes L(a; \delta(\Delta_k; \sigma)).$$

One directly sees that in the GL-type Jacquet module of the left hand side we have an irreducible term in whose support appears exponent $-\alpha$ two times.

Observe $\delta(\Delta_u) \rtimes L(a; \delta(\Delta_k; \sigma)) \leq \delta(\Delta_u) \times L(a) \rtimes \delta(\Delta_k; \sigma)$. For $\delta(\Delta_u) \times L(a) \rtimes \delta(\Delta_k; \sigma)$, the exponent $-\alpha$ which cannot come neither from $M^*_{GL}(L(a))$ nor from $\mu^*(\delta(\Delta_k; \sigma))$. Therefore, it must come from

$$M^*_{GL}(\delta(\Delta_u)) = \sum_{x=-\alpha-1}^{\alpha} \delta([-x, \alpha]) \times \delta([x+1, \alpha]).$$

This implies that we can have the exponent $-\alpha$ at most once in the GL-part of Jacquet module of the right hand side. This contradiction implies that the inequality which we have supposed is false. This implies

$$L(a, \nu^\alpha\rho; \delta(\Delta_\pm; \sigma)) \leq \delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma).$$

Therefore, $\delta(\Delta_u) \times L(a, \Delta_k; \sigma)$ has length at least five. This completes the proof of the lemma.

The second aim of this section is to prove the following

Lemma 5.3. *The multiplicity of*

$$\delta(\Delta_u) \otimes L(a, \Delta_k; \sigma)$$

in

$$\mu^*(\delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma))$$

is at most 4 if card$(\Delta_k) > 1$.

Proof. Denote $\beta := \delta(\Delta_u) \otimes L(a, \Delta_k; \sigma)$. Recall

$$M^*(\delta(\Delta_u)) = M^*(\delta([-\alpha, \alpha])) = \sum_{x=-\alpha-1}^{\alpha} \sum_{y=x}^{\alpha} \delta([-x, \alpha]) \times \delta([y+1, \alpha]) \otimes \delta([x+1, y])$$

Now if we take from $\mu^*(L(a, \Delta_k; \sigma))$ the term $1 \otimes L(a, \Delta_k; \sigma)$, to get β for a sub quotient we need to take from $M^*(\delta(\Delta_u))$ the term $\delta(\Delta_u) \otimes 1$, which shows up there two times. This gives multiplicity two of β.

Now we consider terms from $\mu^*(L(a, \Delta_k; \sigma))$ different from $1 \otimes L(a, \Delta_k; \sigma)$ which can give β after multiplication with a term from $M^*(\delta(\Delta_u))$ (then a term from $M^*(\delta(\Delta_u))$ that can give β for a sub quotient is obviously different from $\delta(\Delta_u) \otimes 1$, which implies that we have $\nu^\alpha \rho$ in the support of the left hand side tensor factor). The above formula for $M^*(\delta(\Delta_u))$ and the set of possible factors of $L(a, \Delta_k; \sigma)$ (which is $\nu^{\pm\alpha}\rho, \nu^{\pm(\alpha+1)}, \dots$) imply that we need to have $\nu^{-\alpha}\rho$ on the left hand side of the tensor product of that term from $\mu^*(L(a, \Delta_k; \sigma))$. For such a term from $\mu^*(L(a, \Delta_k; \sigma))$, considering the support we see that we have two possible terms from $M^*(\delta(\Delta_u))$. They are $\delta([-\alpha+1, \alpha]) \otimes [-\alpha]$ and $\delta([-\alpha+1, \alpha]) \otimes [\alpha]$. Each of them will give multiplicity at most one (use the fact that here on the left and right hand side of $\otimes$ we are in the regular situation). ∎

6. The case of $\text{card}(\Delta_k) = 1$ and $\Delta_{k+1} = \emptyset$

We continue with the notation introduced in the section 4. In this section we assume $\text{card}(\Delta_k) = 1$ and $\Delta_{k+1} = \emptyset$. As we already noted in (4.1), we consider the case when $\text{card}(\Delta_i) > 1$ for some i. Denote maximal such index by k_0. Clearly,

$$k_0 < k.$$

Write

$$\Delta_{k_0} = [\nu^{\alpha+k-k_0}\rho, \nu^c\rho] = [\nu^{\alpha'}\rho, \nu^c\rho],$$
$$\Delta_u = [\nu^{-\alpha'}\rho, \nu^{\alpha'}\rho],$$
$$\Delta = [\nu^{-\alpha'}\rho, \nu^c\rho],$$
$$\Delta_1 = [\alpha, \alpha' - 1],$$
$$b = [\alpha, \alpha' - 1]^t = ([\alpha], [\alpha+1] \dots, [\alpha'-1]) \neq \emptyset.$$

Then

$$\alpha' < c.$$

Let

$$a = (\Delta_1, \Delta_2, \dots, \Delta_{k_0-1}),$$
$$a_1 = (\Delta_1, \Delta_2, \dots, \Delta_{k_0-2}), \quad \text{if } a \neq \emptyset.$$

Then

$$(\Delta_1, \dots, \Delta_k) = (a, \Delta_{k_0}, b).$$

We shall study $L(a,\Delta_{k_0},b)\rtimes\sigma$. The previous lemma implies that in the Grothendieck group we have

$$L(a,\Delta_{k_0},b)\rtimes\sigma = L(a,\Delta_{k_0},b;\sigma)+L(a;\Delta_{k_0},\nu^{\alpha'-1},\dots,\nu^{\alpha+1};\delta(\nu^{\alpha};\sigma)). \tag{6.1}$$

Our first goal in this section is to prove the following

Lemma 6.1. *Then length of the representation $\delta(\Delta_u)\rtimes L(a,\Delta_{k_0},b;\sigma)$ is at least 5 if $k_0<k$ and card$(\Delta_{k_0})>1$.*

First we get that we have two non-equivalent sub quotients

$$L(a,\Delta_{k_0},b;\tau((\Delta_u)_{\pm};\sigma))\le\delta(\Delta_u)\times L(a,\Delta_{k_0},b;\sigma) \tag{6.2}$$

in the same way as in the previous section. Therefore, the length is at least two.

Now we shall prove the following simple

Lemma 6.2. *If $|\Delta_k|=1$, then we have*

$$L(a,\Delta_{k_0}\cup\Delta_u,b,\nu^{\alpha'}\rho)\le\delta(\Delta_u)\times L(a,\Delta_{k_0},b).$$

Proof. Since in general $L(\Delta_1',\Delta_2',\dots,\Delta_m')^t=Z(\Delta_1',\Delta_2',\dots,\Delta_m')$, it is enough to prove the lemma for the Zelevinsky classification.

The highest (non-trivial) derivative of $\mathfrak{s}(\Delta_u)\times Z(a,\Delta_{k_0},b)$ is $\mathfrak{s}(\Delta_u^-)\times Z(a^-,\Delta_{k_0}^-)$. One can easily see that one subquotient of the last representation is $Z(a^-,\Delta^-)$. Therefore, there must exist an irreducible subquotient of $\mathfrak{s}(\Delta_u)\times Z(a,\Delta_{k_0},b)$ whose highest derivative is $Z(a^-,\Delta^-)$. The support and highest derivative completely determine the irreducible representation. One directly sees that this representation is $Z(a,\Delta_{k_0}\cup\Delta_u,b,\nu^{\alpha'}\rho)$. The proof is now complete □

The above lemma implies

$$L(a,\Delta_{k_0}\cup\Delta_u,b,\nu^{\alpha'}\rho;\sigma)\le L(a,\Delta_{k_0}\cup\Delta_u,b,\nu^{\alpha'}\rho)\rtimes\sigma\le\delta(\Delta_u)\times L(a,\Delta_{k_0},b)\rtimes\sigma.$$

By Lemma 3.1 we have for the right hand side $\delta(\Delta_u)\times L(a,\Delta_{k_0},b)\rtimes\sigma=$

$$\delta(\Delta_u)\times L(a,\Delta_{k_0},b;\sigma)+\delta(\Delta_u)\times L(a;\Delta_{k_0},\nu^{\alpha'-1},\dots,\nu^{\alpha+1};\delta(\nu^{\alpha};\sigma)).$$

This implies $L(a,\Delta_{k_0}\cup\Delta_u,b,\nu^{\alpha'}\rho;\sigma)\le$

$$\delta(\Delta_u)\times L(a,\Delta_{k_0},b;\sigma)+\delta(\Delta_u)\times L(a;\Delta_{k_0},\nu^{\alpha'-1},\dots,\nu^{\alpha+1};\delta(\nu^{\alpha};\sigma)).$$

Suppose

$$L(a,\Delta_{k_0}\cup\Delta_u,b,\nu^{\alpha'}\rho;\sigma)\le\delta(\Delta_u)\times L(a;\Delta_{k_0},\nu^{\alpha'-1},\dots,\nu^{\alpha+1};\delta(\nu^{\alpha};\sigma)).$$

Using the properties of the irreducible sub quotients of the standard modules in the Langlands classification, we now conclude in the same way as in the last section that this cannot be the case (the sum of all exponents on the left hand side which are not coming from the tempered representation of the classical group is the same as the sum of exponents of cuspidal representations which show up in the segments of a, in Δ_{k_0}, and $\alpha'-1,\alpha'-2,\dots,\alpha'+1,\alpha'$, while the corresponding sum of the standard modules which come from the right hand side is the sum of exponents of cuspidal representations which show up in the segments of a, in Δ_{k_0}, and $\alpha'-1,\alpha'-2,\dots,\alpha'+1$, which is strictly smaller (for $\alpha>0$) then we have on the left hand side).

This implies

$$L(a,\Delta_{k_0}\cup\Delta_u,b,\nu^{\alpha'}\rho;\sigma)\leq\delta(\Delta_u)\times L(a,\Delta_{k_0},b;\sigma). \tag{6.3}$$

Therefore, $\delta(\Delta_u)\times L(a,\Delta_{k_0},b;\sigma)$ has length at least three.

The following (a little bit longer) step will be to show that $\delta(\Delta_u)\times L(a,\Delta_{k_0},b;\sigma)$ has two additional irreducible sub quotients.

We start this step with an observe that

$$\begin{aligned}L(a,\nu^{\alpha'}\rho,b;\delta(\Delta_\pm;\sigma))\leq&\\ &L(a)\times L(b)\times\nu^{\alpha'}\rho\times\delta(\Delta)\rtimes\sigma\leq L(a)\times L(b)\times\delta(\Delta_{k_0})\times\delta(\Delta_u)\rtimes\sigma\end{aligned} \tag{6.4}$$

If $a=\emptyset$, then formally

$$L(a,b,\nu^{\alpha'}\rho;\delta(\Delta_\pm;\sigma))\leq L(a,\Delta_{k_0})\times L(b)\times\delta(\Delta_u)\rtimes\sigma$$

since $L(a)\times\delta(\Delta_{k_0})=L(\Delta_{k_0})=L(a,\Delta_{k_0})$.

We shall now show that the above inequality holds also if $a\neq\emptyset$. Then staring with (6.4) we get

$$\begin{gathered}L(a,b,\nu^{\alpha'}\rho;\delta(\Delta_\pm;\sigma))\leq L(a)\times\delta(\Delta_{k_0})\times L(b)\times\delta(\Delta_u)\rtimes\sigma=\\ L(a,\Delta_{k_0})\times L(b)\times\delta(\Delta_u)\rtimes\sigma+L(a_1,\Delta_{k_0-1}\cup\Delta_{k_0})\times L(b)\times\delta(\Delta_u)\rtimes\sigma.\end{gathered}$$

Suppose

$$L(a,b,\nu^{\alpha'}\rho;\delta(\Delta_\pm;\sigma))\leq L(a_1,\Delta_{k_0-1}\cup\Delta_{k_0})\times L(b)\times\delta(\Delta_u)\rtimes\sigma.$$

Observe that $L(a,\nu^{\alpha'}\rho,b;\delta(\Delta_\pm;\sigma))\hookrightarrow L(\tilde a,\nu^{-\alpha'}\rho,\tilde b)\rtimes\delta(\Delta_\pm;\sigma)\hookrightarrow L(\tilde a,\nu^{-\alpha'}\rho,\tilde b)\rtimes\delta(\Delta)\times\sigma$.

This implies that the Langlands quotient has in the GL-type Jacquet module an irreducible sub quotient which has exponent c in its Jacquet module, but does not have $c+1$.

Observe that (on the level of semisimplifications) we have

$$\begin{gathered}s_{GL}(L(a_1,\Delta_{k_0-1}\cup\Delta_{k_0})\times L(b)\times\delta(\Delta_u)\rtimes\sigma)\leq s_{GL}(L(a_1)\times\delta(\Delta_{k_0-1}\cup\Delta_{k_0})\times L(b)\times\delta(\Delta_u)\rtimes\sigma)=\\ M^*_{GL}(L(a_1))\times M^*_{GL}(\delta(\Delta_{k_0-1}\cup\Delta_{k_0}))\times M^*_{GL}(L(b))\times M^*_{GL}(\delta(\Delta_u))\rtimes(1\otimes\sigma).\end{gathered}$$

We cannot get any one of exponents c and $c+1$ from $M^*_{GL}((L(a_1))$ or $M^*_{GL}(\delta(\Delta_u))$ or $M^*_{GL}(L(b))$ (consider support as in the previous section). Therefore, it must come from

$$M^*_{GL}(\delta(\Delta_{k_0-1}\cup\Delta_{k_0}))=M^*_{GL}(\delta([\alpha',d]))=\sum_{x=\alpha'-1}^{d}\delta([-x,-\alpha'])\times\delta([x+1,d]),$$

where $c+1 \leq d$. The above formula for $M^*_{GL}(\delta([\alpha', d]))$ implies that whenever we have in the support c, it must come from a segment which ends with d, and therefore, we must have in the support also $c+1$. Therefore, we cannot have only c. In this way we have proved that (in both cases)

$$L(a, \nu^{\alpha'}\rho, b; \delta(\Delta_\pm; \sigma)) \leq L(a, \Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \rtimes \sigma =$$
$$L(a, \Delta_{k_0}, b) \times \delta(\Delta_u) \rtimes \sigma + L(a, [\alpha'-1, c]) \times L([\alpha, \alpha'-2]^t) \times \delta(\Delta_u) \rtimes \sigma.$$

Suppose

$$L(a, \nu^{\alpha'}\rho, b; \delta(\Delta_\pm; \sigma)) \leq L(a, [\alpha'-1, c]) \times L([\alpha, \alpha'-2]^t) \times \delta(\Delta_u) \rtimes \sigma.$$

Observe that

$$L(a, \nu^{\alpha'}\rho, b; \delta(\Delta_\pm; \sigma)) \hookrightarrow \delta(\tilde{\Delta}_1) \times \ldots \times \delta(\tilde{\Delta}_{k_0-1}) \times \nu^{-\alpha'}\rho \times \nu^{-\alpha'+1}\rho \times \ldots \times \nu^{-\alpha}\rho \rtimes \delta(\Delta_\pm; \sigma)),$$

which implies (because of unique irreducible subrepresentation of the right hand side)

$$L(a, \nu^{\alpha'}\rho, b; \delta(\Delta_\pm; \sigma)) \hookrightarrow L(\tilde{a}) \times \delta([-\alpha', -\alpha])^t \rtimes \delta([-\alpha', c]_\pm; \sigma)$$
$$\hookrightarrow L(\tilde{a}) \times \delta([-\alpha', -\alpha])^t \times \delta([-\alpha', c]) \rtimes \sigma.$$

Therefore, we have in the Jacquet module of the left hand side the irreducible representation

$$L(\tilde{a}) \otimes \delta([-\alpha', -\alpha])^t \times \delta([-\alpha', c]) \otimes \sigma.$$

Now we shall examine how we can get this from

$$\mu^*(L(a) \times \delta([\alpha'-1, c]) \times L([\alpha, \alpha'-2]^t) \times \delta(\Delta_u) \rtimes \sigma)$$

a term of the form $\beta \otimes \gamma$ such that the support of β is the same as of $\tilde{a}$. Grading and disjointness of supports "up to a contragredient" imply that we need to take β from $M^*(L(a))$ (we must take $L(\tilde{a}) \otimes 1$). This implies (using transitivity of Jacquet modules) that we need to have

$$\delta([-\alpha', -\alpha])^t \times \delta([-\alpha', c]) \otimes \sigma \leq \mu^*(\delta([\alpha'-1, c]) \times L([\alpha, \alpha'-2]^t) \times \delta(\Delta_u) \rtimes \sigma),$$

which implies

$$\delta([-\alpha', -\alpha])^t \times \delta([-\alpha', c]) \otimes \sigma \leq M^*_{GL}(\delta([\alpha'-1, c]) \times L([\alpha, \alpha'-2]^t) \times \delta(\Delta_u)) \otimes \sigma.$$

Observe that in the multisegment that represents the left hand side, we have $[-\alpha']$. In particular, we have a segment which ends with $-\alpha'$.

Such a segment (regarding ending at $-\alpha'$) we cannot get from $M^*_{GL}(L([\alpha, \alpha'-2]^t))$ (because of the support). Neither we can get it from $M^*_{GL}(\delta(\Delta_u))$ because of the formula:

$$M^*_{GL}(\delta(\Delta_u)) = \sum_{x=-\alpha'-1}^{\alpha'} \delta([-x, \alpha']) \times \delta([x+1, \alpha']).$$

The only possibility is $M^*_{GL}(\delta([\alpha'-1, c]))$. But segments coming from this term end with c or $-\alpha'+1$. So we cannot get $-\alpha'$ for end.

Therefore, we have got a contradiction.

This implies

$$L(a,\nu^{\alpha'}\rho,b;\delta(\Delta_\pm;\sigma)) \leq L(a,\Delta_{k_0},b)\times\delta(\Delta_u)\rtimes\sigma.$$
$$=\delta(\Delta_u)\rtimes L(a,\Delta_{k_0},b;\sigma)+\delta(\Delta_u)\rtimes L(a,\Delta_{k_0},[\alpha+1,\alpha'-1]^t;\delta([\alpha];\sigma)).$$

Suppose

$$L(a,\nu^{\alpha'}\rho,b;\delta(\Delta_\pm;\sigma)) \leq \delta(\Delta_u)\rtimes L(a,\Delta_{k_0},[\alpha+1,\alpha'-1]^t;\delta([\alpha];\sigma)).$$

One directly sees (using the Frobenius reciprocity) that in the GL-type Jacquet module of the left hand side we have an irreducible term in whose support appears exponent $-\alpha$ two times.

This cannot happen on the right hand side. To see this, observe that the right hand side is

$$\leq \delta(\Delta_u)\times L(a,\Delta_{k_0},[\alpha+1,\alpha'-1]^t)\rtimes\delta([\alpha];\sigma)).$$

Observe that we cannot get $-\alpha$ from $L(a,\Delta_{k_0},[\alpha+1,\alpha'-1]^t)$ (consider support, and its contragredient). We cannot get it from $\delta([\alpha];\sigma))$ (since $\mu^*(\delta([\alpha];\sigma))=1\otimes\delta([\alpha];\sigma)+[\alpha]\otimes\sigma$). From the formula for $M^*_{GL}(\delta(\Delta_u)))$ we see that we can get $-\alpha$ at most once (since it is negative).

Therefore, this inequality cannot hold. This implies

$$L(a,\nu^{\alpha'}\rho,b;\delta(\Delta_\pm;\sigma)) \leq \delta(\Delta_u)\rtimes L(a,\Delta_{k_0},b;\sigma). \tag{6.5}$$

Therefore, $\delta(\Delta_u)\rtimes L(a,\Delta_{k_0},b;\sigma)$ has length at least five. The proof of the lemma is now complete.

Our second goal in this section is to prove

Lemma 6.3. *The multiplicity of*

$$\delta(\Delta_u)\otimes L(a,\Delta_{k_0},b;\sigma)$$

in

$$\mu^*(\delta(\Delta_u)\rtimes L(a,\Delta_{k_0},b;\sigma))$$

is at most 4 if $k_0<k$ and card$(\Delta_{k_0})>1$.

Proof. Denote

$$\beta:=\delta(\Delta_u)\otimes L(a,\Delta_{k_0},b;\sigma).$$

If we take from $\mu^*(L(a,\Delta_{k_0},b;\sigma))$ the term $1\otimes L(a,\Delta_{k_0},b;\sigma)$, to get β for a sub quotient, we need to take from $M^*(\delta(\Delta_u))$ the term $\delta(\Delta_u)\otimes 1$ (we can take it two times - see the above formula for $M^*(\delta(\Delta_u))$). In this way we get multiplicity two.

Now we consider in $\mu^*(L(a,\Delta_{k_0},b;\sigma))$ terms different from $1\otimes L(a,\Delta_{k_0},b;\sigma)$ which can give β for a sub quotient.

Observe that by Lemma 3.1

$$L(a,\Delta_{k_0},b;\sigma)\leq L(a,[\alpha'+1,c])\rtimes L([\alpha,\alpha']^t;\sigma).$$

Now the support forces that from $M^*(L(a,[\alpha'+1,c]))$ we must take $1\otimes L(a,[\alpha'+1,c])$. The only possibility which would not give a term of the form $1\otimes -$ is to take from $M^*(L([\alpha,\alpha']^t;\sigma))$ the term $[-\alpha']\otimes L([\alpha,\alpha'-1]^t;\sigma)$ (observe that we need to get a non-degenerate representation on the left hand side of $\otimes$ and use the formula $\mu^*(L([\alpha,\alpha']^t;\sigma))=\sum_{i=0}^{\alpha'-\alpha+1}L([\alpha'-i+1,\alpha']^t)\tilde{}\otimes L([\alpha,\alpha'-i]^t;\sigma)$ which follows directly from the formula for $\mu^*(\delta([\alpha,\alpha'];\sigma)))$.

Now we need to take from $M^*(\Delta_u))$ a term of form $\delta([-\alpha'+1,\alpha'])\otimes -$, for which we have two possibilities (analogously as in the proof of former corresponding lemma; use the formula for $M^*(\Delta_u)))$. Since on the left and right hand side of $\otimes$ we have regular representations (which are always multiplicity one), we get in this way at most two additional multiplicities. Therefore, the total multiplicity is at most 4. □

7. End of proof of Proposition 4.1

We continue with the notation introduced in the section 4.

A direct consequence of the claims that we have proved in the last two sections is the following

Corollary 7.1. *Let $\Delta_{k+1}\neq\emptyset$ and $k\geq 1$. Consider*

$$L(\Delta_1,\dots,\Delta_k;\delta(\Delta_{k+1};\sigma))^t=L(\Delta'_1,\dots,\Delta'_{k'};\sigma).$$

Then card(Δ'_i) >1 for some i. Denote maximal such index by k'_0. Write

$$\Delta_{k_0}=[\nu^{\alpha+k-k_0}\rho,\nu^c\rho]=[\nu^{\alpha'}\rho,\nu^c\rho].$$

Denote

$$\Delta_u=[\nu^{-\alpha'}\rho,\nu^{\alpha'}\rho].$$

Then

1. *The length of $\delta(\Delta_u)^t\rtimes L(\Delta_1,\dots,\Delta_k;\delta(\Delta_{k+1};\sigma))$ is at least 5.*
2. *The multiplicity of of $\delta(\Delta_u)^t\otimes L(\Delta_1,\dots,\Delta_k;\delta(\Delta_{k+1};\sigma))$ in*

$$\mu^*(\delta(\Delta_u)^t\rtimes L(\Delta_1,\dots,\Delta_k;\delta(\Delta_{k+1};\sigma)))$$

is at most 4.

Proof. Denote

$$\tau = L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma)).$$

Now by Lemmas 5.1 and 6.1 we know that

$$\delta(\Delta_u) \rtimes \tau^t$$

is a representation of length at least 5. This implies that

$$\delta(\Delta_u)^t \rtimes \tau$$

has length ≥ 5.

Further, Lemmas 5.3 and 6.3 imply that the multiplicity of $\delta(\Delta_u) \otimes \pi^t$ in $\mu^*(\delta(\Delta_u) \rtimes \pi^t)$ is at most 4. This implies that the multiplicity of $\delta(\tilde{\Delta}_u)^t \otimes \pi \cong \delta(\Delta_u)^t \otimes \pi$ in $\mu^*(\delta(\Delta_u)^t \rtimes \pi)$ is ≤ 4. This completes the proof of the corollary. □

This corollary, together with Lemmas 5.1, 5.3, 6.1 and imply Proposition 4.1.

Later in this paper we shall show how Proposition 4.1 implies in a simple way Theorem **??**. Now we give another proof of the following result of Hanzer, Jantzen and Tadić:

Theorem 7.2. *If γ is an irreducible sub quotient of*

$$\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$$

different from $L([\alpha, \alpha+n]^{(\rho)}; \sigma)$ and $L([\alpha+n]^{(\rho)}, [\alpha+n-1]^{(\rho)}, \ldots, [\alpha]^{(\rho)}, \sigma)$, then

$$L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma))$$

is not unitarizable.

Proof. Chose π as in Proposition 4.1. Suppose that γ is unitarizable. Then $\pi \rtimes \gamma$ is unitarizable. Let τ be a sub quotient of $\pi \rtimes \gamma$. Then $\tau \hookrightarrow \pi \rtimes \gamma$. Now the Frobenius reciprocity implies that $\pi \otimes \gamma$ is in the Jacquet module of τ.

We know that $\pi \rtimes \gamma$ has length ≥ 5. This (and unitarizability) implies that there are (at least) 5 different irreducible subrepresentations of $\pi \rtimes \gamma$. Denote them by $\tau_1, \ldots, \tau_5$. Then

$$\tau_1 \oplus \cdots \oplus \tau_5 \hookrightarrow \pi \rtimes \gamma.$$

Since the Jacquet functor is exact, the first part of the proof implies that the multiplicity of $\pi \otimes \gamma$ in the Jacquet module of $\pi \rtimes \gamma$ is at least 5. This contradicts to the second claim of Proposition 4.1. The proof is now complete. □

8. Jantzen decomposition

In this section we shall recall of the basic results of C. Jantzen from [24]. We shall write them in a slightly different way then in [24]. They are written there for the symplectic and the split odd-orthogonal series of groups. Since the Jantzen's paper is based on the formal properties of the representation theory of these groups (contained essentially in the structure of the twisted Hopf module which exists on the representations of these groups - see [60]), the results of [24] apply also whenever this structure is established. Therefore, it also holds for all the classical p-adic groups considered in [43][12].

A representation $\rho \in \mathcal{C}$[13] is called a factor of an irreducible representation γ of a classical group, if there exists an irreducible subquotient $\tau \otimes \gamma_{cusp}$ of $s_{GL}(\gamma)$ such that ρ is in the support of τ.

We have already used above the well known notion of (cuspidal) support of an irreducible representation of a general linear group introduced by J. Bernstein and A. V. Zelevinsky. Now we shall introduce such notion for classical groups. We shall fix below an irreducible cuspidal representation σ of a classical group. Let $X \subseteq \mathcal{C}$ and suppose that X is self contragredient, i.e. that

$$\tilde{X} = X,$$

where $\tilde{X} = \{\tilde{\rho}; \rho \in X\}$. Following C. Jantzen, one says that an irreducible representation γ of a classical group is supported by $X \cup \{\sigma\}$ if there exist $\rho_1, \dots, \rho_k$ from X such that

$$\gamma \leq \rho_1 \times \dots \times \rho_k \rtimes \sigma.$$

For not-necessarily irreducible representation π of a classical group, one says that it is supported by $X \cup \{\sigma\}$ if each irreducible subquotient of it is supported by that set.

Definition 8.1. *Let*

$$X = X_1 \cup X_2$$

be a partition of a selfcontragredient $X \subseteq \mathcal{C}$. We shall say that this partition is regular if X_1 is self contragredient[14]*, and if among X_1 and X_2 there is no reducibility, i.e. if*

$$\rho \in X_1 \implies \nu\rho \notin X_2.$$

This is equivalent to say that $\rho_1 \times \rho_2$ is irreducible for all $\rho_1 \in X_1$ and $\rho_2 \in X_2$.

For a partition $X = X_1 \cup \dots \cup X_k$ we define to be regular in an analogous way.

[12]In the case of unitary groups one needs to replace usual contragredient by the contragredient twisted by the non-trivial element of the Galois group of the involved quadratic extension (see [43]). The case of disconnected even split orthogonal group is considered in [25].

[13]Recall, $\mathcal{C}$ is the set of all irreducible cuspidal representations of general linear groups.

[14]Then X_2 is also self contragredient

Definition 8.2. *Let π be a representation of S_n supported in $X\cup\{\sigma\}$. Suppose that $X_1\cup X_2$ is a regular partition of a selfcontragredient $X\subseteq\mathcal{C}$. Write $\mu^*(\pi)=\sum_i\beta_i\otimes\gamma_i$, a sum of irreducible representations in $R\otimes R[S]$. Let $\mu^*_{X_1}(\pi)$ denote the sum of every $\beta_i\otimes\gamma_i$ in $\mu^*(\pi)$ such that the support of β_i is contained in X_1 and the support of γ_i is contained in $X_2\cup\{\sigma\}$.*

Now we recall below the main results of [24]. As we have already mentioned, our presentation is slightly different from the presentation in [24]. In the rest of this section, $X_1\cup X_2$ will be a regular partition of a selfcontragredient $X\subseteq\mathcal{C}$.

Lemma 8.3. *If π has support contained in $X\cup\{\sigma\}$, then $\mu^*_{X_1}(\pi)$ is nonzero.*

Definition 8.4. *Suppose β is a representation of a general linear group supported in X. Write $M^*(\beta)=\sum_i\tau_i\otimes\tau_i'$, a sum of irreducible representations in $R\otimes R$. Let $M^*_{X_1}(\beta)$ denote the sum of every summand $\tau_i\otimes\tau_i'$ in $M^*(\beta)$ such that the support of τ_i is contained in X_1 and the support of τ_i' is contained in X_2.*

Proposition 8.5. *Suppose β is a representation of a general linear group with the support contained in X and γ a representation of S_k with the support contained in $X\cup\{\sigma\}$. Then,*

$$\mu^*_{X_1}(\beta\rtimes\gamma)=M^*_{X_1}(\beta)\rtimes\mu^*_{X_1}(\gamma).$$

Corollary 8.6. *Suppose β has the support contained in X_1 and γ has the support contained in $X_2\cup\{\sigma\}$. Then*

(1)
$$\mu^*_{X_1}(\beta\rtimes\gamma)=M^*_{GL}(\beta)\otimes\gamma.$$

(2) *Write*
$$s_{GL}(\gamma)=\Xi\otimes\sigma$$
in the Grothendieck group[15]. *Then*
$$\mu^*_{X_2}(\beta\rtimes\gamma)=\Xi\otimes\beta\rtimes\sigma.$$

Definition 8.7. *Suppose π is an irreducible representation of S_n supported in $X\cup\{\sigma\}$. Fix $i\in\{1,2\}$. Then there exists an irreducible $\beta_i\otimes\gamma_i$ with β_i supported on X_{3-i} and γ_i supported on $X_i\cup\{\sigma\}$ such that*

$$\pi\hookrightarrow\beta_i\rtimes\gamma_i.$$

The representation γ_i is uniquely determined by the above requirement, and it is denoted by

$$X_i(\pi).$$

Further,

$$\mu^*_{X_{3-i}}(\pi)\le\mu^*_{X_{3-i}}(\beta_i\rtimes\gamma_i)=M^*_{GL}(\beta_i)\otimes\gamma_i. \tag{8.1}$$

Now we shall recall of the key theorem from the Jantzen's paper [24]:

[15]Clearly, Ξ does not need to be irreducible.

Theorem 8.8. *(Jantzen) Suppose that $X_1 \cup X_2$ is a regular partition of a selfcontragredient subset X of $\mathcal{C}$, and σ an irreducible cuspidal representation of S_r. Let $Irr(X_i;\sigma)$ denote the set of all irreducible representations of all S_n, $n \geq 0$, supported on $X_i \cup \{\sigma\}$, and similarly for $Irr(X;\sigma)$.*

Then the map

$$\begin{array}{ccc} Irr(X;\sigma) & \longrightarrow & Irr(X_1;\sigma) \times Irr(X_2;\sigma), \\ \pi & \longmapsto & (X_1(\pi), X_2(\pi)) \end{array}$$

is a bijective correspondence. Denote the inverse mapping by

$$\Psi_{X_1,X_2}.$$

For $\gamma_i \in Irr(X_i;\sigma)$ these bijective correspondence have the following properties:

(1) *If γ_i is a representation of S_{n_i+r}, then*
$$\pi = \Psi_{X_1,X_2}(\gamma_1,\gamma_2)$$
is a representation of $S_{n_1+n_2+r}$

(2) $\widetilde{\Psi_{X_1,X_2}(\gamma_1,\gamma_2)} = \Psi_{X_1,X_2}(\widetilde{\gamma}_1,\widetilde{\gamma}_2)$ *and $X_i(\tilde{\pi}) = \widetilde{X_i(\pi)}$, where $\tilde{}$ denotes contragredient.*

(3) $\Psi_{X_1,X_2}(\gamma_1,\gamma_2)^t = \Psi_{X_1,X_2}({\gamma_1}^t,{\gamma_2}^t)$ *and $X_i(\pi^t) = X_i(\pi)^t$, where ^{t} denotes the involution of Aubert-Schneider-Stuhler .*

(4) *Suppose that*
$$s_{GL}(\gamma_i) = \sum_j c_j(X_i)\tau_j(X_i) \otimes \sigma,$$
where $\tau_j(X_i)$ is an irreducible representation and $c_j(X_i)$ its multiplicity. Then
$$\mu^*_{X_i}(\Psi_{X_1,X_2}(\gamma_1,\gamma_2)) = \sum_j c_j(X_i)\tau_j(X_i) \otimes \gamma_{3-i}$$

(5) *Let $\beta = \beta(X_1) \times \beta(X_2)$ be an irreducible representation of a general linear group with support of $\beta(X_i)$ contained in X_i, $i = 1,2$, and $\Psi = \Psi_{X_1,X_2}(\gamma_1,\gamma_2)$ an irreducible representation of S_k with support contained in $X \cup \{\sigma\}$. (We allow the possibility that $\beta(X_i) = 1$ or $\gamma_i = \sigma$.) Suppose*
$$\beta(X_i) \rtimes \gamma_i = \sum_j m_j(X_i)\gamma_j(X_i;\sigma),$$
with $\gamma_j(X_i;\sigma)$ irreducible and $m_j(X_i)$ its multiplicity. Then,
$$\beta \rtimes \Psi = \sum_{j_1,j_2} (m_{j_1}(X_1)m_{j_2}(X_2))\Psi_{X_1,X_2}(\gamma_{j_1}(X_1;\sigma),\gamma_{j_2}(X_2;\sigma)).$$

(6) $\Psi_{X_1,X_2}(\gamma_1,\gamma_2)$ *is tempered (resp. square-integrable) if and only if γ_1,γ_2 are both tempered (resp. square-integrable).*

(7) *Suppose, in the subrepresentation setting in "tempered" formulation of the Langlands classification,*

$$\gamma_i = L(\nu^{\alpha_1}\tau_1(X_i), \dots, \nu^{\alpha_\ell}\tau_\ell(X_i); T(X_i;\sigma))$$

for $i = 1, 2$ (n.b. recall that $\tau_j(X_i)$ may be the trivial representation of $GL(0, F)$; $T(X_i;\sigma)$ may just be σ). Then,

$$\begin{aligned}\Psi_{X_1,X_2}(\gamma_1, \gamma_2) \;&= L(\nu^{\alpha_1}\tau_1(X_1) \times \nu^{\alpha_1}\tau_1(X_2), \\ &\dots, \nu^{\alpha_\ell}\tau_\ell(X_1) \times \nu^{\alpha_\ell}\tau_\ell(X_2); \Psi_{X_1,X_2}(T(X_1;\sigma), T(X_2;\sigma))).\end{aligned}$$

In the other direction, if

$$\pi = L(\nu^{\alpha_1}\tau_1(X_1) \times \nu^{\alpha_1}\tau_1(X_2), \dots, \nu^{\alpha_\ell}\tau_\ell(X_1) \times \nu^{\alpha_\ell}\tau_\ell(X_2); T(X;\sigma)),$$

then

$$X_i(\pi) = L(\nu^{\alpha_1}\tau_1(X_i), \dots, \nu^{\alpha_\ell}\tau_\ell(X_i); X_i(T(X;\sigma))).$$

(In the quotient setting of the Langlands classification, the same results hold.)

(8) *Suppose,*

$$\mu^*(\gamma_i) = \sum_j n_j(X_i)\eta_j(X_i) \otimes \theta_j(X_i;\sigma),$$

with $\eta_j(X_i) \otimes \theta_j(X_i;\sigma)$ irreducible and $n_j(X_i)$ its multiplicity. Then,

$$\begin{aligned}&\mu^*(\Psi_{X_1,X_2}(\gamma_1, \gamma_2)) \\ &\quad= \sum_{j_1,j_2}(n_{j_1}(X_1)n_{j_2}(X_2))(\eta_{j_1}(X_1) \times \eta_{j_2}(X_2)) \otimes \Psi_{X_1,X_2}(\theta_{j_1}(X_1;\sigma), \theta_{j_2}(X_2;\sigma)).\end{aligned}$$

(9) *Let $X = X_1 \cup X_2 \cup X_3$ be a regular partition and $\pi \in Irr(X;\sigma)$. Then*

$$X_1\big((X_1 \cup X_2)(\pi)\big) = X_1\big((X_1 \cup X_3)(\pi)\big).$$

In the other direction we have

$$\Psi_{X_1\cup X_2,X_3}\big(\Psi_{X_1,X_2}(\pi_1, \pi_2), \pi_3\big) = \Psi_{X_1,X_2\cup X_3}\big(\pi_1, \Psi_{X_2,X_3}(\pi_2, \pi_3)\big)$$

for $\pi_i \in Irr(X_i;\sigma)$.

Remark 8.9. (1) *Let β_i be an irreducible representation of a general linear group supported in X_i, $i = 1, 2$, and let γ_i be an irreducible representation of a classical p-adic group supported in $X_i \cup \{\sigma\}$, $i = 1, 2$. Then (5) of the above theorem implies*

$$(\beta_1 \times \beta_2) \rtimes \Psi_{X_1,X_2}(\gamma_1, \gamma_2) \text{ is irreducible } \iff \text{ both } \beta_i \rtimes \gamma_i \text{ are irreducible.}$$

(2) *One can express the above theorem without the last claim, in a natural way for a regular partition in more than two pieces.*

9. Cuspidal lines

Let ρ be an irreducible unitarizable cuspidal representation of a general linear group. Denote

$$X_\rho = \{\nu^x\rho; x \in \mathbb{R}\} \cup \{\nu^x\tilde{\rho}; x \in \mathbb{R}\},$$
$$X_\rho^c = \mathcal{C}\backslash X_\rho.$$

For an irreducible representation π of a classical p-adic group take any finite set of different classes $\rho_1, \dots, \rho_k \in \mathcal{C}_u$ such that $\rho_i \not\cong \rho_j$ for any $i \neq j$, and that π is supported in

$$X_{\rho_1} \cup \dots \cup X_{\rho_k} \cup \{\sigma\}.$$

Then π is uniquely determined by

$$(X_{\rho_1}(\pi), \dots, X_{\rho_k}(\pi)).$$

Now we have a natural

Preservation of unitarizability question: *Let π be an irreducible weakly real representation of a classical p-adic group*[16]*. Is π unitarizable if and only if all $X_{\rho_i}(\pi)$ are unitarizable?*

10. Proof of the main result

Theorem 10.1. *Suppose that θ is an irreducible unitarizable representation of a classical group, and suppose that the infinitesimal character of some $X_\rho(\theta)$ is the same as the infinitesimal character of a generalized Steinberg representation supported in $X_\rho \cup \{\sigma\}$ with $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}$*[17]*. Then $X_\rho(\theta)$ is the generalized Steinberg representation, or its Aubert-Schneider-Stuhler dual.*

In particular, if char$(F) = 0$, then $X_\rho(\theta)$ is unitarizable.

Proof. Denote

$$\theta_\rho = X_\rho(\theta), \qquad \theta_\rho^c = X_\rho^c(\theta).$$

Then

$$\theta = \Psi_{X_\rho, X_\rho^c}(\theta_\rho, \theta_\rho^c).$$

Suppose that θ_ρ is neither the generalized Steinberg representation, nor it is its Aubert-Schneider-Stuhler dual. Now Proposition 4.1 implies that there exists a selfcontragredient unitarizable representation π of a general linear group supported in X_ρ such that the length of

$$\pi \rtimes \theta_\rho$$

[16]We do not need to assume π to be weakly real in the above question. Theorem 2.1 (or 2.2) implies that this is an equivalent to the above question.

[17]As we already noted, this is know if char$(F) = 0$.

is at least 5, and that the multiplicity of $\pi\otimes\theta_\rho$ in the Jacquet module of $\pi\rtimes\theta_\rho$ is at most 4.

Consider now

$$\pi\rtimes\theta=\pi\rtimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c).$$

Then this representation is of length ≥ 5 (take in (5) of Jantzen theorem $\beta(X_\rho)=\pi,\beta(X_\rho^c)=1$, and multiply it by the representation $\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c)$).

We shall now use the assumption that $\theta=\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c)$ is unitarizable. From the fact that the length of $\pi\otimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c)$ is at least 5 and the exactness of the Jacquet module functor, it follows that the multiplicity of $\pi\otimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c)$ in $\mu^*(\pi\rtimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c))$ is at least five.

By the definition of θ_ρ, we can chose an irreducible representation ϕ of a general linear group supported in X_ρ^c such that

$$\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c)\hookrightarrow\phi\rtimes\theta_\rho.$$

By the Frobenius reciprocity, $\phi\otimes\theta_\rho$ is a sub quotient of the Jacquet module of $\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c)$. Denote its multiplicity by k. This implies that the multiplicity of $\pi\otimes\phi\otimes\theta_\rho$ in $\mu^*(\pi\rtimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c))$ is at least $5k$.

Recall that the support of π is in X_ρ and the support of ϕ is in X_ρ^c. Let Π be an irreducible representation of a general linear group which has in its Jacquet module $\pi\otimes\phi$. Then $\Pi\cong\pi'\times\phi'$, where the support of π' is in X_ρ and the support of ϕ' is in X_ρ^c. Further, π and π' are representations of the same group (as well as ϕ and ϕ'). Frobenius reciprocity implies that $\pi'\otimes\phi'$ is in the Jacquet module of Π. Further, the formula $m^*(\Pi)=m^*(\pi)\times m^*(\phi)$ implies that if we have in the Jacquet module of Π an irreducible representation of the form $\pi''\otimes\phi''$, where the support of π'' is in X_ρ and the support of ϕ'' is in X_ρ^c, then $\pi''\cong\pi'$, $\phi''\cong\phi'$, and the multiplicity of $\pi'\times\phi'$ in the Jacquet module of Π is one.

This first implies that $\Pi\cong\pi\times\phi$, then that the only irreducible representation of a general linear group which has in its Jacquet module $\pi\otimes\phi$ is $\pi\times\phi$, and further that the multiplicity of $\phi\otimes\pi$ in the Jacquet module $\pi\times\phi$ is one.

This and the transitivity of the Jacquet modules imply that the multiplicity of $\phi\otimes\pi\otimes\theta_\rho$ in the Jacquet module of $\mu^*(\pi\rtimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c))$ is at least $5k$.

Now we examine in a different way the multiplicity of $\phi\otimes\pi\otimes\theta_\rho$ in the Jacquet module of $\mu^*(\pi\rtimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c))$. Observe that $\phi\otimes\pi\otimes\theta_\rho$ must be a sub quotient of a Jacquet module of the following part

$$\mu^*_{X_\rho^c}(\pi\rtimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c))=(1\otimes\pi)\rtimes\mu^*_{X_\rho^c}(\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c))$$

of $\mu^*(\pi\rtimes\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c))$. Recall that by (8.1), $\mu^*_{X_\rho^c}(\Psi_{X_\rho,X_\rho^c}(\theta_\rho,\theta_\rho^c))$ is of the form $*\otimes\theta_\rho$. If we want to get $\phi\otimes\pi\otimes\theta_\rho$ from a term from here, it must be $\phi\otimes\theta_\rho$. Recall that we have this term with multiplicity k here. Therefore, we need to see the multiplicity of $\phi \otimes \pi \otimes \theta_\rho$ in the Jacquet module of $k \cdot (1 \otimes \pi) \rtimes (\phi \otimes \theta_\rho) = k \cdot (\phi \otimes \pi \rtimes \theta_\rho)$. We know that this multiplicity is at most $4k$. Therefore, $5k \leq 4k$ (and $k \geq 1$). This is a contradiction.

Therefore, θ_ρ is the generalized Steinberg representation or its Aubert-Schneider-Stuhler dual. The generalized Steinberg representation is unitarizable (since it is square integrable). Further in characteristic zero, [17] (or [40] and [42]) implies that its Aubert-Schneider-Stuhler dual is unitarizable. Therefore, θ_ρ is unitarizable if we are in the characteristic zero. $\square$

11. Irreducible generic and irreducible unramified representations

We consider in this section quasi-split classical p-adic groups (see [32] for more details). One can find in [32] more detailed exposition of the facts about irreducible generic representations and unitarizable subclasses that we shall use here. We shall recall here only very briefly of some of that facts.

Let γ be an irreducible representation of a classical group. Let $X_1 \cup X_2$ be a regular partition of $\mathcal{C}$. Now [47] directly implies that γ is generic if and only if $X_1(\gamma)$ and $X_2(\gamma)$ are generic. Therefore,

$$\gamma \text{ is generic if and only if all } X_\rho(\gamma) \text{ are generic, } \rho \in \mathcal{C}_u. \tag{11.1}$$

Analogous statement holds for temperness by (6) of Theorem 8.8.

Recall that by (5) of Theorem 8.8, if the support of some irreducible representation β of a general linear group is contained in $X_{\rho'}$, then holds

$$\beta \rtimes \gamma \text{ is irreducible} \iff \beta \rtimes X_{\rho'}(\gamma) \text{ is irreducible.} \tag{11.2}$$

Denote by $\mathcal{C}_u'$ any subset of $\mathcal{C}_u$ satisfying:

$$\mathcal{C}_u' \cup (\mathcal{C}_u')\tilde{} = \mathcal{C}_u \text{ and } \rho \in \mathcal{C}_u' \cap (\mathcal{C}_u')\tilde{} \implies \rho \cong \tilde{\rho}.$$

Let π be an irreducible generic representation of a classical group. We can write π uniquely as

$$\pi \cong \delta_1 \times \cdots \times \delta_k \rtimes \tau \tag{11.3}$$

where the δ_i's are irreducible essentially square-integrable representations of general linear groups which satisfy

$$e(\delta_1) \geq \cdots \geq e(\delta_k) > 0, \tag{11.4}$$

and τ is a generic irreducible tempered representation of a classical group.

For $\rho' \in \mathcal{C}_u'$ chose some irreducible representation $\Gamma_{\rho'}^c$ of a general linear group such that

$$\tau \hookrightarrow \Gamma_{\rho'}^c \rtimes X_{\rho'}(\tau),$$

and that $\Gamma^c_{\rho'}$ is supported out of $X_{\rho'}$. Observe that

$$\pi \cong \Big(\prod_{\rho\in\mathcal{C}'_u}\Big(\prod_{\mathrm{supp}(\delta_i)\subseteq X_\rho}\delta_i\Big)\Big)\rtimes\tau \hookrightarrow \Big(\prod_{\rho\in\mathcal{C}'_u}\Big(\prod_{\mathrm{supp}(\delta_i)\subseteq X_\rho}\delta_i\Big)\Big)\times\Gamma^c_{\rho'}\rtimes X_{\rho'}(\tau)$$

$$\cong \Big(\prod_{\rho\in\mathcal{C}'_u\setminus\{\rho'\}}\Big(\prod_{\mathrm{supp}(\delta_i)\subseteq X_\rho}\delta_i\Big)\Big)\times\Gamma^c_{\rho'}\times\Big(\prod_{\mathrm{supp}(\delta_i)\subseteq X_{\rho'}}\delta_i\Big)\rtimes X_{\rho'}(\tau).$$

One easily sees that there exists an irreducible sub quotient $\Pi^c_{\rho'}$ of

$$\Big(\prod_{\rho\in\mathcal{C}'_u\setminus\{\rho'\}}\Big(\prod_{\mathrm{supp}(\delta_i)\subseteq X_\rho}\delta_i\Big)\Big)\times\Gamma^c_{\rho'}$$

such that

$$\pi\hookrightarrow\Pi^c_{\rho'}\times\Big(\prod_{\mathrm{supp}(\delta_i)\subseteq X_{\rho'}}\delta_i\Big)\rtimes X_{\rho'}(\tau).$$

Since $\Pi^c_{\rho'}$ is supported out of $X_{\rho'}$ and $\Big(\prod_{\mathrm{supp}(\delta_i)\subseteq X_{\rho'}}\delta_i\Big)\rtimes X_{\rho'}(\tau)$ is irreducible and supported in $X_{\rho'}\cup\{\sigma\}$, we get that

$$X_{\rho'}(\pi)=\Big(\prod_{\mathrm{supp}(\delta_i)\subseteq X_{\rho'}}\delta_i\Big)\rtimes X_{\rho'}(\tau). \tag{11.5}$$

Let $\pi\cong\delta_1\times\cdots\times\delta_k\rtimes\tau$ be as in (11.3). Then for any square-integrable representation δ of a general linear group denote by $\mathcal{E}_\pi(\delta)$ the multiset of exponents $e(\delta_i)$ for those i such that $\delta_i^u\cong\delta$. We denote below by $\mathbf{1}_G$ the trivial one-dimensional representation of a group G. Now we recall of the solution of the unitarizability problem for irreducible generic representations of classical p-adic groups obtained in [32].

Theorem 11.1. *Let π be given as in (11.3). Then π is unitarizable if and only if for all irreducible square integrable representations δ of general linear groups hold*

1. $\mathcal{E}_\pi(\tilde\delta)=\mathcal{E}_\pi(\delta)$, *i.e. π is Hermitian.*
2. *If either $\delta\not\cong\tilde\delta$ or $\nu^{\frac12}\delta\rtimes\mathbf{1}_{G_0}$ is reducible then $0<\alpha<\frac12$ for all $\alpha\in\mathcal{E}_\pi(\delta)$.*
3. *If $\tilde\delta\cong\delta$ and $\nu^{\frac12}\delta\rtimes\mathbf{1}_{G_0}$ is irreducible then $\mathcal{E}_\pi(\delta)$ satisfies Barbasch' conditions, i.e. we have $\mathcal{E}_\pi(\delta)=\{\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_l\}$ with*
$$0<\alpha_1\le\cdots\le\alpha_k\le\frac12<\beta_1<\cdots<\beta_l<1$$
such that
   (a) $\alpha_i+\beta_j\neq1$ *for all* $i=1,\dots,k$, $j=1,\dots,l$; $\alpha_{k-1}\neq\frac12$ *if* $k>1$.
   (b) $\#\{1\le i\le k:\alpha_i>1-\beta_1\}$ *is even if* $l>0$.
   (c) $\#\{1\le i\le k:1-\beta_j>\alpha_i>1-\beta_{j+1}\}$ *is odd for* $j=1,\dots,l-1$.
   (d) $k+l$ *is even if* $\delta\rtimes\tau$ *is reducible.*

Observe that (11.2) implies that if $\mathrm{supp}(\delta_i) \subset X_{\rho'}$, then

$$\delta_i \rtimes \tau \text{ is irreducible } \iff \delta_i \rtimes X_{\rho'}(\tau) \text{ is irreducible.} \tag{11.6}$$

Let π be a generic representation. We can then present it by the formula (11.3)

Suppose that π is unitarizable. This implies that π satisfies the above theorem. Now from (11.6), the above theorem implies that $X_{\rho'}(\pi) \cong \left(\prod_{\mathrm{supp}(\delta_i) \subseteq X_{\rho'}} \delta_i \right) \rtimes X_{\rho'}(\tau)$ is unitarizable (we need (11.6) only for (d) of (3) in the above theorem).

Suppose now that all $X_{\rho'}(\pi) = \left(\prod_{\mathrm{supp}(\delta_i) \subseteq X_{\rho'}} \delta_i \right) \rtimes X_{\rho'}(\tau)$, $\rho \in \mathcal{C}'_u$, are unitarizable. Then each of them satisfy the above theorem. Now the above theorem and (11.6) imply that π is unitarizable.

Therefore, we have proved the following

Corollary 11.2. *For an irreducible generic representation π of a classical group holds*

$$\pi \textit{ is unitarizable } \iff \textit{ all } X_\rho(\pi),\ \rho \in \mathcal{C}_u, \textit{ are unitarizable.} \quad \square$$

In a similar way, using the classification of the irreducible unitarizable unramified representations of split classical p-adic groups in [45] (or as it is stated in [67]), we get that the above fact holds for irreducible unramified representations of classical p-adic groups.

12. Question of independence

Let ρ and σ be irreducible unitarizable cuspidal representations of a general linear and a classical group respectively. If there exists a non-negative $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}$ such that

$$\nu^\alpha \rho \rtimes \sigma$$

reduces. When we fix ρ and σ as above, to shorten notation then this $\alpha_{\rho,\sigma}$ will be denoted also by α.

By a $\mathbb{Z}$-segment in $\mathbb{R}$ we shall mean a subset of form $\{x, x+1, \dots, x+l\}$ of $\mathbb{R}$. We shall denote this subset by $[x, x+l]$. For such a segment Δ, we denote

$$\Delta^{(\rho)} = \{\nu^x \rho; x \in \Delta\}.$$

We shall fix two pairs ρ_i, σ_i as above, such that

$$\alpha_{\rho_1,\sigma_1} = \alpha_{\rho_2,\sigma_2}$$

and denote it by

$$\alpha.$$

We shall construct a natural bijection

$$E_{1,2} : Irr(X_{\rho_1};\sigma_1) \to Irr(X_{\rho_2};\sigma_2),$$

which will be canonical, except in the case when $\alpha = 0$. First we shall define $E_{1,2}$ on the irreducible square integrable representations.

A classification of irreducible square integrable representations of classical p-adic groups modulo cuspidal data is completed in [43]. We shall freely use notation of that paper, and also of [64]. We shall very briefly recall of parameters of irreducible square integrable representations in $Irr(X_\rho;\sigma)$ (one can find more details in [64], sections 16 and 17). Below (ρ,σ) will denote (ρ_1,σ_1) or (ρ_2,σ_2).

An irreducible square integrable representation $\pi \in Irr(X_\rho;\sigma)$ is parameterized by Jordan blocks $Jord_\rho(\pi) = \{\Delta_1^{(\rho)},\dots,\Delta_k^{(\rho)}\}$, where Δ_i are $\mathbb{Z}$-segments contained in $\alpha+\mathbb{Z}$, and by a partially defined function $\epsilon_\rho(\pi)$ (partial cuspidal support is σ). Since $\{\Delta_1^{(\rho)},\dots,\Delta_k^{(\rho)}\}$ and $\{\Delta_1,\dots,\Delta_k\}$ are in a natural bijective correspondence, we can view $\epsilon_\rho(\pi)$ as defined (appropriately) on $\{\Delta_1,\dots,\Delta_k\}$ (which means that $\epsilon_\rho(\pi)$ is independent of particular ρ). In sections 16 and 17 of [64], it is explained how π and the triple

$$(\{\Delta_1^{(\rho)},\dots,\Delta_k^{(\rho)}\},\epsilon_\rho(\pi),\sigma)$$

are related. In this case we shall write

$$\pi \longleftrightarrow (\{\Delta_1^{(\rho)},\dots,\Delta_k^{(\rho)}\},\epsilon_\rho(\pi),\sigma). \tag{12.1}$$

Take irreducible square integrable representations $\pi_i \in Irr(X_\rho;\sigma)$, $i=1,2$. Suppose

$$\pi_1 \longleftrightarrow (\{\Delta_1^{(\rho_1)},\dots,\Delta_k^{(\rho_1)}\},\epsilon_{\rho_1}(\pi_1),\sigma_1). \tag{12.2}$$

Then we define

$$E_{1,2}(\pi_1)=\pi_2$$

if

$$\pi_2 \longleftrightarrow (\{\Delta_1^{(\rho_2)},\dots,\Delta_k^{(\rho_2)}\},\epsilon_{\rho_1}(\pi_1),\sigma_2).$$

For defining $E_{1,2}$ on the whole $Irr(X_{\rho_1};\sigma_1)$, the key step is an extension of $E_{1,2}$ from the square integrable classes to the tempered classes. For this, we shall use parameterization of irreducible tempered representations obtained in [68][18].

Let $\pi \in Irr(X_\rho,;\sigma)$ be square integrable and let $\delta := \delta(\Delta^{(\rho)})$ be an irreducible (unitarizable) square integrable representation of a general linear group, where Δ is a segment in $\alpha+\mathbb{Z}$ such that $\delta \rtimes \pi$ reduces (one directly reads from the invariants (12.1) when this happens). Now Theorem 1.2 of [68] defines the irreducible tempered subrepresentation π_δ of $\delta \rtimes \pi$. The other irreducible summand is denoted by $\pi_{-\delta}$.

[18]Another possibility would be to use the Jantzen's parameterization obtained in [26] (we do not know if using [26] would result with the same mapping $E_{1,2}$).

Let $\pi \in Irr(X_\rho, ; \sigma)$ be square integrable, let $\delta_i := \delta(\Delta_i^{(\rho)})$ be different irreducible (unitarizable) square integrable representations of general linear groups, where Δ_i are $\mathbb{Z}$-segments contained in $\alpha + \mathbb{Z}$ such that all $\delta_i \rtimes \pi$ reduce, and let $j_i \in \{\pm 1\}$, $i = 1, \dots, n$. Then there exists a unique (tempered) irreducible representation π' of a classical group such that

$$\pi' \hookrightarrow \delta_1 \times \ldots \times \delta_{i-1} \times \delta_{i+1} \times \ldots \times \delta_n \rtimes \pi_{j_i \delta_i},$$

for all i. Then we denote

$$\pi' = \pi_{j_1\delta_1,\ \dots, j_n\delta_n}.$$

In the situation as above we define

$$E_{1,2}(\pi_{j_1\delta(\Delta_1^{(\rho_1)}),\ \dots, j_n\delta(\Delta_1^{(\rho_1)})}) = E_{1,2}(\pi)_{j_1\delta(\Delta_1^{(\rho_2)}),\ \dots, j_n\delta(\Delta_1^{(\rho_2)})}.$$

Let additionally $\Gamma_1^{(\rho)}, \dots, \Gamma_m^{(\rho)}$ be segments of cuspidal representations such that for each i, either Γ_i is among Δ_j's, or $\delta(\Gamma_i^{(\rho)}) \rtimes \pi$ is irreducible, and $-\Gamma_i = \Gamma_i$. Then the tempered representation

$$\delta(\Gamma_1^{(\rho)}) \times \ldots \times \delta(\Gamma_m^{(\rho)}) \rtimes \pi_{j_1\delta(\Delta_1^{(\rho)}),\ \dots, j_n\delta(\Delta_1^{(\rho)})} \tag{12.3}$$

is irreducible. We define

$$E_{1,2}(\delta(\Gamma_1^{(\rho_1)}) \times \ldots \times \delta(\Gamma_m^{(\rho_1)}) \rtimes \pi_{j_1\delta(\Delta_1^{(\rho_1)}),\ \dots, j_n\delta(\Delta_1^{(\rho_1)})}) = \\ \delta(\Gamma_1^{(\rho_2)}) \times \ldots \times \delta(\Gamma_m^{(\rho_2)}) \rtimes E_{1,2}(\pi_{j_1\delta_1,\ \dots, j_n\delta_n}).$$

In this way we have define $E_{1,2}$ on the subset of all the tempered classes in $Irr(X_{\rho_1}; \sigma)$.

Let now π be any element of $Irr(X_{\rho_1}; \sigma)$. Write

$$L(\Delta_1^{(\rho_1)}, \dots, \Delta_k^{(\rho_1)}; \tau)$$

as a Langlands quotient (Δ_i are $\mathbb{Z}$ segments in $\mathbb{R}$ and τ is a tempered class in $Irr(X_{\rho_1}; \sigma)$). Then we define

$$E_{1,2}(L(\Delta_1^{(\rho_1)}, \dots, \Delta_k^{(\rho_1)}; \tau)) = L(\Delta_1^{(\rho_2)}, \dots, \Delta_k^{(\rho_2)}; E_{1,2}(\tau)).$$

Independence of unitarizability question: *Let ρ_1, ρ_2, σ_1 and σ_2 be irreducible cuspidal representations as above. Suppose $\alpha_{\rho_1,\sigma_1} = \alpha_{\rho_2,\sigma_2}$ and construct the mapping $E_{1,2}$ as above. Let $\pi \in Irr(X_{\rho_1}; \sigma)$. Is π unitarizable if and only if $E_{1,2}(\pi)$ is unitarizable?*

One can also ask if the other important representation theoretic data are preserved by $E_{1,2}$ (Jacquet modules, irreducibilities of parabolically induced representations, Kazhdan-Lusztig multiplicities etc.).

Remark 12.1. *In this remark we consider irreducible generic representations, and assumptions on the groups are the same as in the section 11. We continue with the previous notation. Let $\delta := \delta(\Delta^{(\rho)})$be an irreducible (unitarizable) square integrable representation of a general linear group, where Δ is a segment in $\alpha + \mathbb{Z}$.*

Then we know that $\nu^{1/2}\delta(\Delta^{(\rho)}) \rtimes \mathbf{1}_{S_0}$ reduces if and only if

(1) *card*(Δ) *is odd if* $\alpha \notin \mathbb{Z}$*;*
(2) *card*(Δ) *is even if* $\alpha \in \mathbb{Z}$*.*

Therefore the conditions of reducibility of $\nu^{1/2}\delta(\Delta^{(\rho)}) \rtimes \mathbf{1}_{S_0}$ *in (2) and irreducibility in (3) of Theorem 7.2 does not depend on* ρ*, but only on* Δ *and* α*.*

Further, let τ *be the representation in* (12.3)*. Now* $\delta(\Delta^{(\rho)}) \rtimes \tau$ *is reducible if and only if*

(i) $\alpha \in \Delta$*;*
(ii) $\Delta \notin \{\Delta_1, \dots, \Delta_n\}$ *(recall that* $\Delta_1, \dots \Delta_n$ *form the Jordan block of* π *along* ρ*);*
(1) $\Delta \notin \{\Gamma_1, \dots, \Gamma_m\}$*.*

Obviously, these conditions again does not depend on ρ_i*, but on* $\alpha = \alpha_{\rho,\sigma}$ *and the parameters which are preserved by* $E_{1,2}$*. Therefore now Theorem 7.2 implies that the above Independence of unitarizability question has positive answer for the irreducible generic representations, i.e. the unitarizability in this case does not depend on particular* ρ *and* σ*, but only on* $\alpha = \alpha_{\rho,\sigma}$*.*

13. The case of unitary groups

We shall now comment the case of unitary groups. Then we have a quadratic extension F' of F and the non-trivial element Θ of the Galois group. Let π be a representation of $GL(n, F')$. Then the representation $g \mapsto \tilde{\pi}(\Theta(g))$ will be called the F'/F-contragredient of π and denoted by $\check{\pi}$.

The results of this paper hold also for the unitary groups if we replace everywhere representations of general linear groups over F by representations of general linear groups over F', and contragredients of representations of general linear groups by F'/F-contragredients. The proofs of the statements are the same (with one exception - see below), after we apply the above two changes everywhere.

The only difference is that the unitarizability of the Aubert-Schneider-Stuhler involution of the generalized Steinberg representation we do not get from [17]. The unitarizability of the Aubert-Schneider-Stuhler involution of a general irreducible square integrable representation of a classical group over a p-adic field of characteristic zero follows from [40], [37] (proposition 4.2 there) and (in the non-quasi split case) [42] (Theorem 4.1 there).

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