PARABOLICALLY INDUCED REPRESENTATIONS AND UNITARIZABILITY FOR $Sp(2,F)$

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Let G be a reductive p-adic group. The unitarizability problem is the problem of identifying of the subset $\hat{G}$ of all unitarizable classes in the non-unitary dual $\tilde{G}$. Important step in studying unitarizability is organizing Hermitian representations into continuous families. Roughly, in this way the study of unitarizability reduces to a study of a smaller number of continuous families, and relations among them. Using this approach, one can apply in the study of unitarizability strong arguments, which otherwise could not be applied. The most natural such continuous families are families of parabolically induced representations. For their construction is crucial knowledge of reducibility of parabolic induction, and further, understanding of irreducible subquotients at the reducibility points.

Up to now, we have discussed the unitarizability problem as if we knew the non-unitary dual. Except for $GL(n)$ and few other low rank groups, this is not the case (for these groups, we usually know also a solution of the unitarizability problem). In general, regarding the non-unitary dual, we have Langlands classification. It classifies irreducible representations in terms of irreducible tempered ones of Levi subgroups. The later representations we do not know (in general). Therefore, we need also classification of irreducible square integrable representations, and further of irreducible tempered ones. Both of these problems are very much connected to understanding of the parabolic induction (the first one, with the non-unitary parabolic induction, and the second one with the unity induction).

The above discussion explains why for the unitarizability problem, as well as for the problem of the non-unitary dual is important to understand the parabolic induction. In these notes, we shall explain the splitting of the parabolically induced representations of $Sp(2,F)$, and obtain as a consequence of this classification of irreducible square integrable, tempered and unitarizable representations of this group (modulo cuspidal irreducible representations). The results that we shall present belong mainly to F. Rodier, P.J. Sally and the author of these notes. Let us note that the group $Sp(2,F)$ is a relatively simple group, and that also a number of other mathematicians have related results. This paper is expanded version of the author's original notes written for the talk given at the conference "Représentations du groupe p-adique $Sp(4)$" (C.I.R.M. Marseille-Luminy, 1998).

We shall start with introducing some notation.

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1. General linear groups

Let F be a p-adic field. We shall assume $\mathrm{char}\,(F) \neq 2$. For two admissible representations π_1 and π_2 of $GL(n_1, F)$ and $GL(n_2, F)$, J. Bernstein and A.V. Zelevinsky have denote by $\pi_1 \times \pi_2$ the representation of $GL(n_1 + n_2, F)$ parabolically induced by $\pi_1 \otimes \pi_2$ from the parabolic subgroup

$$\begin{bmatrix} GL(n_1) & * \\ 0 & GL(n_2) \end{bmatrix}.$$

Recall that

$$(\pi_1 \times \pi_2) \times \pi_3 \cong \pi_1 \times (\pi_2 \times \pi_3) \tag{1-1}$$

and that the composition series of $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ coincide, i.e. that for the semi simplifications hold

$$\mathrm{s.s.}(\pi_1 \times \pi_2) = \mathrm{s.s.}(\pi_2 \times \pi_1). \tag{1-2}$$

Denote $\nu = |\det|_F : GL(n, F) \to \mathbb{R}_+^\times$, where $|\ \ |_F$ denotes the modulus character of F. For each irreducible essentially square integrable representation δ of $GL(n, F)$, there exists $e(\delta) \in \mathbb{R}$ such that $\delta = \nu^{e(\delta)} \delta^u$ with δ^u unitarizable.

2. Symplectic groups

Let

$$J_n = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & 1 \\ 0 & 0 & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & 0 & 0 \\ 1 & 0 & \cdot & \cdot & 0 & 0 \end{bmatrix} \in GL(n),$$

and

$$Sp(n, F) = Sp(n) = \left\{ g \in GL(2n, F);\ {}^t g \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\}.$$

In the above formula we have denoted by ${}^t g$ the transposed matrix of g. The transposed matrix of g with respect to the other diagonal will be denoted by ${}^\tau g$.

For $0 \leq k \leq n$, the group

$$M_{(k)} = \left\{ \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & {}^\tau g \end{bmatrix} ; g \in GL(k), h \in Sp(n-k, F) \right\}$$

is a Levi subgroup of a standard parabolic subgroup , which we denote by $P_{(k)}$ (this parabolic is maximal if $k > 0$). Note that we have obvious isomorphism $M_{(k)} \cong GL(k) \times Sp(n-k)$. Using this isomorphism, for admissible representations π of $GL(k)$ and σ of $Sp(n-k)$, we define $\pi \rtimes \sigma$ to be the representation of $Sp(n)$ parabolically induced by $\pi \otimes \sigma$

from $P_{(k)}$. Note that operations $\times$ and $\rtimes$ are sufficient for describing parabolically induced representations of symplectic groups (from irreducible representations of Levi factors of standard parabolic subgroups). Elementary properties of $\rtimes$ are

$$(\pi_1 \times \pi_2) \rtimes \sigma \cong \pi_1 \rtimes (\pi_2 \rtimes \sigma) \qquad \text{(induction in stages)}, \tag{2-1}$$

$$\text{s.s.}(\pi \rtimes \sigma) = \text{s.s.}(\tilde{\pi} \rtimes \sigma) \tag{2-2}$$

(induction from associate parabolics and associate representations).

The normalized Jacquet module of a representation π of $Sp(n)$ with respect to $P_{(k)}$ will be denoted by

$$s_{(k)}.$$

3. Langlands classification

Now we shall briefly recall the Langlands classification for symplectic groups. Denote by D the set of all equivalence classes of irreducible essentially square integrable representations of all $GL(n), n \geq 1$. Let $D_+ = \{\delta \in D; e(\delta) > 0\}$. Take $\delta_1, .., \delta_k \in D_+$ and an irreducible tempered representation τ of some symplectic group. Let p be a permutation of $\{1, .., k\}$ such that

$$e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \cdots \geq e(\delta_{p(k)}).$$

Then the representation

$$\delta_{p(1)} \times \cdots \times \delta_{p(k)} \rtimes \tau$$

has a unique irreducible quotient, which we denote by

$$L(\delta_1, \dots, \delta_k, \tau).$$

This quotient doesn't depend on p (up to an isomorphism). Each irreducible representation π of a symplectic group is isomorphic to some $L(\delta_1, \dots, \delta_k, \tau)$ as above. Further, τ is uniquely determined (up to an isomorphism) by π, and $\delta_1, \dots, \delta_k$ are determined by π up to an order.

4. Principal series

Principal series representations of symplectic groups are representations of the form

$$\chi_1 \times \ldots \times \chi_k \rtimes 1, \tag{4-1}$$

where χ_i are characters of $F^\times + GL(1)$ (we can work with essentially tempered representations bellow, instead of characters). Note that 1 in the formula (4-1) denotes the trivial representation of $Sp(0)$. Having in mind the relations (1-1), (1-2), (2-1) and (2-2) satisfied by $\times$ and by $\rtimes$, evident necessary conditions for the principal series (4-1) to be irreducible are

(i) $\chi_i \times \chi_j$ is irreducible for any $i < j$;
(ii) $\chi_i \times \tilde{\chi}_j$ is irreducible for any $i < j$;
(iii) $\chi_i \rtimes 1$ is irreducible for any i.

Otherwise, in the case of $k > 1$ we would have reducibility already for induction to some proper Levi subgroup (clearly, for $k = 1$ the condition (iii) must hold in the case of irreducibility).

The above conditions $(i)-(iii)$ are moreover enough for the irreducibility of the principal series (4-1). We shall briefly illustrate proof that they imply the irreducibility in the case $k = 2$. The relations (1-2), (2-1) and (2-2) which satisfy $\times$ and $\rtimes$, tell us that it is enough to consider the case $e(\chi_1) \geq e(\chi_2) \geq 0$. For simplicity, assume that we have $e(\chi_2) > 0$. Then

$$\chi_1 \times \chi_2 \rtimes \tau \cong \chi_1 \times \tilde{\chi}_2 \rtimes 1 \cong \tilde{\chi}_2 \times \chi_1 \rtimes 1 \cong \tilde{\chi}_2 \times \tilde{\chi}_1 \rtimes 1 \cong \tilde{\chi}_1 \times \tilde{\chi}_2 \rtimes 1.$$

Note that the long intertwining operator from the Langlands classification is an element of

$$\text{Hom}_{Sp(2)}(\chi_1 \times \chi_2 \rtimes 1, \tilde{\chi}_1 \times \tilde{\chi}_2 \rtimes 1), \tag{4-2}$$

and it has irreducible image (the Langlands quotient). Further, the space of intertwinings (4-2) is one-dimensional. This implies that $\tilde{\chi}_1 \times \tilde{\chi}_2 \rtimes 1$ is irreducible, and therefore (using relations (1-2), (2-1) and (2-2) satisfied by $\times$ and $\rtimes$), that $\chi_1 \times \chi_2 \rtimes 1$ is irreducible. Analogously proceeds the proof in the case $e(\chi_1) > 0$, $e(\chi_2) = 0$. If $e(\chi_1) = e(\chi_2)$, then we are in the case of unitary principal series, and then [K] implies the irreducibility.

Now the well known facts from the representation theory of $GL(2)$ and $SL(2)$ for the reducibility of principal series representations of these groups imply

4.1. Theorem ([T1], 7.1). *The principal series representation*

$$\chi_1 \times \cdots \times \chi_k \rtimes 1$$

is irreducible if and only if the following conditions hold

(1) χ_i *is not of order two* $(i = 1, \ldots, k)$*;*
(2) $\chi_i \neq \nu^{\pm 1}$ $(i = 1, \ldots, k)$*;*
(3) $\chi_i \neq \nu^{\pm 1} x_j^{\pm 1}$ $(1 \leq i < j \leq k$*; all possible combinations of* $+$ *and* $-$ *are allowed)*

Now we shall direct our attention to $Sp(2)$. If a principal series $\chi_1 \times \chi_2 \rtimes 1$ is irreducible, then one can write directly the Langlands parameter of this principal series representations. We shall focus now our attention to the composition series of principal series representations in the case of reducibility. From the above theorem follows that it is enough to know the reducibility and composition series of the following induced representations

$$\begin{array}{ll} \chi St_{GL(2)} \rtimes 1, & \chi 1_{GL(2)} \rtimes 1; \\ \chi \rtimes St_{Sp(1)}, & \chi \rtimes 1_{GL(2)}; \\ \chi \rtimes T_\xi^i, & i = 1, 2, \end{array}$$

where ξ is a character of order two, and $\xi \rtimes 1 = T_\xi^1 \oplus T_\xi^2$ is decomposition into a sum of tempered irreducible representations (for a reductive group G, St_G and 1_G denote respectively the Steinberg and the trivial representations of G). These representations, together with principal series representations, exhaust all representations parabolically induced from irreducible representations supported in the minimal parabolic subgroups (these representations we need also in the study of unitarizability).

5. A proof of irreducibility

We shall illustrate how one gets reducibility points on the example of

$$\chi \rtimes St_{Sp(1)}.$$

First we shall deal with the irreducibility. Write semi simplifications of Jacquet modules of $\chi \rtimes St_{Sp(1)}$ with respect to the maximal parabolic subgroups (this follows from a general result of and J. Bernstein and A.V. Zelevinsky, and W. Casselman):

$$\begin{array}{ccccc}
\text{s.s.}(s_{(1)}(\chi \rtimes St_{Sp(1)}))) & \longleftarrow & \chi \rtimes St_{Sp(1)} & \longrightarrow & \text{s.s.}(s_{(2)}(\chi \rtimes St_{Sp(1)}))) \\
\| & & & & \| \\
\chi \otimes St_{Sp(1)} + \chi^{-1} \otimes St_{Sp(1)} & & & & \\
+ \ \nu \otimes \chi \rtimes 1 & & & & \chi \times \nu \otimes 1 \ + \chi^{-1} \times \nu \otimes 1.
\end{array}$$

Assume $\chi^2 \neq 1_{F^\times}$ and $\chi \neq \nu^{\pm 1}, \nu^{\pm 2}$. Then all five representations in the above two sums are irreducible, in particular $\nu \otimes \chi \rtimes 1$, $\chi \times \nu \otimes 1$ and $\chi^{-1} \times \nu \otimes 1$ are irreducible. Now we shall concentrate our attention to the Jacquet modules of the last three representations:

$$\begin{array}{cccc}
—— + —— + \nu \otimes \chi \rtimes 1 & \chi \times \nu \otimes 1 & + & \chi^{-1} \times \nu \otimes 1 \\
\downarrow & \downarrow & & \downarrow \\
—— + —— + \nu \otimes \chi \otimes 1 + \nu \otimes \chi^{-1} \otimes 1 & \begin{array}{c} \chi \otimes \nu \otimes 1 + \\ \underline{\nu \otimes \chi \otimes 1} \end{array} & + & \begin{array}{c} \chi^{-1} \otimes \nu \otimes 1 + \\ \underline{\nu \otimes \chi^{-1} \otimes 1} \end{array}
\end{array}$$

Let π be an irreducible subquotient of $\chi \rtimes St_{Sp(1)}$ which has $\nu \otimes \chi \rtimes 1$ for a subquotient of the Jacquet module with respect to $P_{(1)}$. Then π must have $\nu \otimes \chi \otimes 1$ and $\nu \otimes \chi^{-1} \otimes 1$ as subquotients of the Jacquet module with respect to the minimal parabolic subgroup (see the above diagram). These characters are underlined members on the right hand side of the above diagram. Since these underlined members are coming from different subquotients of the Jacquet module with respect to $P_{(2)}$, we see that π must have $\chi \times \nu \otimes 1$ and $\chi^{-1} \times \nu \otimes 1$ in the Jacquet module. Since these two representations form the whole Jacquet module of $\chi \rtimes St_{Sp(1)}$, the exactness of the Jacquet modules implies $\pi = \chi \rtimes St_{Sp(1)}$. Thus, $\chi \rtimes St_{Sp(1)}$ is irreducible.

In the case of $\chi = \nu^{\pm 1}$ we have irreducibility, but this is more delicate to prove, and we shall not discuss this here (see Proposition 5.1 of [T3], or [R]).

6. A proof of reducibility

The representation $\chi \rtimes St_{Sp(1)}$ is reducible if $\chi = \nu^{\pm 2}$ or $\chi^2 = 1_{F^\times}$. We shall explain in the case $\chi = 1_{F^\times}$ how one can proof the reducibility. First recall that we have a natural partial order on semi simplifications of representations of finite length (in particular, the following simple fact holds: if π' is a subquotient of π, then $\text{s.s.}(\pi') \leq \text{s.s.}(\pi)$). From the representation theory of $SL(2)$ and $GL(2)$ we get directly

(6-1) $$\text{s.s.}(1_{F^\times} \rtimes St_{Sp(1)}) \leq \text{s.s.}(1_{F^\times} \times \nu \rtimes 1),$$

(6-2) $$\text{s.s.}(\nu^{1/2} St_{GL(2)} \rtimes 1) \leq \text{s.s.}(1_{F^\times} \times \nu \rtimes 1),$$

and further

$$\text{(6-3)} \qquad \text{s.s.}(s_{(2)}(1_{F^\times} \rtimes St_{Sp(1)})) \nleq \text{s.s.}(s_{(2)}(\nu^{1/2} St_{GL(2)} \rtimes 1)),$$

$$\text{(6-4)}\ \text{s.s.}(s_{(2)}(1_{F^\times} \rtimes St_{Sp(1)})) + \text{s.s.}(s_{(2)}(\nu^{1/2} St_{GL(2)} \rtimes 1)) \nleq \text{s.s.}(s_{(2)}(1_{F^\times} \times \nu \rtimes 1)).$$

The relations (6-1) – (6-4) and the exactness of the Jacquet modules imply the reducibility of $1_{F^\times} \rtimes St_{Sp(1)}$.

In a similar way, one shows the reducibility also in the other cases.

7. Reducibility points and composition series

First we shall write (among others) the lengths of the representations $\chi \rtimes St_{Sp(1)}$ which we have considered in the last section at the reducibility points, and the Langlands parameters of the irreducible subquotients at these points.

7.1. Proposition. *The representation $\chi \rtimes St_{Sp(1)}$ is reducible if and only if $\chi \rtimes 1_{Sp(1)}$ is reducible. We have the reducibility if and only if $\chi^2 = 1_{F^\times}$ or $\chi = \nu^{\pm 2}$. The following description of the composition series at the reducibility points holds.*

(1) *Let $\chi = \nu^2$. Then*

$$\begin{aligned} s.s.(\nu^2 \rtimes St_{Sp(1)}) &= L(\nu^2, St_{Sp(1)}) + St_{Sp(2)} \\ s.s.(\nu^2 \rtimes 1_{Sp(1)}) &= L(\nu^2, \nu, 1) + L(\nu^{3/2} St_{GL(2)}, 1). \end{aligned}$$

(2) *Let $\chi^2 = 1_{F^\times}, \chi \neq 1_{F^\times}$. Then $\chi \rtimes St_{Sp(1)}$ is a direct sum of two irreducible tempered representations (they can be described more precisely: one is a unique common irreducible subquotient of $\chi \rtimes St_{Sp(1)}$ and $\nu \rtimes T^1_\chi$, and the other is the unique common irreducible subquotient of $\chi \rtimes St_{Sp(1)}$ and $\nu \rtimes T^2_\chi$). Further*

$$\chi \rtimes 1_{Sp(1)} = L(\nu, T^1_\chi) \oplus L(\nu, T^2_\chi)$$

(Recall that $\chi \rtimes 1 = T^1_\chi \oplus T^2_\chi$ is a decomposition into a sum of irreducible representations.)

(3) *The representation $1_{F^\times} \rtimes St_{Sp(1)}$ is a sum of two inequivalent irreducible tempered representations (one can be described as a unique irreducible subquotient of $1_{F^\times} \rtimes St_{Sp(1)}$ and $\nu^{1/2} St_{GL(2)} \rtimes 1$, and the other as a unique irreducible subquotient of $1_{F^\times} \rtimes St_{Sp(1)}$ and $\nu^{1/2} 1_{GL(2)} \rtimes 1$). Further,*

$$1_{F^\times} \rtimes 1_{Sp(1)} = L(\nu, 1_{F^\times} \rtimes 1) \oplus L(\nu^{1/2} St_{GL(2)}, 1).$$

A reference for the above and the following two propositions is Proposition 5.4 of [SaT]. The description of the composition series in these propositions follows from the third section of [SaT] and Lemma 6.2 of [T1].

7.2. Proposition. *Let ξ be a fixed character of $F^\times$ of order two. Suppose that $\chi \notin \{\nu^{\pm 1}, \xi\nu^{\pm 1}\}$ and $\chi \neq \xi'$ for any character ξ' of $F^\times$ of order two different from ξ. Then and only then $\chi \rtimes T^i_\xi$ is irreducible. We have the following description of the Langlands parameters of the irreducible subquotients at the reducibility points.*

(1) *If ξ' is of order two and $\xi \neq \xi'$, then $\xi' \rtimes T_\xi$ splits into a direct sum of two inequivalent tempered representations (for a more precise description of these representations see (iii) of Theorem 5.2 in [SaT]).*

(2) *The representation $\nu \rtimes T^i_\xi$ has length two. One irreducible subquotient is $L(\nu, T^i_\xi)$, while the other one is an irreducible tempered representation (this tempered representation is described in (2) of Proposition 7.1 as a unique common irreducible subquotient of $\xi \rtimes St_{Sp(1)}$ and $\nu \rtimes T^i_\xi$).*

(3) *The representation $\xi\nu \rtimes T^i_\xi$ contains a unique irreducible subrepresentation, which we denote by*

$$\delta([\xi, \nu\xi], 1)_{T^i_\xi}.$$

This representation is square integrable. Further,

$$\begin{aligned} s.s.(\nu\xi \rtimes T^i_\xi) = & L(\xi\nu, T^i_\xi) + L(\nu^{1/2}\xi St_{GL(2)}, 1) \\ & + \delta([\xi, \nu\xi], 1)_{T^i_\xi}. \end{aligned}$$

7.3. Proposition. *The representation $\chi St_{GL(2)} \rtimes 1$ reduces if and only if $\chi 1_{GL(2)} \rtimes 1$ reduces. The reducibility happens if and only if $\chi = \nu^{\pm 3/2}$ or $\chi = \nu^{\pm 1/2}\xi$ for some character ξ of $F^\times$ satisfying $\xi^2 = 1_{F^\times}$. We have*

(1) $s.s.(\nu^{3/2} St_{GL(2)} \rtimes 1) = L(\nu^{3/2} St_{GL(2)}, 1) + St_{Sp(2)}$,
$s.s.(\nu^{3/2} 1_{GL(2)} \rtimes 1) = L(\nu^2, \nu, 1) + L(\nu^2, St_{Sp(1)})$.

(2) $s.s.(\nu^{1/2} St_{GL(2)} \rtimes 1) = L(\nu^{1/2} St_{GL(2)}, 1)$ + *irreducible tempered representation (this representation is described in (3) of Proposition 7.1 as a unique common irreducible subquotient of* $\nu^{1/2} St_{GL(2)} \rtimes 1$ and $1_{F^\times} \rtimes St_{Sp(1)}$*),*
$s.s.(\nu^{1/2} 1_{GL(2)} \rtimes 1) = L(\nu, St_{Sp(1)})$ + *irreducible tempered representation (this representation is described in (3) of Proposition 7.1 as a unique irreducible subquotient of* $\nu^{1/2} 1_{GL(2)} \rtimes 1$ *and* $1_{F^\times} \rtimes St_{Sp(1)}$*).*

(3) *If ξ is of order two, then*
$s.s.(\xi\nu^{1/2} St_{GL(2)} \rtimes 1) = \delta([\xi, \nu\xi], 1)_{T^1_\xi} + \delta([\xi, \nu\xi], 1)_{T^1_\xi} + L(\nu^{1/2}\xi St_{GL(2)}, 1)$,
$s.s.(\xi\nu^{1/2} 1_{GL(2)} \rtimes 1) = L(\nu\xi, T^1_\xi) + L(\nu\xi, T^2_\xi) + L(\nu^{1/2}\xi St_{GL(2)}, 1)$.
The representations $\delta([\xi, \nu\xi], 1)_{T^1_\xi}$ and $\delta([\xi, \nu\xi], 1)_{T^1_\xi}$ are not equivalent.

7.4. Remark. *Let ξ be a character of $F^\times$ of order two.*
(1) The claim (iii) of the above proposition implies the following three properties of the principal series representation $\pi = \nu\xi \times \xi \rtimes 1$ of $Sp(2)$ hold.

(i) *π has total length 6.*
(ii) *π has two inequivalent square integrable subquotients.*
(iii) *π has a subquotient of multiplicity two.*
(iv) *One can show that each irreducible subquotient of π is unitarizable (see (vii) and (viii) of Theorem 5.3 in [SaT]).*

Already from the example of this principal series representation we can notice the substantial difference which exists between the representation theory of general linear and symplectic groups.

(2) The standard module $\nu^{1/2}\xi St_{GL(2)} \rtimes 1$ contains both representations $\delta([\xi,\nu\xi]1)_{T_\xi^1}$ and $\delta([\xi,\nu\xi]1)_{T_\xi^2}$ as irreducible subrepresentations (recall that these subrepresentations are inequivalent). Therefore, this standard module (induced by a generic irreducible representation) can not have injective Whittaker model. Recall that a result of H. Jacquet and J.A. Shalika, each standard module of a general linear group has an injective Whittaker model. This is another significant difference with the representation theory of general linear groups. Both of the differences between the representation theory of general linear and symplectic groups that we mention, already shows up for $Sp(2)$.

After the above description of reducibilities of principal series representations, it is easy to wrote all square integrable representations supported in the minimal parabolic subgroup:

7.5. Proposition ([SaT], 5.1). *The Steinberg representation $St_{Sp(2)}$ and the representations $\delta([\xi,\nu\xi],1)_{T_\xi^i}$, $i = 1,2$, where ξ runs over all characters of $F^\times$ of order two, are square integrable. They are inequivalent and they exhaust all the irreducible square integrable representations of $Sp(2)$ supported in the minimal parabolic subgroup.*

Further, one can get easily from Propositions 7.1, 7.2 and 7.3 the classification of irreducible tempered representations of $Sp(2)$ which are supported in the minimal parabolic subgroup (see Theorem 5.2 of [SaT]).

8. Some comments regarding higher symplectic groups

A broad application of the ideas that we have outlined for studying reducibility and the composition series can be found in [T3] and [J] (the second paper contains the most complete results in that direction, while the first paper also considers general principles).

The simple square integrable representations $\delta([\xi,\nu\xi],1)_{T_\xi^i}$ that we have defined in the last section can be widely generalized. We shall give just one simple example example of their generalization. For a segment $\Delta = [\rho,\nu^k\rho] = \{\rho,\nu\rho,\nu^2\rho,\dots,\nu^k\rho\}$ of irreducible cuspidal representations of general linear groups, we shall denote by $\delta(\Delta)$ the essentially square integrable representation corresponding to it (it is a unique irreducible subrepresentations of $\nu^k\rho \times \nu^{k-1} \times \cdots \times \rho$).

8.1. Theorem. *Let ρ be an irreducible cuspidal representation of a general linear group. Denote by ω_ρ the central character of ρ. Suppose*

$$\rho \cong \tilde\rho \quad \text{and} \quad \omega_\rho \not\cong 1_{F^\times}.$$

Fix integers n and m satisfying $0 \le n < m$. Denote

$$\Delta = [\nu^{-n}\rho, \nu^m\rho].$$

Then $\delta(\Delta\cap\tilde\Delta)\rtimes 1$ decomposes into a direct sum of two inequivalent irreducible tempered representations $\tau_1 \oplus \tau_2$. Further, the representation $\delta(\Delta\setminus\tilde\Delta)\rtimes\tau_i$ (which is a standard module) has a unique irreducible subrepresentation, which we denote by $\delta(\Delta,1)\tau_i$. The representations $\delta(\Delta,1)\tau_i$ are square integrable and $\delta(\Delta,1)_{\tau_1} \not\cong \delta(\Delta,1)_{\tau_2}$.

8.2. Remark. *Let us turn our attention for a moment to the simplest case, when ρ is an unramified character of $F^\times$ (of order two). Then the space of Iwahori fixed vectors in $\delta(\Delta,1)_{\tau_i}$ has dimension*

$$\sum_{k=0}^{n}\binom{m+n+1}{k}.$$

One can construct square integrable representations $\delta(\Delta,\sigma)_{\tau_i}$, similar to the above ones, in a much wider generality (see [T4]). Further, one can further attach square integrable representations $\delta(\Delta_1,\dots,\Delta_k,\sigma)_{\tau_j}$ to several segments, using above representations $\delta(\Delta,\sigma)_{\tau_i}$ (see [T5]).

It is not hard to describe which of the representations $\delta(\Delta_1,\dots,\Delta_k,\sigma)\tau_j$ is generic (σ needs to be a generic representation). G. Muić has described them, and he has shown that each generic irreducible square integrable representation of a symplectic group can be obtained in this way ([M1] and [M2]). Actually, G. Muić has obtained much more (construction of generic dual modulo cuspidals). Let us note that generic irreducible square integrable representation make a very small part of the representations $\delta(\Delta_1,\dots,\Delta_k,\sigma)\tau$.

9. Unitary dual of $Sp(2)$ supported in the minimal parabolic subgroup

Inducing irreducible unitarizable representations of $M_{(1)}$ and $M_{(2)}$ supported in the minimal parabolic subgroup and decomposing them into irreducible ones, one shall get irreducible unitarizable representations of $Sp(2)$ supported in the minimal parabolic. Besides there irreducible unitarizable representations, one has square integrable representations and the trivial representation, which are obviously unitarizable. The question is if this is the whole unitary dual supported in the minimal parabolic subgroup. The answer is no. Not too many unitarizable representations are out of this list. The main news is a two parameter family of complementary series, induced from the minimal parabolic subgroup (and representations in its closure). The precise description of the unitarizable representations of $Sp(2)$ supported in the minimal parabolic subgroup is given in Theorem 5.2 of [SaT]:

9.2. Theorem (F. Rodier, P.J. Sally and M. T.). *Let χ and ξ be unitary characters of $F^\times$. Suppose $\xi^2=1_{F^\times}$. Let $\beta,\beta_1,\beta_2>0$. The following groups of irreducible representations of $Sp(2)$ are unitarizable, they are supported in the minimal parabolic subgroups and they exhaust all the irreducible unitarizable representations of $Sp(2)$ supported in the minimal parabolic subgroups.*

(i) *Irreducible tempered representations of $Sp(2)$ supported in the minimal parabolic subgroup (they are described in Theorem 5.2 of [SaT]).*
(ii) $L((\nu^2,\nu,1))=1_{Sp(2)}$.
(iii) $L((\nu^\beta\chi,\nu^\beta\chi^{-1},1))$, $\beta\leqslant 1/2$, $\chi^2\neq 1_{F^\times}$.
(iv) $L((\nu^\beta,\chi\rtimes 1))$, $\beta\leqslant 1$, *χ is not of order two.*
(v) $L((\nu^\beta,T_\xi^i))$, $\xi\neq 1_{F^\times}$, $\beta\leqslant 1$, $i\in\{1,2\}$.
(vi) $L((\nu^{\beta_1}\xi,\nu^{\beta_2}\xi,1))$, $\beta_1+\beta_2\leqslant 1$, $\beta_1\geqslant\beta_2$.
(vii) $L((\nu^\beta\xi,T_\xi^i))$, $\beta\leqslant 1$, $\xi\neq 1_{F^\times}$, $i\in\{1,2\}$.
(viii) $L((\nu^\beta\xi St_{GL(2)},1))$, $\beta\leqslant 1/2$. □

The proof of the theorem goes in the usual way. First one restricts attention to the Hermitian representations. Using the results about reducibility of parabolic induction, one organizes Hermitian representations into continuous families, and with the help of the intertwining operators, one gets continuous families of $Sp(2)$-invariant Hermitian forms. Once one has this done, the rest is easy in this case (there are no new isolated representations in the unitary dual, except the obvious ones: irreducible square integrable representations and $1_{Sp(2)}$). In [SaT] we have first solved the unitarizability problem for $GSp(2)$, and from it we have obtained the solution for $Sp(2)$.

10. Other parabolic subgroups

At the end, we shall recall of the reducibility results for the parabolically induced representations supported in non-minimal proper parabolic subgroups. From these results, one gets directly classifications of irreducible square integrable, tempered and unitarizable representations supported in these parabolic subgroups.

$\boxed{P_{(2)}}$

Let $\rho = \nu^\alpha \rho_0$ be an irreducible cuspidal representation of $GL(2)$, with ρ_0 unitarizable and $\alpha \in \mathbb{R}$. To have reducibility, one must have

$$\rho_0 \cong \tilde{\rho_0}$$

Conversely, $\rho_0 \cong \tilde{\rho_0}$ implies reducibility for some $\alpha \in \mathbb{R}$. If $\omega_{\rho_0} \neq 1_{F^\times}$, then the above type of condition (for the group $GSp(2)$ implies that $\mathrm{Ind}^{GSp(2)}(\nu^\alpha \rho_0 \otimes 1_F^\times)$ is irreducible for all $\alpha \in \mathbb{R}$. Considering the restriction of this representation to $Sp(2)$, this implies easily that the representation

$$\rho_0 \rtimes 1$$

of $Sp(2)$ reduces (it reduces into a sum of two inequivalent tempered representations), and that $\nu^\alpha \rho_0 \rtimes 1$ is irreducible for $\alpha \in \mathbb{R}^\times$. Here we do not get complementary series, neither we get square integrable subquotients.

If $\omega_{\rho_0} = 1_{F^\times}$, then there is no such an easy way to get reducibility points. F. Shahidi has shown that $\alpha = \pm 1/2$ is the only reducibility point (Proposition 6.1 of [Sh]; he has obtained above the reducibility using his L-functions methods). His result is proved for $\mathrm{char}\,(F) = 0$.

At the reducibility point $1/2$ we get a square integrable subquotient which we denote by $\delta(\nu^{1/2}, \rho)$ (recall $\omega_\rho = 1_{F^\times}$).

Here complementary series are

$$\nu^\beta \rho \rtimes 1, \quad 0 \leq \beta < 1/2 \quad (\omega_\rho = 1_{F^\times}).$$

Description of the tempered and the unitary dual corresponding to this parabolic subgroup is now clear (note that $L(\nu^{1/2}, \rho)$ is unitarizable for $\omega_\rho = 1_{F^\times}$, since this representation is at the closure of the complementary series).

$\boxed{P_{(1)}}$

For this parabolic subgroup, the cuspidal reducibilities were described by J.-L. Waldspurger in [W] (for $GSp(2)$), and later by F. Shahidi. We shall now recall of them. Let σ be an irreducible cuspidal representations of $Sp(1)$, and let $\chi = \nu^\alpha \chi_0$ be a character of $F^\times$, where $\alpha \in \mathbb{R}$ and χ_0 is a unitary character. The reducibility of $\chi \rtimes \sigma$ implies $\chi_0^2 = 1_{F^\times}$. Conversely, $\chi_0^2 = 1_{F^\times}$ implies reducibility for same $\alpha \in \mathbb{R}$.

For $a \in F^\times$ consider the representation

$$\sigma_a : g \mapsto \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Denote

$$F_\sigma^\times = \{a \in F^\times, \sigma \cong \sigma_a\}$$

(each $\varphi \in (F^\times / F_\sigma^\times)^\wedge$ satisfies $\varphi^2 = 1_{F^\times}$).

The list of all reducibility points of $\nu^\alpha \chi_0 \rtimes 1$ is:

(i) $\chi_0 = 1_{F^\times}$ and $\alpha = 0$;
(ii) $\chi_0^2 = 1_{F^\times},\ \chi_0 \notin (F^\times / F_\sigma^\times)^\wedge$ and $\alpha = 0$;
(iii) $\chi_0 \in (F^\times / F_\sigma^\times)^\wedge \setminus \{1_{F^\times}\}$ and $\alpha = \pm 1$.

Only the case (ii) can be settled elementary, similarly as one case of the reducibility for $P_{(2)}$.

We get irreducible square integrable subquotients only in the case (iii) (the square integrable representation occurring in this reducibility is denoted by $\delta(\nu\chi_0, \sigma)$). Here we get also complementary series

$$\nu^\alpha \chi_0 \rtimes \sigma, 0 \le \alpha < 1$$

(again $L(\nu\chi_0, \sigma)$ is unitarizable, since it is in the closure of the complementary series). In the other two cases, we do not get complementary series. At the reducibility points at these cases, we get a direct sums of two inequivalent irreducible tempered representations of $Sp(2)$.

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