ON SQUARE INTEGRABLE REPRESENTATIONS OF CLASSICAL p-ADIC GROUPS

Marko Tadić

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Introduction

Let F be a local non-archimedean field. We shall assume $\operatorname{char} F \neq 2$. Irreducible square integrable representations of a reductive groups are basic for understanding of the Plancherel measure. Besides this, their classification is an important step toward classification of the non-unitary duals. The aim of this paper is to present a construction of a wide family of non-cuspidal irreducible square integrable representations of $Sp(n, F)$ and $SO(2n+1, F)$.

To describe our result, we shall first introduce some notation. The modulus character of F is denoted by $|\ \ |_F$. Set $\nu = |\det|_F$. Basing on the fact that Levi factor of a maximal parabolic subgroup of a general linear group is a product of two smaller general linear groups, using the parabolic induction Bernstein and Zelevinsky defined multiplication $\times$ among representations of general linear groups (see [Z1], or the first section). Let $\mathcal{C}$ be the set of all equivalence classes of irreducible cuspidal representations of all $GL(p, F)$, $p \geq 1$. For $\rho \in \mathcal{C}$ and $n \geq 0$, the set $[\rho, \nu^n\rho] = \{\rho, \nu\rho, \dots, \nu^n\}$ is called a segment in $\mathcal{C}$. The set of all such segments is denoted by $\mathcal{S}(\mathcal{C})$. For $\Delta = [\rho, \nu^n\rho] \in \mathcal{S}(\mathcal{C})$ the representation $\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \nu\rho \times \rho$ contains a unique irreducible subrepresentation, which we denote by $\delta(\Delta)$. This subrepresentation is essentially square integrable and $\Delta \mapsto \delta(\Delta)$ is a bijection of $\mathcal{S}(\mathcal{C})$ onto the set of all classes of irreducible essentially square integrable representations of general linear groups (see [Z1]). For an irreducible essentially square integrable representation of a general linear group there exists a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Set $\delta^u = \nu^{-e(\delta)}\delta$.

We fix one of families $Sp(m, F)$ $(m \geq 0)$ or $SO(2m+1, F)$ $(m \geq 0)$ of classical groups. The group of rank m from the fixed family will be denoted by S_m. Levi factor of a maximal parabolic subgroup of S_m is isomorphic to $GL(k, F) \times S_{m-k}$, with $1 \leq k \leq m$. Now similarly as in the case of general linear groups, using the parabolic induction, one can introduce multiplication $\rtimes$ among representations of general linear groups and representations of groups S_m's. Products are representations of groups S_m (see the first section).

[1] Two minor modifications has been made to the paper after this date. Namely, we have droped two expectations that we had in the first version (understanding of the field forced us to change these expectations).

Now we can state our main result. To simplify the exposition, we shall present it in the non-degenerate case when $\mathrm{char}(F) = 0$ (these assumptions have roots in [Sh1]). Later, we shall describe the generality that we consider in the paper.

I.1. Theorem. *Suppose char $(F) = 0$. Let $\Delta_1, \dots, \Delta_k \in \mathcal{S}(\mathcal{C})$ satisfy $e(\delta(\Delta_i)) > 0$ for $1 \le i \le k$, and let σ be a non-degenerate irreducible cuspidal representation of S_q. Suppose that*

1. *$\delta(\Delta_i) \rtimes \sigma$ reduces, and if $\Delta_i \cap \tilde{\Delta}_i \neq \emptyset$, then $\delta(\Delta_i \cap \tilde{\Delta}_i) \rtimes \sigma$ reduces ($1 \le i \le k$).*
2. *If $\Delta_i \cap \Delta_j \neq \emptyset$, for some $1 \le i < j \le k$, then either $\Delta_i \cup \tilde{\Delta}_i \gneqq \Delta_j \cap \tilde{\Delta}_j$, or $\Delta_j \cup \tilde{\Delta}_j \gneqq \Delta_i \cap \tilde{\Delta}_i$.*

Denote $l = \mathrm{card}\{i; 1 \le i \le k$ and $\Delta_i \cap \tilde{\Delta}_i \neq \emptyset\}$. Then:
(i) Each irreducible subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$ has multiplicity one. There exist exactly 2^l irreducible subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$.
(ii) $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ decomposes into sum $\oplus_{j=1}^{2^l} \tau_j$ of 2^l inequivalent irreducible (tempered) representations. Each representation $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau_j$ has a unique irreducible subrepresentation, which we denote by

$$\delta(\Delta_1, \dots, \Delta_k, \sigma)_{\tau_j}.$$

Representations $\delta(\Delta_1, \dots, \Delta_k, \sigma)_{\tau_j}$ are square integrable.
(iii) $\{\delta(\Delta_1, \dots, \Delta_k, \sigma)_{\tau_j}; i = 1, \dots, 2^l\}$ is just the set of all irreducible representations of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$.

For further understanding of representations $\delta(\Delta_1, \dots, \Delta_k, \sigma)_{\tau_j}$, it is important to know when $\delta(\Delta) \rtimes \sigma$ reduces. The following theorem reduces this problem to the cuspidal case:

I.2. Theorem ([T7]). *Suppose char $(F) = 0$. Let $\Delta \in \mathcal{S}(\mathcal{C})$ and let σ be a non-degenerate irreducible cuspidal representation of S_q. Then,*

$$\delta(\Delta) \rtimes \sigma \text{ reduces if and only if } \rho \rtimes \sigma \text{ reduces for some } \rho \in \Delta.$$

About reducibility in the cuspidal case, we have the following nice result of F. Shahidi:

I.3. Theorem (F. Shahidi, [Sh1], [Sh2]). *Assume char $(F) = 0$. Let $\rho \in \mathcal{C}$, let σ be a non-degenerate irreducible cuspidal representation of S_q and let $\beta \in \mathbb{R}$. If $\rho \ncong \tilde{\rho}$, then $\nu^\beta \rho \rtimes \sigma$ is irreducible. Suppose $\rho \cong \tilde{\rho}$. Then:*

$$\text{(C)} \qquad \text{There exists } \alpha_0 \in \{0, 1/2, 1\} \text{ such that } \nu^{\pm\alpha_0}\rho_0 \rtimes \sigma \text{ reduce,}$$
$$\text{and } \nu^\beta \rho \rtimes \sigma \text{ is irreducible for } |\beta| \neq \alpha_0.$$

The above theorem implies that (ρ, σ) satisfies exactly one of the following three conditions:

(C0) $\rho \rtimes \sigma$ reduces and $\nu^\beta \rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}^\times$;

(C1/2) $\nu^{\pm 1/2}\rho \rtimes \sigma$ reduce and $\nu^\beta \rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R} \backslash \{\pm 1/2\}$;

(C1) $\nu^{\pm 1}\rho \rtimes \sigma$ reduce and $\nu^\beta \rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R} \backslash \{\pm 1\}$.

It is not yet determined which of the above three conditions satisfies general (ρ, σ). F. Shahidi has determined this in a number of cases ([Sh2]). J.-L. Waldspurger has settled earlier one such case ([W]). G. Muić has settled recently some new cases ([M]).

Regarding condition (1) in Theorem I.1, one easily sees from above two theorems that reducibility of $\delta(\Delta) \rtimes \sigma$ and $\Delta \cap \tilde{\Delta} \neq \emptyset$ implies reducibility of $\delta(\Delta \cap \tilde{\Delta}) \rtimes \sigma$ in most cases. The only exception is the case when $\Delta = [\rho, \nu^k \rho]$ such that ρ is unitarizable, (ρ, σ) satisfies (C1) and $k \geq 1$.

The following theorem of Goldberg gives a significant reduction of the problem of parameterization of irreducible tempered representations which we get parabolically inducing irreducible square integrable representations of groups S_m.

I.4. Theorem (D. Goldberg, [G]). *Suppose char* $(F) = 0$. *Let* $\delta_1, \dots, \delta_k$ *be irreducible square integrable representations of general linear groups, and let* π *be an irreducible square integrable representations of* S_q. *Let* a *be a number of inequivalent* δ_i *such that* $\delta_i \rtimes \pi$ *reduces. Then* $\delta_1 \times \cdots \times \delta_k \rtimes \pi$ *is a multiplicity one representation of length* 2^a.

This theorem reduces the tempered representations to the case $\delta_i \rtimes \pi$. The claim in (ii) of Theorem I.1 that $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ decomposes into sum of 2^l inequivalent irreducible representations, is a special case of the above much more general result.

Regarding parameterization of irreducible tempered representations, let us say that we have determined in most cases reducibility of $\delta \rtimes \delta(\Delta_1, \dots, \Delta_k, \sigma)_{\tau_j}$ where δ is an irreducible square integrable representation of a general linear group (this work is not yet available in a form of preprint).

Let us now explain the role of non-degeneracy of σ in Theorem I.1. We need it to know that pairs (ρ, σ) ($\rho \in \mathcal{C}$ selfdual), with which we work in the theorem, satisfy condition (C). Our approach in the paper is converse. We do not assume that σ is non-degenerate. Instead, we work with pairs (ρ, σ) which satisfy condition (C) (in this way we do not need in the paper Theorems I.2, I.3 and I.4).

Let us say a few words about the methods that we use in the construction. We have constructed in [T4] the structure which enables us to obtain in a simple way composition series of Jacquet modules of parabolically induced representations. The fact that Levi factors of maximal parabolic subgroups of S_m are isomorphic to products of general linear groups and groups S_q, enables us to use the full power of Bernstein's and Zelevinsky's theory from [Z1] in the representations theory of $Sp(n, F)$ and $SO(2n + 1, F)$.

Although our work in this paper deals with representations of groups $Sp(n, F)$ and $SO(2n + 1, F)$, this work is also directed to other classical groups. Namely, we expect that a significant part of present work will apply to other series of classical groups, once when the structure of representations of these groups over representations of general linear groups will be clarified (this work is under the way).

The first two sections of this paper introduce notation and recalls of some previous results that we use often in the paper. The most of this paper deals with construction of representations $\delta(\Delta_1, \dots, \Delta_k, \sigma)_{\tau_j}$ in the case of $k = 1$ (we also obtain a number of interesting information about representations $\delta(\Delta)_{\tau_j}$). We proceed with construction in three separate cases: (C1/2), (C0) and (C1). Although the general strategy of construction of $\delta(\Delta)_{\tau_j}$ in all three cases is the same, there is a plenty of delicate details which are different in these cases. This is the reason to treat these cases independently. Sections 3, 4, 5, 6 and 7 present construction of representations $\delta(\Delta,\sigma)_{\tau_j}$ corresponding to different type of reducibility that can occur . The eight section is an observation in which way representations $\delta(\Delta,\sigma)_{\tau_j}$ are complete. In the ninth section is construction of representations $\delta(\Delta_1,\dots,\Delta_k,\sigma)_{\tau_j}$ for $k>1$.

D. Vogan has shown us where he expects to have square integrable representations for symplectic groups (having in mind the local Langlands philosophy). This was one of the motivations to construct such representations using the techniques developed in [T5]. The other motivation for our work is getting of a parameterization of the non-unitary dual (convenient for the work on the unitarizability problem).

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1. Preliminaries

We fix in this paper a local non-archimedean field F of characteristic different from two. At the beginning of this section we shall recall of the standard notation from the representation theory of $GL(n,F)$ (see [Z1] for complete definitions). The minimal parabolic subgroup of $GL(n,F)$ consisting of all upper triangular matrices in $GL(n,F)$ is fixed. Parabolic subgroups of $GL(n,F)$ which contain this minimal parabolic subgroup will be called standard parabolic subgroups of $GL(n,F)$.

Let π_i be an admissible representation of $GL(n_i,F)$, for $i=1,2$. Then $\pi_1\times\pi_2$ denotes the representation of $GL(n_1+n_2,F)$ which parabolically induces the representation $\pi_1\otimes\pi_2$ of a suitable standard parabolic subgroup. Then $\pi_1\times(\pi_2\times\pi_3)\cong(\pi_1\times\pi_2)\times\pi_3$.

If G is a reductive group over F, then there is always a natural order on the Grothendieck group of the category of all admissible representations of G of finite length. We shall denote by $\tilde{G}$ the set of all equivalence classes of irreducible admissible representations of G. The set of unitarizable classes in $\tilde{G}$ is denoted by $\hat{G}$.

Denote the Grothendieck group of the category of all admissible representations of $GL(n,F)$ of finite length by R_n. The canonical mapping from the objects of the category to R_n is denoted by s.s. (the image forms a cone of positive elements). Set $R=\oplus_{n\geq 0}R_n$. One lifts the above multiplication to a multiplication $\times$ on R. The induced mapping $R\otimes R\to R$ is denoted by m.

Take an admissible representation π of $GL(n,F)$ of finite length. Let $\alpha=(n_1,\dots,n_k)$ be an ordered partition of n. Take the standard parabolic subgroup P_α^{GL} of $GL(n,F)$ whose Levi factor M_α^{GL} is naturally isomorphic to $GL(n_1,F)\times\ldots\times GL(n_k,F)$. The Jacquet module of π with respect to P_α^{GL} is denoted by $r_\alpha(\pi)$. Consider s.s. $(r_\alpha(\pi))\in R_{n_1}\otimes\ldots\otimes R_{n_k}$. Set

$$m^*(\pi)=\sum_{k=0}^{n}\text{s.s. }\left(r_{(k,n-k)}(\pi)\right)\in R\otimes R.$$

One lifts m^* $\mathbb{Z}$-linearly to all of R.

For a matrix g denote by ${}^t g$ (resp. ${}^\tau g$) the transposed matrix of g (resp. the transposed matrix of g with respect to the second diagonal). For a representation π of $GL(n,F)$, ${}^\tau\pi^{-1}$ denotes the representation $g \mapsto \pi({}^\tau g^{-1})$. We denote by $\tilde{\pi}$ the contragredient representation of π. We have ${}^\tau\pi^{-1} \cong \tilde{\pi}$ for irreducible π.

Let π be an irreducible admissible representation of $GL(n,F)$. If π is a subquotient of $\rho_1 \times \cdots \times \rho_k$ where ρ_i are irreducible cuspidal representations of $GL(n_i,F)$, then we shall call the multiset $(\rho_1,\dots,\rho_k)$ the support of π. We write $\mathrm{supp}(\pi) = (\rho_1,\dots,\rho_k)$. If π is of finite length and if any irreducible subquotient π' of π has $\mathrm{supp}(\pi') = (\rho_1,\dots,\rho_k)$, then we say that π has a support and we shall write $\mathrm{supp}(\pi) = (\rho_1,\dots,\rho_k)$. Suppose $\pi \in R_n,\ \pi > 0$. Similarly as above, we define if π has support (there is a natural order on R_n's).

We now introduce a similar notation for two series of classical groups (see [T2] and [T4]). The $n \times n$ matrix having 1's on the second diagonal and all other entries 0, will be denoted by J_n. The identity matrix is denoted by I_n. For a $2n \times 2n$ matrix S set

$$^\times S = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^t S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}.$$

The group $Sp(n,F)$ consists of all $2n \times 2n$ matrices over F which satisfy ${}^\times S\, S = I_{2n}$. We define $Sp(0,F)$ to be the trivial group. Fix the minimal parabolic subgroup $P_{\min}$ in $Sp(n,F)$ consisting of all upper triangular matrices in the group.

We denote by $SO(2n+1,F)$ the group of all $(2n+1)\times(2n+1)$ matrices X of determinant one with entries in F, which satisfy ${}^\tau X\, X = I_{2n+1}$. Fix the minimal parabolic subgroup $P_{\min}$ in $SO(2n+1,F)$ consisting of all upper triangular matrices in the group.

In the sequel, we denote by S_n either the group $Sp(n,F)$ of $SO(2n+1,F)$. Parabolic subgroups which contain the minimal parabolic subgroup which we have fixed, will be called standard parabolic subgroups.

For $p_i \times p_i$ matrices $X_i,\ i = 1,\dots,k$, the quasi diagonal $(p_1 + \cdots + p_k) \times (p_1 + \cdots + p_k)$ matrix which has on the quasi diagonal matrices $X_1,\cdots,X_k$, is denoted by q-diag $(X_1,\cdots,X_k)$.

Let $\alpha = (n_1,\dots,n_k)$ be an ordered partition of some non-negative integer $m \le n$ into positive integers. If $m = 0$, then the only partition will be denoted by (0). Set

$$M_\alpha = \left\{ \text{q-diag}\,(g_1,\cdots,g_k,h,\ {}^\tau g_k^{-1},\cdots,\ {}^\tau g_1^{-1}); g_i \in GL(n_i,F), h \in S_{n-m} \right\}$$

Then $P_\alpha = M_\alpha P_{\min}$ is a standard parabolic subgroup of S_n. The unipotent radical of P_α is denoted by N_α. Since M_α is naturally isomorphic to $GL(n_1,F) \times \ldots \times GL(n_k,F) \times S_{n-m}$, we have a natural bijection

$$\tilde{M_\alpha} \quad \leftrightarrow \quad GL(n_1,F)\tilde{} \times \cdots \times GL(n_k,F)\tilde{} \times \tilde{S}_{n-m}.$$

Let π be an admissible representation of $GL(n,F)$ and let τ be a similar representations of S_m. We denote by $\pi \rtimes \sigma$ the representation of S_{n+m} which parabolically induces the representation $\pi \otimes \sigma$ of $P_{(n)}$. Here $\pi \otimes \sigma$ maps q-diag$(g, h, {}^\tau g^{-1}) \in M_{(n)}$ to $\pi(g) \otimes \sigma(h)$. For admissible representations π, π_1, π_2 of general linear groups and for a similar representation σ of S_m hold

$$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma, \tag{1-1}$$

$$(\pi \rtimes \sigma)^{\sim} \cong \tilde{\pi} \rtimes \tilde{\sigma}. \tag{1-2}$$

The Grothendieck group of the category of all admissible representations of S_n of finite length is denoted by $R_n(S)$. Denote $R(S) = \oplus_{n \geq 0} R_n(S)$. We lift the multiplication $\rtimes$ to a multiplication $\rtimes : R \times R(S) \to R(S)$ in a usual way. In this way $R(S)$ becomes an R-module. Denote the contragredient involution on R and $R(S)$ by $^{\sim}$. For $\pi \in R$ and $\sigma \in R(S)$ we have

$$\pi \rtimes \sigma = \tilde{\pi} \rtimes \sigma. \tag{1-3}$$

Let $\mu : R \otimes R(S) \to R(S)$ be the $\mathbb{Z}$-bilinear mapping which satisfies $\mu(\pi \otimes \sigma) = \text{ s.s.}(\pi \rtimes \sigma)$, for $\pi \in R, \sigma \in R(S)$. Since we have natural orders on Grothendieck groups, there is a natural order on R, $R(S)$ and $R \otimes R(S)$.

Let σ be a smooth representation of S_n of finite-length and let $\alpha = (n_1, \dots, n_k)$ be an ordered partition of $0 \leq m \leq n$. The Jacquet module of σ for P_α is denoted by $s_\alpha(\sigma)$. We may consider $\text{s.s.}\,(s_\alpha(\sigma)) \in R_{n_1} \otimes \cdots \otimes R_{n_k} \otimes R_{n-m}(S)$. Define a $\mathbb{Z}$-linear mapping $\mu^* : R(S) \to R \otimes R(S)$ on the basis of irreducible admissible representations by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}\left(s_{(k)}(\sigma)\right).$$

Denote by $s : R \otimes R \to R \otimes R$ the mapping $s(\sum_i x_i \otimes y_i) = \sum_i y_i \otimes x_i$. For $r_1 \otimes r_2 \in R \otimes R$ and $r \otimes t \in R \otimes R(S)$ set $(r_1 \otimes r_2) \rtimes (r \otimes t) = (r_1 \times r) \otimes (r_2 \rtimes t)$. Extend $\rtimes$ $\mathbb{Z}$-bilinearly to $\rtimes : (R \otimes R) \times (R \otimes R(S)) \to R \otimes R(S)$. Set

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*.$$

Then

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma) \tag{1-4}$$

for an admissible representation π of $GL(n, F)$ of finite length and a similar representation of S_m.

Let $\pi \otimes \sigma$ be an admissible representation of $GL(n, F) \times S_m$. We say that $\pi \otimes \sigma$ has GL-support if π has support and if σ is an irreducible cuspidal representation. Then we write

$$\text{supp}_{GL}(\pi \otimes \sigma) = \text{supp}(\pi).$$

Let $\pi \otimes \sigma \in R_n \otimes R_m$. Suppose that $\pi > 0$ and that σ is an irreducible cuspidal representation. Similarly as above, one defines if $\pi \otimes \sigma \in R_n \otimes R_m(S)$ has a GL-support.

Suppose that τ is an irreducible admissible representation of S_m. Then there exists an irreducible cuspidal representations ρ_i of $GL(n_i, F)$, $i = 1, \dots, k$, and an irreducible cuspidal representation σ of $S_{m-(n_1+\dots+n_k)}$ such that τ is a subquotient of $\rho_1 \times \dots \times \rho_k \rtimes \sigma$. We define

$$\text{depth}_{GL}(\tau) = n_1 + \dots + n_k.$$

If τ is a an admissible representation of S_m of finite length such that $\text{depth}_{GL}(\tau') = d$ for any irreducible subquotient τ' of τ, then we say that τ has a depth and we write $\text{depth}_{GL}(\pi) = d$. In a similar way we define depth of $\tau \in R_n(S)$, $\tau > 0$. If an admissible representation τ of finite length has a depth, then we denote

$$s_{GL}(\tau) = s_{(\text{depth}_{GL}(\tau))}(\tau).$$

In a similar way we define $s_{GL}(\tau)$ for $\tau \in R_n(S)$, $\tau > 0$, if τ has a depth.

2. Square integrability, Langlands' classification

The set of all equivalence classes of irreducible cuspidal representations of all $GL(p, F)$, $p \geq 1$, will be denoted by $\mathcal{C}$. Let $\rho \in \mathcal{C}$ and let n be a non-negative integer. The set $[\rho, \nu^n\rho] = \{\rho, \nu\rho, \nu^2\rho, \dots, \nu^n\rho\}$ is called a segment in irreducible cuspidal representations of general linear groups, or a segment in $\mathcal{C}$. The set of all segments in $\mathcal{C}$ will be denoted by $\mathcal{S}(\mathcal{C})$. The representation $\nu^n\rho \times \nu^{n-1}\rho \times \dots \times \nu\rho \times \rho$ has a unique irreducible subrepresentation which we denote by $\delta([\rho, \nu^n\rho])$. The representation $\delta([\rho, \nu^n\rho])$ is an essentially square integrable representation and $\Delta \mapsto \delta(\Delta)$ is a bijection of $\mathcal{S}(\mathcal{C})$ onto the set of all equivalence classes of essentially square integrable representations of all $GL(k, F)$, $k \geq 0$.

If $n < 0$, then we define $[\rho, \nu^n\rho]$ to be the empty set $\emptyset$, and we take $\delta(\emptyset)$ to be $1 \in R$. Now we have from [Z1]

$$m^*\left(\delta([\rho, \nu^n\rho])\right) = \sum_{k=-1}^{n} \delta([\nu^{k+1}\rho, \nu^n\rho]) \otimes \delta([\rho, \nu^k\rho]). \tag{2-1}$$

This formula implies that $s\left(m^*\left(\delta([\rho, \nu^n\rho])\right)\right) = \sum_{k=-1}^{n} \delta([\rho, \nu^k\rho]) \otimes \delta([\nu^{k+1}\rho, \nu^n\rho])$. We have $r_{(m)^{n+1}}\left(\delta([\rho, \nu^n\rho])\right) = \nu^n\rho \otimes \nu^{n-1}\rho \otimes \dots \otimes \rho$, where $(m)^{n+1}$ denotes $(m, m, \dots, m) \in \mathbb{Z}^{n+1}$.

Let X be a set. We shall denote by M(X) the set of all finite multisets in X (more details regarding this notation one can find in [Z1] and [Z2]). The addition among multisets is defined by $(x_1, \dots, x_k) + (x'_1, \dots, x'_{k'}) = (x_1, \dots, x_k, x'_1, \dots, x'_{k'})$. If $a, b, c \in M(X)$ and $a + b = c$, then we shall denote a also by $c - b$.

For an irreducible essentially square integrable representation δ of $GL(m, F)$ one can find a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Set $\delta^u = \nu^{-e(\delta)}\delta$. Then $\delta = \nu^{e(\delta)}\delta^u$ where $e(\delta) \in \mathbb{R}$ and δ^u is unitarizable.

An irreducible representation π of a reductive p-adic group G is called essentially square integrable, if there exists a continuous (not necessarily unitary character) $\chi : G \to \mathbb{C}^\times$ such that $\chi\pi$ is a square integrable representation (i.e., $\chi\pi$ has a unitary central character, and for any matrix coefficient ϕ of $\chi\pi$, $|\phi|$ is a square integrable function on G modulo center).

We denote by D the set of all equivalence classes of the irreducible essentially square integrable representations of $GL(n,F)$'s when $n \geq 1$. Let $d = (\delta_1, \dots, \delta_k) \in M(D)$ where $M(D)$ denotes the set of all finite multisets in D. Take a permutation p of the set $\{1, \dots, k\}$ such that $e(\delta_{p(1)}) > e(\delta_{p(2)}) \cdots > e(\delta_{p(k)})$. The representation $\delta_{p(1)} \times \cdots \times \delta_{p(k)}$ has a unique irreducible quotient which we denote by $L(d)$. Then $d \mapsto L(d)$ is Langlands' classification for general linear groups. We shall usually write $L(d) = L((\delta_1, \dots, \delta_k))$ simply as $L(\delta_1, \dots, \delta_k)$.

In this paper we shall use several times the following well-known fact proved by A.V. Zelevinsky ([Z1]). For two segments Δ', $\Delta'' \in \mathcal{S}(\mathcal{C})$ one says that they are linked if $\Delta' \cup \Delta'' \in \mathcal{S}(\mathcal{C})$ and $\Delta' \cup \Delta'' \not\in \{\Delta', \Delta''\}$. Let $\Delta_1, \dots, \Delta_k \in CalS(\mathcal{C})$. If there exist $1 \leq i < j \leq k$ such that Δ_i and Δ_j are linked, then we shall write

$$(\Delta_1, \Delta_2, \dots, \Delta_{i-1}, \Delta_i \cup \Delta_j, \Delta_{i+1}, \dots, \Delta_{j-1}, \Delta_i \cap \Delta_j, \Delta_{j+1}, \dots, \Delta_{k-1}, \Delta_k) \\ \prec (\Delta_1, \Delta_2, \dots, \Delta_{k-1}, \Delta_k).$$

Generate by $\prec$ a partial order on $\mathcal{S}(\mathcal{C})$. Denote the obtained partial order by $\leq$. Let $\Delta_1', \dots, \Delta_{k'}' \in \mathcal{S}(\mathcal{C})$. Then $L(\delta(\Delta_1'), \dots, \delta(\Delta_{k'}'))$ is a subquotient of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k)$ if and only if $(\Delta_1', \dots, \Delta_{k'}') \leq (\Delta_1, \dots, \Delta_k)$. Suppose that $(\Delta_1', \dots, \Delta_{k'}') \leq (\Delta_1, \dots, \Delta_k)$ and suppose that among Δ_i', Δ_j', $\ 1 \leq i \neq j \leq k'$ there do not exist linked segments. Then $\delta(\Delta_1') \times \cdots \times \delta(\Delta_{k'}')$ is irreducible and it has multiplicity one in $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k)$.

Suppose that $\Delta_i, \Delta_j' \in \mathcal{S}(\mathcal{C})$, $1 \leq i \leq k, 1 \leq j \leq k'$. If Δ_i is not linked to any Δ_j', for $1 \leq i \leq k, 1 \leq j \leq k'$, then $L(\delta(\Delta_1), \dots, \delta(\Delta_k)) \times L(\delta(\Delta_1'), \dots, \delta(\Delta_{k'}'))$ is irreducible and $L(\delta(\Delta_1), \dots, \delta(\Delta_k)) \times L(\delta(\Delta_1'), \dots, \delta(\Delta_{k'}')) = L(\delta(\Delta_1), \dots, \delta(\Delta_k)\delta(\Delta_1'), \dots, \delta(\Delta_{k'}'))$.

Set $D_+ = \{\delta \in D; e(\delta) > 0\}$. Let $T(S)$ be the set of all equivalence classes of the irreducible tempered admissible representations of S_n's for all $n \geq 0$. Take $t = ((\delta_1, \dots, \delta_n), \tau) \in M(D_+) \times T(S)$ ($M(D_+)$ denotes the set of all finite multisets in D_+). Choose a permutation p of the set $\{1, 2, \dots, n\}$ such that $e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \ldots \geq e(\delta_{p(n)})$. The representation $\delta_{p(1)} \times \delta_{p(2)} \times \ldots \times \delta_{p(n)} \rtimes \tau$ has a unique irreducible quotient which we denote by $L(t)$. This is Langlands' classification for groups S_m. The mapping $t \mapsto L(t)$ is a one-to-one parameterization of all irreducible representations of groups S_m by $M(D_+) \times T(S)$. We shall usually write $L(t) = L(((\delta_1, \dots, \delta_n), \tau))$ simply as $L((\delta_1, \dots, \delta_n), \tau)$ or $L(\delta_1, \dots, \delta_n, \tau)$.

We recall the Casselman square integrability criterion in the case of S_n (see [C], and also [T2]). Consider the standard inner product on $\mathbb{R}^n$. Denote

$$\beta_i = (\underbrace{1, 1, \dots, 1}_{i \text{ times}}, 0, 0, \dots, 0) \in \mathbb{R}^n, \ 1 \leq i \leq n.$$

Let π be a non-cuspidal irreducible admissible representation of S_n. Take α such that $s_\alpha(\pi)$ has a cuspidal subquotient ($s_\alpha(\pi) \neq 0$). Write $\alpha = (n_1, \dots, n_\ell)$ and let $n_1 + \cdots + n_\ell = m$. Take an irreducible subquotient σ of s_α and decompose $\sigma = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_\ell \otimes \rho$ where $\rho_i \in GL(n_i, F)\tilde{}$, $\rho \in \tilde{S}_{n-m}$. Define

$$e_*(\sigma) = (\underbrace{e(\rho_1), \dots, e(\rho_1)}_{n_1 \text{ times}}, \dots, \underbrace{e(\rho_\ell), \dots, e(\rho_\ell)}_{n_\ell \text{ times}}, \underbrace{0, \dots, 0}_{n-m \text{ times}})$$

If π is square integrable, then

$$(e_*(\sigma), \beta_{n_1}) > 0, (e_*(\sigma), \beta_{n_1+n_2}) > 0, \cdots, (e_*(\sigma), \beta_m) > 0.$$

Conversely, if all above inequalities hold for any α and σ as above, then π is square integrable. If instead of >0 holds the weaker condition ≥ 0 in all above relations, then π is tempered.

Now we shall recall of the square integrable representations of Steinberg type ([T5]).

2.1. Theorem. *Fix an irreducible unitarizable cuspidal representation ρ of $GL(\ell, F)$ and fix a similar representation σ of S_m. Suppose that $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha > 0$. Then $\rho \cong \tilde{\rho}$. The representation $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \nu^{\alpha+1}\rho \times \nu^\alpha\rho \rtimes \sigma$ has a unique irreducible subrepresentation which we denote by $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ $(n \geq 0)$. We have $s_{(\ell)^{n+1}}(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)) = \nu^{\alpha+n}\rho \otimes \nu^{\alpha+n-1}\rho \otimes \cdots \otimes \nu^{\alpha+1}\rho \otimes \nu^\alpha\rho \otimes \sigma$ (here $(\ell)^{n+1} = (\ell, \ell, \dots, \ell) \in \mathbb{Z}^{n+1}$) and*

$$\mu^*\left(\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)\right) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^\alpha\rho, \nu^{\alpha+k}\rho], \sigma)$$

The representation $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)$ is square integrable and we have $\delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \sigma)\tilde{}$ $\cong \delta([\nu^\alpha\rho, \nu^{\alpha+n}\rho], \tilde{\sigma})$.

We take $\delta(\emptyset, \sigma)$ in the above formula to be just σ.

Let ρ be an irreducible unitarizable cuspidal representation of $GL(p, F)$ and let σ be an irreducible cuspidal representation of S_q. It is well-known that if $\nu^\alpha\rho \rtimes \sigma$ reduces for same $\alpha \in \mathbb{R}$, then $\rho \cong \tilde{\rho}$. One proves this in a similar way as in the GSp-case in [T2] (here the proof is even much simpler then there). The converse of this fact holds: if $\rho \cong \tilde{\rho}$, then $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha \in \mathbb{R}$. The argument is following. Suppose that $\rho \cong \tilde{\rho}$ and that $\rho \rtimes \sigma$ does not reduce. Then one can chose $\alpha_0 > 0$ such that $\nu^\alpha\rho \rtimes \sigma$ is irreducible for $0 \leq \alpha < \alpha_0$,. These representations are unitarizable (they form a complementary series). Since matrix coefficients of unitarizable representations are bounded and the Jacquet module is s.s.$(s_{(p)}(\nu^\alpha\rho \rtimes \sigma)) = \nu^\alpha\rho \otimes \sigma + \nu^{-\alpha}\rho \otimes \sigma$, the connection of asymptotic of matrix coefficients and Jacquet modules in [C] implies that there must exist $\alpha_0 > 0$ such that $\nu^{\alpha_0}\rho \rtimes \sigma$ reduces (one can even get an explicit upper bound for such α_0).

An admissible representation ρ shall be called selfdual if $\rho \cong \tilde{\rho}$. If representation is selfdual, then it is unitarizable. Let $\rho \in \mathcal{C}$ be selfdual, and let σ be an irreducible cuspidal representation of S_q. In this paper we shall deal with pairs (ρ, σ) which satisfy the following condition.

(C) There exists $\alpha_0 \in \{0, 1/2, 1\}$ such that $\nu^{\alpha_0}\rho \rtimes \sigma$ reduces,
and $\nu^\beta\rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}$, $|\beta| \neq \alpha_0$.

The condition (C) holds for any ρ, if $q = 0$ ([Sh2]).

If (ρ, σ) as above satisfies (C), then it satisfies exactly one of the following three conditions:

(C0) $\rho \rtimes \sigma$ reduces and $\nu^\beta \rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}^\times$;

(C1/2) $\nu^{1/2}\rho \rtimes \sigma$ reduces and $\nu^\beta \rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}\backslash\{\pm 1/2\}$;

(C1) $\nu\rho \rtimes \sigma$ reduces and $\nu^\beta \rho \rtimes \sigma$ is irreducible for $\beta \in \mathbb{R}\backslash\{\pm 1\}$

(we follow the notation of the Jantzen's paper [J]).

The following fact proved in [T5] explains why only selfdual irreducible cuspidal representations of general linear groups are interesting for the construction of irreducible square integrable representations of groups S_m.

2.2. Proposition. *Let $\rho_1, \rho_2, \dots, \rho_n \in \mathcal{C}$, and let σ be an irreducible cuspidal representation of S_q. Suppose that $\rho_1 \times \rho_2 \times \dots \times \rho_n \rtimes \sigma$ contains a square integrable subquotient. Then all ρ_i^u are selfdual representations.*

In [T5] we have got a number of other conditions which must be satisfied by $\rho_1, \rho_2, \dots, \rho_n$ and σ as above.

3. Reducibility at 1/2, I

We fix an irreducible unitarizable cuspidal representation ρ of $GL(p, F)$ and an irreducible cuspidal representation σ of S_q. We shall assume in this section that $\nu^{1/2}\rho \rtimes \sigma$ reduces (thus $\rho \cong \tilde{\rho}$), and that $\nu^\alpha \rho \cong \tilde{\rho}$ is irreducible for $\alpha \in \mathbb{R}\backslash\{\pm 1/2\}$. In other words, we assume that (ρ, σ) satisfies (C1/2).

3.1. Lemma. *Suppose that $m_1, m_2, \dots, m_k$ are integers which satisfy $m_1 \geq m_2 \geq m_3 \geq \dots \geq m_{k-1} \geq m_k \geq 0$. Let $\Delta_i = [\nu^{1/2}\rho, \nu^{m_i+1/2}\rho]$ and*

$$\tau = \nu^{1/2}\rho \times \nu^{3/2}\rho \times \dots \times \nu^{m_1+1/2}\rho \times \nu^{1/2}\rho \times \nu^{3/2}\rho \times \dots \times \nu^{m_2+1/2}\rho \times \cdots \\ \times \nu^{1/2}\rho \times \nu^{3/2}\rho \times \dots \times \nu^{m_k+1/2}\rho \rtimes \sigma.$$

Then
(i) $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \otimes \sigma$ is a subquotient of $s_{GL}(\tau)$. The multiplicity in $s_{GL}(\tau)$ is one.
(ii) There exists a unique irreducible subquotient π of τ such that $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \otimes \sigma$ is a subquotient of $s_{GL}(\pi)$. The multiplicity of π in τ is one and π is the unique irreducible subrepresentation of $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \rtimes \sigma$.

Proof. From (3-5) we get by induction

$$\begin{aligned} s_{GL}(\tau) = \sum & \nu^{e_{(1,1/2)}1/2}\rho \times \nu^{e_{(1,3/2)}3/2}\rho \times \dots \times \nu^{e_{(1,m_1+1/2)}(m_1+1/2)}\rho \\ & \times \nu^{e_{(2,1/2)}1/2}\rho \times \nu^{e_{(2,3/2)}3/2}\rho \times \dots \times \nu^{e_{(2,m_2+1/2)}(m_2+1/2)}\rho \times \dots \\ & \dots \times \nu^{e_{(k,1/2)}1/2}\rho \times \nu^{e_{(k,3/2)}3/2}\rho \times \dots \times \nu^{e_{(k,m_k+1/2)}(m_k+1/2)}\rho \otimes \sigma \end{aligned}$$

where the sum runs over all possible $e_{(i,j+1/2)} \in \{\pm 1\}, 1 \leq i \leq k, 0 \leq j \leq m_i$. Now (i) follows directly. Further, (i) implies that the multiplicity of π in τ is one. The Frobenius reciprocity implies that every irreducible subrepresentation of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$ has $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \otimes \sigma$ for a quotient of suitable Jacquet module. Therefore, there exists a unique irreducible subrepresentation, and it is π. $\square$

Note that above definition in the case of $k = 1$ agrees with the definition of square integrable representation of Steinberg type (see Theorem 2.1), which was denoted by $\delta(\Delta_1) = \delta([\nu^{1/2}\rho, \nu^{m_1+1/2}\rho])$. If $k = 2$, then we shall denote the representation defined in the lemma by

$$\delta([\nu^{-1/2-m_2}\rho, \nu^{m_1+1/2}\rho], \sigma).$$

The tempered representations which we consider in the following theorem play an important role in the construction of irreducible square integrable representations.

3.2. Theorem. *Let $n \in \mathbb{Z}$, $n \geq 0$, and suppose that $\nu^{1/2}\rho \rtimes \sigma$ reduces. Then:*
(i) $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma$ and $\delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho], \sigma)$ contain a unique common irreducible subquotient. That subquotient is $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$.

(ii)
$$\text{s.s.}\left(s_{((2n+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)\right)\right) = \sum_{k=0}^{n+1} \delta([\nu^{-k+1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{k+1/2}\rho, \nu^{n+1/2}\rho]) \otimes \sigma.$$

(iii) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma$ is a direct sum of two irreducible inequivalent subrepresentations. One of them is $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$. Denote the other one by $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$. We have s.s. $\left(s_{((2n+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)\right)\right) + \delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho])^2 \otimes \sigma = $ s.s. $\left(s_{((2n+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)\right)\right)$.
(iv) Representations $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$ and $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$ are tempered. They are not square integrable.

(v)
$$\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)\tilde{} \cong \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \tilde{\sigma}),$$
$$\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)\tilde{} \cong \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \tilde{\sigma}).$$

Proof. From (2-1) and (1-4) we obtain

(3-1)
$$\begin{aligned}
&\text{s.s.}\left(s_{((2n+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma\right)\right) \\
&= \sum_{k=-n-1}^{n+1} \delta([\nu^{-k+1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{k+1/2}\rho, \nu^{n+1/2}\rho]) \otimes \sigma \\
&= \delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho])^2 \otimes \sigma + 2\sum_{k=1}^{n+1} \delta([\nu^{-k+1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{k+1/2}\rho, \nu^{n+1/2}\rho]) \otimes \sigma
\end{aligned}$$

From this we can conclude that $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$ is a multiplicity one representation of length ≤ 2 (use the Frobenius reciprocity and the fact that $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$ is completely reducible, because this representation is unitarizable). We look further at

$$\text{(3-2)}\qquad \text{s.s.}\left(s_{((2n+2)p)}\left(\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma)\right)\right)$$
$$=\left[\sum_{k=0}^{n+1}\delta([\nu^{-k+1/2}\rho,\nu^{-1/2}\rho])\times\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\right]\times\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma$$

We shall now write all common irreducible subquotients of (3-1) and (3-2). They are

$$\text{(3-3)}\qquad \delta([\nu^{-k+1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma,\ \ k=0,1,\dots,n+1.$$

Multiplicities in (3-1) of above representation are all two, except of the first one (for $k=0$), which is one. The multiplicities of above representation in (3-2) are all 1. Write now

$$\text{(3-4)}\qquad \text{s.s.}\left(s_{((2n+2)p)}\left(\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\rtimes\sigma\right)\right)$$
$$=\left[\sum_{k=0}^{n+1}\delta([\nu^{-k+1/2}\rho,\nu^{-1/2}\rho])\times\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\right]^2\otimes\sigma.$$

We shall determine multiplicities of representations from (3-3) in (3-4). Note that if we look at a fixed representation from (3-3), then the cuspidal representations which appear in the support form a segment which ends with $\nu^{n+1/2}\rho$. In general, the support is of the form $(\nu^{1/2-k}\rho,\nu^{1/2-k+1}\rho,\dots,\nu^{1/2+n}\rho)+(\nu^{k+1/2}\rho,\nu^{k+3/2}\rho,\dots,\nu^{1/2+n}\rho)$ where $k=0,1,\dots,n,n+1$. It is now easy to conclude that the multiplicities of representation from (3-3) in (3-4) are the same as the multiplicities in (3-1).

Note that $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\leq\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\rtimes\sigma$ and

$$\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma)\leq\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\rtimes\sigma.$$

We always consider inequalities as above, as inequalities between semi simplifications in the Grothendieck group of the corresponding category of smooth representations of finite length. We obtain easily from (3-1) and (3-2)

$$\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma\not\leq\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma),$$
$$\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma)\not\leq\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$$

(if we would have somewhere above inequality, then the inequality would hold between all Jacquet modules, what can not be by (3-1) and (3-2)). This, and the multiplicities of representations of (3-3) in (3-1), (3-2) and (3-4) imply that there exists a unique common irreducible subquotient of $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho])\rtimes\sigma$ and $\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\rtimes\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho],\sigma)$. Since this common irreducible subquotient must have in the Jacquet module $\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])^2\otimes\sigma$ (as a subquotient), it must be $\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)$. Therefore, (i) holds. The calculation of multiplicities implies (ii). Now (iii) follows from (ii) and (3-1). Further, (iv) is a consequence of the square integrability criterion. Finally, we get (v) using the characterization in (i). $\square$

3.3. Theorem. *Let $n, m \in \mathbb{Z}$, $m > n \geq 0$. Suppose that (ρ, σ) satisfies (C1/2). Then:*

(*i*)
$$\text{s.s.}\left(s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)\right)\right)$$
$$= \sum_{k=0}^{n+1} \delta([\nu^{1/2-k}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n+1/2}\rho]) \otimes \sigma.$$

(ii) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ is square integrable.
(iii) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ is a unique common irreducible subquotient of $\nu^{m+1/2}\rho \rtimes \delta([\nu^{-n-1/2}\rho, \nu^{m-1/2}\rho], \sigma)$ and $\nu^{n+1/2}\rho \rtimes \delta([\nu^{-n+1/2}\rho, \nu^{m+1/2}\rho], \sigma)$.
(iv) $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)\tilde{} \cong \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \tilde{\sigma})$.

Proof. Write

$$\text{s.s.}\left(s_{((n+m+2)p)}\left(\nu^{n+1/2}\rho \rtimes \delta([\nu^{-n+1/2}\rho, \nu^{m+1/2}\rho], \sigma)\right)\right) \tag{3-5}$$
$$= (\nu^{n+1/2}\rho + \nu^{-n-1/2}\rho) \times \left[\sum_{k=0}^{n} \delta([\nu^{1/2-k}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n-1/2}\rho])\right] \otimes \sigma,$$

$$\text{s.s.}\left(s_{((n+m+2)p)}\left(\nu^{m+1/2}\rho \rtimes \delta([\nu^{-n-1/2}\rho, \nu^{m-1/2}\rho], \sigma)\right)\right) \tag{3-6}$$
$$= (\nu^{m+1/2}\rho + \nu^{-m-1/2}\rho) \times \left[\sum_{k=0}^{n+1} \delta([\nu^{1/2-k}\rho, \nu^{m-1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n+1/2}\rho])\right] \otimes \sigma.$$

Common irreducible subquotients of (3-5) and (3-6) can not contain in the GL-supports $\nu^{-m-1/2}\rho$ (see (3-5)). Also, the representations in the GL-supports of each common irreducible subquotient will form a segment which ends with $\nu^{m+1/2}\rho$ (see (3-6) and use the above remark about $\nu^{-m-1/2}\rho$). We shall now write all pairs from (3-5) and (3-6) which can have common irreducible subquotients. They are

$$\begin{gathered} \nu^{m+1/2}\rho \times \delta([\nu^{-n-1/2}\rho, \nu^{m-1/2}\rho]) \otimes \sigma \quad \text{and} \\ \nu^{-n-1/2}\rho \times \delta([\nu^{-n+1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma; \\ \nu^{m+1/2}\rho \times \delta([\nu^{1/2-k}\rho, \nu^{m-1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n+1/2}\rho]) \otimes \sigma \quad \text{and} \\ \nu^{n+1/2}\rho \times \delta([\nu^{1/2-k}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{1/2+k}\rho, \nu^{n-1/2}\rho]) \otimes \sigma, \text{ for } k = 0, 1, \dots, n. \end{gathered} \tag{3-7}$$

We can now write easily the common irreducible factors of (3-5) and (3-6) from (3-7). They are

$$\delta([\nu^{1/2+k}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{1/2-k}\rho, \nu^{m+1/2}\rho]) \otimes \sigma, \ k = 0, 1, \dots, n+1. \tag{3-8}$$

Multiplicities of the representations from (3-8) in (3-5) and (3-6) are all equal to one.

We further consider

$$\text{s.s.}\Big(s_{((n+m+2)p)}\Big(\nu^{n+1/2}\rho\times\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m-1/2}\rho],\sigma\Big)\Big)$$
$$=(\nu^{n+1/2}\rho+\nu^{-n-1/2}\rho)\times(\nu^{m+1/2}\rho,\nu^{-m-1/2}\rho)$$
$$\times\Big[\sum_{k=0}^{n}\delta([\nu^{1/2-k}\rho,\nu^{m-1/2}\rho])\times\delta([\nu^{1/2+k}\rho,\nu^{n-1/2}\rho])\Big]\otimes\sigma. \tag{3-9}$$

We want to see multiplicities of representations from (3-8) in (3-9). We need to consider only the following terms in the sum

$$\nu^{-n-1/2}\rho\times\nu^{m+1/2}\rho\times\delta([\nu^{-n+1/2}\rho,\nu^{m-1/2}\rho])\otimes\sigma,$$
$$\nu^{n+1/2}\rho\times\nu^{m+1/2}\rho\times\delta([\nu^{1/2-k}\rho,\nu^{m-1/2}\rho])\times\delta([\nu^{1/2+k}\rho,\nu^{n-1/2}\rho])\otimes\sigma,\ k=0,1,\dots,n.$$

It is easy to get now that all multiplicities are 1.

From the definition of representations $\delta([\nu^{-n'-1/2}\rho,\nu^{m'+1/2}\rho],\sigma)$ we get

$$\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho],\sigma)\le\nu^{n+1/2}\rho\times\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m-1/2}\rho],\sigma),$$
$$\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m-1/2}\rho],\sigma)\le\nu^{n+1/2}\rho\times\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m-1/2}\rho],\sigma).$$

This, together with the multiplicities that we have computed, implies that if we write $\le$ instead of $=$ in (i), then such inequality holds. For the opposite inequality we shall first prove

$$\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\le s_{((n+m+2)p)}(\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma). \tag{3-10}$$

To prove this, observe that

$$\delta([\nu^{-m-1/2}\rho,\nu^{m+1/2}\rho],\sigma)\le\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$$

(one checks that the subquotient of the Jacquet module of $\delta([\nu^{-m-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ which characterizes this representation must be in the Jacquet module of the right hand side). Thus

$$\delta([\nu^{-m-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$$
$$\le s_{((2m+2)p)}(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)).$$

The formula for the above Jacquet module and the inequality (i) that we have already proved, imply (3-10) now.

We shall use now (3-10). The representation on the left hand side of (3-10) must be a direct summand of the Jacquet module on the right hand side of (3-10) (see the central characters and use the inequality $\le$ from (i) which we have proved). Thus for $n>0$

$$\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)\hookrightarrow\nu^{m+1/2}\rho\times\cdots\times\nu^{-n+1/2}\rho\times\nu^{-n-1/2}\rho\rtimes\sigma$$
$$\cong\nu^{m+1/2}\rho\times\cdots\times\nu^{-n+1/2}\rho\times\nu^{n+1/2}\rho\rtimes\sigma.$$

Using the Frobenius reciprocity and comparing with GL-supports of representations in (3-7), we can conclude that $\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho])\times\nu^{n+1/2}\rho\otimes\sigma$ is in the Jacquet module. Proceeding in the same way we shall get all other members except $\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma$. The last representation is by definition in the Jacquet module of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$. This finishes the proof of (i). The square integrability criterion and (i) imply (ii) (use [Z1]). Now it is easy to get (iii) from (i) and our previous considerations. One gets (iv) by induction using the characterization in (iii), and Theorems 2.1 and 3.2. □

3.4. Remark. It seems that it would be equally convenient to use the representations $\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{n+1/2}\rho],\sigma)$ and $\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ for the upper estimate of the Jacquet module of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ in the last proof.

4. Reducibility at 1/2, II

As in the previous section, we fix an irreducible unitarizable cuspidal representation ρ of $GL(p,F)$ and an irreducible cuspidal representation σ of S_q such that (ρ,σ) satisfies (C1/2).

4.1. Lemma. *Let $n,m\in\mathbb{Z}$, $m\geq n\geq 0$. Then:*
(i) For $k=1,2,\dots,n,n+1$, multiplicity of $\delta([\nu^{k+1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-k+1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma\right)$ is 2. In particular, the multiplicities of

$$\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma \quad \text{and} \quad \delta([\nu^{3/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$$

in $s_{GL}\left(\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma\right)$ are both 2.
(ii) Multiplicity of $\delta([\nu^{3/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ in

$$s_{((n+m+2)p)}(\nu^{-n-1/2}\rho\times\nu^{-n+1/2}\rho\times\nu^{-n+3/2}\rho\times\cdots\times\nu^{m+1/2}\rho\rtimes\sigma)$$

is 2.
(iii) If π is an irreducible subquotient of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma$ such that

$$\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\leq s_{((n+m+2)p)}(\pi),$$

then $2\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\not\leq s_{((n+m+2)p)}(\pi)$.

Proof. The claim (i) follows from the following formula

$$\text{s.s.}\left(s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma\right)\right)$$
$$=\sum_{i=-m-1}^{n+1}\delta([\nu^{i+1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-i+1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$$

(use (1-4) and (2-1) to get the formula). The claim (ii) follows from the first formula in the proof of Lemma 3.1.

We know that $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ is a subquotient of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma$, and that this irreducible representation satisfies two conditions from (iii) (see Theorems 3.2 and 3.3). This, together with (i) and (ii), implies (iii). □

4.2. Theorem. *Let $n, m \in \mathbb{Z}$, $m \geq n \geq 0$.*
(i) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ contains exactly two irreducible subquotients π which satisfy $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{((n+m+2)p)}(\pi)$. One of these subquotients is $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$. The other one we denote by

$$\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma).$$

Then $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma) \not\cong \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$.
(ii) The multiplicity of $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ in

$$\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \nu^{-n+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma$$

is one.
(iii) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ can be characterized as a unique irreducible subquotient π of $\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \nu^{-n+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma$ which satisfies conditions

$$\delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{((n+m+2)p)}(\pi)$$
$$\delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \not\leq s_{((n+m+2)p)}(\pi).$$

(iv)
$$\begin{aligned} &s.s.\left(s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)\right)\right) \\ &= \sum_{i=0}^{n} \delta([\nu^{-i-1/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{i+3/2}\rho, \nu^{n+1/2}\rho]) \otimes \sigma. \end{aligned}$$

(v) If $m > n$, then the representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ is square integrable.
(vi) $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)\tilde{} \cong \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \tilde{\sigma})$.

We define $\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ to be $\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho], \sigma)$. This convention is useful bellow in the proofs by induction.

Proof. From the previous lemma, one directly gets (i) and (ii).

Recall that the multiplicity of $\delta([\nu^{3/2}\rho, \nu^{n+1/2}\rho]) \times \delta([\nu^{-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ in

$$s_{((n+m+2)p)}(\nu^{-n-1/2}\rho \times \nu^{-n+1/2}\rho \times \nu^{-n+3/2}\rho \times \cdots \times \nu^{m+1/2}\rho \rtimes \sigma)$$

and $s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma\right)$ is 2 in both cases, while multiplicity in

$$s_{((n+m+2)p)}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)\right)$$

is 1.

We now prove (iii) and (iv) by induction on $n+m$. For $n = m$ we know that both claims hold (Theorem 3.2). Therefore, it is enough to consider the case $n < m$. We assume this, and we assume that the claims (iii) and (iv) hold for m', n' such that $m' + n' < m + n$.

From Theorem 3.3 and the previous lemma we see that there exists a unique subquotient π of $\nu^{-n-1/2}\rho\times\nu^{-n+1/2}\rho\times\cdots\times\nu^{m+1/2}\rho\rtimes\sigma$ such that

$$\delta([\nu^{3/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\leq s_{((n+m+2)p)}(\pi),$$
$$\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\not\leq s_{((n+m+2)p)}(\pi).$$

The previous lemma implies that π is a subquotient of $\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\rtimes\sigma$. Otherwise, $\delta([\nu^{3/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ would have multiplicity at least 3 in $s_{((n+m+2)p)}(\nu^{-n-1/2}\rho\times\nu^{-n+1/2}\rho\times\cdots\times\nu^{m+1/2}\rho\rtimes\sigma)$, what can not be by the previous lemma.

If $n>0$, then (1-4) and Theorems 3.2 and 3.3 imply

$$\begin{aligned}\text{(4-1)}\quad &\delta([\nu^{3/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\\ &\qquad\leq s_{((n+m+2)p)}\left(\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)\right),\end{aligned}$$

$$\begin{aligned}\text{(4-2)}\quad &\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\\ &\qquad\not\leq s_{((n+m+2)p)}\left(\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)\right);\end{aligned}$$

and

$$\begin{aligned}\text{(4-3)}\quad &\delta([\nu^{3/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\\ &\qquad\leq s_{((n+m+2)p)}\left(\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m-1/2}\rho]_-,\sigma)\right),\end{aligned}$$

$$\begin{aligned}\text{(4-4)}\quad &\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\\ &\qquad\not\leq s_{((n+m+2)p)}\left(\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m-1/2}\rho]_-,\sigma)\right).\end{aligned}$$

We shall now consider the case $n=0$. Observe that (iii) is obvious for $n=0$. We shall now prove (iv) by induction with respect to m. For $n=0$, the formulas (4-3) and (4-4) hold (and also (4-1) holds, but (4-2) does not hold). This implies that $\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$ is a subquotient of $\nu^{m+1/2}\rho\rtimes\delta([\nu^{-1/2}\rho,\nu^{m-1/2}\rho]_-,\sigma)$. Now the inductive assumption and (1-4) imply

$$\begin{aligned}\text{(4-5)}\quad &s_{((m+2)p)}\left(\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)\right)\\ &\qquad\leq(\nu^{m+1/2}\rho+\nu^{-m-1/2}\rho)\times\delta([\nu^{-1/2}\rho,\nu^{m-1/2}\rho])\otimes\sigma.\end{aligned}$$

Note that

$$\begin{aligned}\text{(4-6)}\quad &\text{s.s.}\left(s_{((m+2)p)}\left(\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\rtimes L(\nu^{1/2}\rho,\sigma)\right)\right)\\ &=\left[\sum_{i=-1}^{m}\delta([\nu^{-i-1/2}\rho,\nu^{-1/2}\rho])\times\delta([\nu^{i+3/2}\rho,\nu^{m+1/2}\rho])\right]\times\nu^{-1/2}\rho\otimes\sigma.\end{aligned}$$

The above formula and Lemma 4.1 imply that $\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$ is a subquotient of $\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\rtimes L(\nu^{1/2}\rho,\sigma)$. This implies

$$s_{((m+2)p)}\left(\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-\sigma)\right)\leq s_{((m+2)p)}\left(\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])\rtimes L(\nu^{1/2}\rho,\sigma)\right)$$

From this and formulas (5-5) and (4-6), now one can easily get the following estimate

$$s_{((m+2)p)}\left(\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)\right)\leq \delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma.$$

Obviously, in the above relation the equality must hold. This finishes the proof for $n=0$.

Suppose now $n>0$. Relations (4-1), (4-2), (4-3) and (4-4) imply that π is a subquotient of $\nu^{n+1/2}\rho\rtimes\delta([\nu^{-n+1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$ and $\nu^{m+1/2}\rho\rtimes\delta([\nu^{-n-1/2}\rho,\nu^{m-1/2}\rho]_-,\sigma)$. Now in the same way as in the proof of Theorem 3.3, one gets

$$s_{((n+m+2)p)}(\pi)\leq\sum_{i=0}^{n}\delta([\nu^{-i-1/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{i+3/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma. \tag{4-7}$$

One checks directly that $\delta([\nu^{3/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ has multiplicity ≥ 1 in $s_{((2m+2)p)}\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\pi\right)$. Since

$$\delta([\nu^{1/2}\rho,\nu^{m+1/2}\rho])^2\otimes\sigma\not\leq s_{((2m+2)p)}\left(\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\pi\right)$$

(we can see it from (4-7)), we conclude that

$$\delta([\nu^{-m-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)\leq\delta([\nu^{n+3/2}\rho,\nu^{m+1/2}\rho])\rtimes\pi.$$

From (4-7) and (1-4) follow easily that $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma\leq s_{((n+m+2)p)}(\pi)$. Now in the same way as in the end of proof of Theorem 3.3 (see the last section of that proof), one gets

$$s_{((n+m+2)p)}(\pi)\geq\sum_{i=0}^{n}\delta([\nu^{-i-1/2}\rho,\nu^{m+1/2}\rho])\times\delta([\nu^{i+3/2}\rho,\nu^{n+1/2}\rho])\otimes\sigma$$

The above two inequalities for $s_{((n+m+2)p)}(\pi)$ imply that in (4-7) we have an equality. This implies that $\pi=\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$, what is the claim of (iii). Now (iv) is obvious. Further, (iv) implies (v).

One can get (vi) considering $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho])\otimes\sigma$ in the Jacquet module of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$, using (i) of Lemma 4.1, Theorems 3.2, 3.3, and Corollary 4.2.5 of [C]. We could also get (vi) using Proposition 3.6 of [J]. This finishes the proof of the theorem. □

The following theorem gives a simple characterization of $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho],\sigma)$ and $\delta([\nu^{-n-1/2}\rho,\nu^{m+1/2}\rho]_-,\sigma)$.

4.3. Theorem. *Let $n, m \in \mathbb{Z}$, $m > n \geq 0$. Then*
(i) $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ and $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$ are (isomorphic to) irreducible subrepresentations of $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$. Further, $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ does not contain any other irreducible subrepresentation.
(ii) The representation $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ (resp. $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$) is a unique irreducible subrepresentation of $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$ (resp. $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$).

Proof. Denote $\pi = \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ (resp. $\pi_- = \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$). Now (i) of Theorem 3.3 (resp. (iv) of Theorem 4.2) and Theorem 7.3.2 of [C] imply that $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ is a direct summand in $s_{GL}(\pi)$ (resp. $s_{GL}(\pi_-)$). Frobenius reciprocity implies that there exists an embedding $\phi : \pi \hookrightarrow \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$ (resp. $\phi_- : \pi_- \hookrightarrow \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$). Suppose that π' is an irreducible subrepresentation of $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma$, such that $\mathrm{Im}(\phi) \cap \pi' = \{0\}$ and $\mathrm{Im}(\phi_-) \cap \pi' = \{0\}$. Frobenius reciprocity implies that $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ is a quotient of $s_{GL}(\pi')$. Therefore, multiplicity of $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ in $s_{GL}\left(\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma\right)$ is at least 3 (we use also here the last claim of (i) in Theorem 4.2). This multiplicity is 2 by (i) of Lemma 4.1. This contradiction completes the proof of (i).

In the same way as before, one checks that multiplicity of $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ in $s_{GL}\left(\delta([\nu^{n+1/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma\right)$ is 2, and

$$\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{GL}\left(\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)\right),$$

$$\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma \leq s_{GL}\left(\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)\right).$$

Therefore, multiplicity of $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ in the right hand sides of the above two inequalities is 1. Further, $\delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{m+1/2}\rho]) \otimes \sigma$ is a subquotient of $s_{GL}\left(\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)\right)$

Since $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \hookrightarrow \delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho])$ ([Z1]), we have $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma$. The last representation is isomorphic to $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma) \oplus \delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$. Now we can conclude that π embeds into $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$ and π_- into $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \rtimes \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$. It remains to see the uniqueness of the irreducible subrepresentations in (ii). Frobenius reciprocity implies that it is enough to show that multiplicity of $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \otimes \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$, and also of $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \otimes \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$ in $\mu^*\left(\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma\right)$, is 1. For this, one needs only to prove that the multiplicity is ≤ 1 (Frobenius reciprocity implies that the converse inequalities hold). In the continuation of this paper we shall prove a much more general fact about uniqueness of irreducible subrepresentation (Proposition 9.2, (ii)), which implies the second claim in (ii). Therefore, we shall only sketch here the proof that the multiplicity is ≤ 1. Write

$$M^*\left(\delta([\nu^{3/2+n}\rho, \nu^{1/2+m}\rho])\right) = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*\left(\delta([\nu^{3/2+n}\rho, \nu^{1/2+m}\rho])\right) \tag{4-8}$$

$$= (m \otimes 1) \circ (\sim \otimes m^*) \circ s\Big(\sum_{a=n}^{m} \delta([\nu^{a+3/2}\rho, \nu^{1/2+m}\rho]) \otimes \delta([\nu^{3/2+n}\rho, \nu^{1/2+a}\rho]) \Big)$$

$$= (m \otimes 1) \circ (\sim \otimes m^*) \Big(\sum_{a=n}^{m} \delta([\nu^{3/2+n}\rho, \nu^{1/2+a}\rho]) \otimes \delta([\nu^{a+3/2}\rho, \nu^{1/2+m}\rho]) \Big)$$

$$= \sum_{a=n}^{m} \sum_{b=a}^{m} \delta([\nu^{-1/2-a}\rho, \nu^{-3/2-n}\rho]) \times \delta([\nu^{b+3/2}\rho, \nu^{1/2+m}\rho]) \otimes \delta([\nu^{3/2+a}\rho, \nu^{1/2+b}\rho]).$$

Compute now $\mu^* \left(\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma\right)$ using (1-4). To obtain $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \otimes \tau$ in $\mu^* \left(\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \times \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma\right)$ when we compute it using (4-1), we must take from (4-8) the term corresponding to $a = n$. From

$$\mu^* \left(\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma\right) \leq \mu^*\Big(\Big(\prod_{i=-n-1/2}^{n+1/2} \nu^i\rho\Big) \rtimes \sigma\Big)$$

$$= \Big(\prod_{i=-n-1/2}^{n+1/2} (1 \otimes \nu^i\rho + \nu^i\rho \otimes 1 + \nu^{-i}\rho \otimes 1)\Big) \rtimes (1 \otimes \sigma)$$

(the above product runs over $i \in (1/2) + \mathbb{Z}, -n - 1/2 \leq i \leq n + 1/2$), we get directly that b must be n. Thus, $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \otimes \tau$ can appear as a subquotient only from the term $\delta([\nu^{n+3/2}\rho, \nu^{m+1/2}\rho]) \otimes \delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]) \rtimes \sigma$ (which corresponds to $a = b = n$). Now (iii) of Theorem 3.2 implies our claim about multiplicities. This finishes the proof of the theorem. $\square$

4.4. Proposition. *Let $n \in \mathbb{Z}$, $n \geq 0$ and $\alpha \in \mathbb{R}$.*
(i) Assume that (ρ, σ) satisfies (C1/2). Suppose that $\nu^\alpha\delta([\rho, \nu^n\rho]) \rtimes \sigma$ contains an irreducible square integrable subquotient, say π. Then π is equivalent either to a representation listed in Theorem 2.1, or Theorem 3.3, or Theorem 4.2.
(ii) If $\rho \not\cong \tilde{\rho}$, then $\nu^\alpha\delta([\rho, \nu^n\rho]) \rtimes \sigma$ can not contain a square integrable subquotient.

Proof. Suppose that $\nu^\alpha\delta([\rho, \nu^n\rho]) \rtimes \sigma$ contains a square integrable subquotient.

If $\nu^\alpha\delta([\rho, \nu^n\rho])$ is unitarizable, obviously we can not get a square integrable subquotient (this follows directly from the Frobenius reciprocity). Therefore, we can assume that $\nu^\alpha\delta([\rho, \nu^n\rho])$ is not unitarizable.

If $\nu^\alpha\delta([\rho, \nu^n\rho]) \rtimes \sigma$ is irreducible, then it is not square integrable (the Langlands quotient coming from a proper parabolic subgroup, is never square integrable). Therefore, we can assume that $\nu^\alpha\delta([\rho, \nu^n\rho]) \rtimes \sigma$ reduces. Theorem 9.1 of [T7] implies $\rho \cong \tilde{\rho}$ and

$$\nu^\alpha\delta([\rho, \nu^n\rho]) \in \Big\{\delta([\nu^{-n-1/2}\rho, \nu^{-1/2}\rho]), \delta([\nu^{-n+1/2}\rho, \nu^{1/2}\rho]), \delta([\nu^{-n+3/2}\rho, \nu^{3/2}\rho]),$$

$$\dots, \delta([\nu^{-1/2}\rho, \nu^{n-1/2}\rho]), \delta([\nu^{1/2}\rho, \nu^{n+1/2}\rho])\Big\}.$$

Suppose that this is the case (and $\nu^\alpha\delta([\rho,\nu^n\rho])$ is not unitarizable, as we already have assumed). Note that at each reducibility point the Langlands quotient is not square integrable. Recall that $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ and $\nu^{-\alpha}\delta([\nu^{-n}\rho,\rho])\rtimes\sigma$ have the same Jordan-Hölder series (see (1-3)). Further, note that by Proposition 3.6 of [J], applying the involution constructed in [A2] (one can apply also [ScSt]), these representations have multiplicity one, and they have length 3, except if

$$\nu^\alpha\delta([\rho,\nu^n\rho])\in\left\{\delta([\nu^{-n-1/2}\rho,\nu^{-1/2}\rho]),\delta([\nu^{1/2}\rho,\nu^{n+1/2}\rho])\right\},$$

when the length is two. This implies the proposition. $\square$

5. Reducibility at 0

In this section we fix an irreducible unitarizable cuspidal representation ρ of $GL(p,F)$ and an irreducible cuspidal representation σ of S_q. We shall assume that $\rho\rtimes\sigma$ reduces (then $\rho\cong\tilde\rho$) and that $\nu^\alpha\rho\rtimes\sigma$ does not reduce for $\alpha\in\mathbb{R}^\times$ (in other words, we assume that (ρ,σ) satisfies (C0)).

From the Jacquet module $s_{(p)}(\rho\rtimes\sigma)$ one gets that $\rho\rtimes\sigma$ is a sum of two irreducible representations. Further, the Frobenius reciprocity implies that $\rho\rtimes\sigma$ is a multiplicity one representation. Write $\rho\rtimes\sigma=\tau_1\oplus\tau_2$ where τ_1 and τ_2 are irreducible ($\tau_1\not\cong\tau_2$).

5.1. Lemma. *The representation $\nu\rho\rtimes\tau_i$ contains a unique irreducible subrepresentation, which we denote by $\delta([\rho,\nu\rho]_{\tau_i},\sigma)$. This subrepresentation is square integrable and it is the only square integrable subquotient of $\nu\rho\rtimes\tau_i$. We have*

$$\mu^*\left(\delta([\rho,\nu\rho]_{\tau_i},\sigma)\right)=1\otimes\delta([\rho,\nu\rho]_{\tau_i},\sigma)+\nu\rho\otimes\tau_i+\delta([\rho,\nu\rho])\otimes\sigma,$$
$$\delta([\rho,\nu\rho]_{\tau_i},\sigma)\tilde{}\cong\delta([\rho,\nu\rho]_{\tilde\tau_i},\tilde\sigma),\qquad\delta([\rho,\nu\rho]_{\tau_1},\sigma)\not\cong\delta([\rho,\nu\rho]_{\tau_2},\sigma).$$

Proof. We have epimorphisms $\nu\rho\rtimes\tau_i\twoheadrightarrow L(\nu\rho,\tau_i)$. Writing the above formula for contragredients $\tilde\tau_i$ and passing to contragredients, one gets monomorphisms $L(\nu\rho,\tilde\tau_i)\tilde{}\hookrightarrow\nu^{-1}\rho\rtimes\tau_i$. Since $L(\nu\rho,\tilde\tau_i)\tilde{}\cong L(\nu\rho,\tau_i)$, the Frobenius reciprocity implies that there exist epimorphisms

$$s_{(p)}\left(L(\nu\rho,\tau_i)\right)\twoheadrightarrow\nu^{-1}\rho\otimes\tau_i. \tag{5-1}$$

Further, we have an epimorphism $\delta([\rho,\nu\rho])\rtimes\sigma\twoheadrightarrow L\left(\delta([\rho,\nu\rho]),\sigma\right)$. Similarly as before we get an epimorphism

$$s_{(2p)}\left(L\left(\delta([\rho,\nu\rho]),\sigma\right)\right)\twoheadrightarrow\delta([\nu^{-1}\rho,\rho])\otimes\sigma. \tag{5-2}$$

Write now using (1-4)

$$\mu^*(\nu\rho\rtimes\tau_i)=1\otimes\nu\rho\rtimes\tau_i+[\nu\rho\otimes\tau_i+\nu^{-1}\rho\otimes\tau_i+\rho\otimes\nu\rho\rtimes\sigma]+[\nu\rho\times\rho\otimes\sigma+\nu^{-1}\rho\times\rho\otimes\sigma]. \tag{5-3}$$

From the above formula we see that $\nu\rho\times\rho\rtimes\sigma$ is a representation of length ≤ 6 because

$$\nu\rho\times\rho\rtimes\sigma=\nu\rho\rtimes\tau_1\oplus\nu\rho\rtimes\tau_2. \tag{5-4}$$

Also, from (5-1) and (5-3) one gets that each $L(\nu\rho, \tau_i)$ has multiplicity one in $\nu\rho \times \rho \rtimes \sigma$. Further, there is an exact sequence of representation.

$$0 \to \delta([\rho, \nu\rho]) \rtimes \sigma \xrightarrow{\alpha} \nu\rho \times \rho \rtimes \sigma \xrightarrow{\beta} L(\nu\rho, \rho) \rtimes \sigma \longrightarrow 0.$$

We have

$$\mu^*(\delta([\rho, \nu\rho]) \rtimes \sigma) = 1 \otimes \delta([\rho, \nu\rho]) \rtimes \sigma$$
$$+ [\nu\rho \otimes \tau_1 + \nu\rho \otimes \tau_2 + \rho \otimes \nu\rho \rtimes \sigma] + \left[\delta([\rho, \nu\rho]) \otimes \sigma + \rho \times \nu\rho \otimes \sigma + \delta([\nu^{-1}\rho, \rho]) \otimes \sigma\right].$$

From (5-2) and the above formula for $\mu^*(\delta([\rho, \nu\rho]) \rtimes \sigma)$ we can conclude that $\rho \otimes \nu\rho \rtimes \sigma$ is a subquotient of $s_{(p)}\left(L\left(\delta([\rho, \nu\rho]), \sigma\right)\right)$. Write further

$$\mu^*\left(L(\nu\rho, \rho) \rtimes \sigma\right) = 1 \otimes L(\nu\rho, \rho) \rtimes \sigma + \tag{5-5}$$
$$[\nu^{-1}\rho \otimes \tau_1 + \nu^{-1}\rho \otimes \tau_2 + \rho \otimes \nu\rho \rtimes \sigma] + \left[L(\nu\rho, \rho) \otimes \sigma + \nu^{-1}\rho \times \rho \otimes \sigma + L(\rho, \nu^{-1}\rho) \otimes \sigma\right].$$

Now we claim that $\nu\rho \rtimes \tau_i$ has $L\left(\delta([\rho, \nu\rho]), \sigma\right)$ for a subquotient. To prove that, it is enough to prove that there exists a non-zero intertwining $\delta([\rho, \nu\rho]) \rtimes \sigma \longrightarrow \nu\rho \rtimes \tau_i$. We shall show that now. Consider the composition $\delta([\rho, \nu\rho]) \rtimes \sigma \hookrightarrow \nu\rho \times \rho \rtimes \sigma \xrightarrow{\mathrm{pr}_i} \nu\rho \rtimes \tau_i$ where pr_i denotes the projection of $\nu\rho \times \rho \rtimes \sigma$ onto $\nu\rho \rtimes \tau_i$ with respect to the decomposition (5-4). Denote it by φ_i. If $\varphi_i \neq 0$, then our claim holds. Therefore, suppose that $\varphi_i = 0$. This implies that there exists an epimorphism of $L(\rho, \nu\rho) \rtimes \sigma \cong (\nu\rho \times \rho \rtimes \sigma) \big/ (\delta([\rho, \nu\rho]) \rtimes \sigma)$ onto $\nu\rho \rtimes \tau_i$. This implies the existence of an epimorphism also on the level of each Jacquet module. Formulas (5-5) and (5-3) imply that this is not possible. This finishes the proof of our claim.

We can now conclude that $L\left(\delta([\rho, \nu\rho]), \sigma\right)$ has multiplicity two in $\nu\rho \times \rho \rtimes \sigma$. Further, it is easy to get that the following equality holds in the Grothendieck group

$$L(\nu\rho, \rho) \rtimes \sigma = L(\nu\rho, \tau_1) + L(\nu\rho, \tau_2) + L\left(\delta([\rho, \nu\rho]), \sigma\right)$$

(use (5-1), (5-3) and (5-5) to see that we have the first two summands; the last summand follows from (5-2) and (5-3)).

Note that no one of three irreducible subquotients that we considered up to now has $\nu\rho \otimes \tau_i$ for a subquotient in suitable Jacquet module (see (5-5)).

Consider $\nu\rho \rtimes \tau_i$ and $\delta([\rho, \nu\rho]) \rtimes \sigma$ as a subrepresentations in $\nu\rho \times \rho \rtimes \sigma$. Then from the Jacquet modules one can conclude that their intersection is non-zero. Moreover, there exists an irreducible subquotient of the intersection which has $\nu\rho \otimes \tau_i$ in the suitable Jacquet module. Denote it $\delta([\rho, \nu\rho]_{\tau_i}, \sigma)$. Then $\mu^*\left(\delta([\rho, \nu\rho]_{\tau_i}, \sigma)\right) = 1 \otimes \delta([\rho, \nu\rho]_{\tau_i}, \sigma) + \nu\rho \otimes \tau_i + \delta([\rho, \nu\rho]) \otimes \sigma$. This representation is square integrable by the square integrability criterion. All the claims of the lemma follow directly now. $\square$

In the sequel we shall also use the following notation:

$$\delta([\rho, \rho]_{\tau_i}, \sigma) = \tau_i, \qquad \delta(\emptyset_{\tau_i}, \sigma) = \sigma.$$

5.2. Theorem. *Suppose that (ρ,σ) satisfies (C0). Write $\rho\rtimes\sigma=\tau_1\oplus\tau_2$ where τ_1 and τ_2 are irreducible. For $m\geq 1$ the representation $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu\rho\times\tau_i$ contains a unique irreducible subrepresentation, which we denote $\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$. Then:*
(i) $\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$ is square integrable.
(ii) $\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)\tilde{}\cong\delta([\rho,\nu^m\rho]_{\tilde\tau_i},\tilde\sigma)$.

(iii)
$$\mu^*\left(\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)\right)=\sum_{k=0}^{n+1}\delta([\nu^k\rho,\nu^m\rho])\otimes\delta([\rho,\nu^{k-1}\rho]_{\tau_i},\sigma).$$

(iv) We may characterize $\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$ as a unique irreducible subquotient π of $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu\rho\times\tau_i$ for which $\delta([\rho,\nu^m\rho])\otimes\sigma$ is a subquotient of $s_{(p(m+1))}(\pi)$.
(v) $\delta([\rho,\nu^m\rho]_{\tau_1},\sigma)\not\cong\delta([\rho,\nu^m\rho]_{\tau_2},\sigma)$.

Proof. Since $\mu^*(\tau_i)=1\otimes\tau_i+\rho\otimes\sigma$, we get inductively

$$s_{((m+1)p)}(\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^2\rho\times\nu\rho\rtimes\tau_i)=\sum_{(\varepsilon_i)\in\{\pm1\}^m}\nu^{\varepsilon_m m}\rho\times\cdots\times\nu^{\varepsilon_2 2}\rho\times\nu^{\varepsilon_1}\rho\times\rho\otimes\sigma.$$

From this one sees that $s_{(p)^{m+1}}(\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^2\rho\times\nu\rho\rtimes\tau_i)$ is a multiplicity one representation. One gets easily that $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^2\rho\times\nu\rho\rtimes\tau_i$ has a unique irreducible subrepresentation now.

Lemma 5.1 implies that the theorem holds for $m=1$ (for (iv) see (5-3)). We proceed by induction now. Suppose that the theorem holds up to $m\geq 1$. Consider $\nu^{m+1}\rho\rtimes\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$. The inductive assumption implies

$$\begin{aligned}&\text{s.s.}\left(s_{((m+2)p)}\left(\nu^{m+1}\rho\rtimes\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)\right)\right)\\&\qquad=\nu^{m+1}\rho\times\delta([\rho,\nu^m\rho])\otimes\sigma+\nu^{-(m+1)}\rho\times\delta([\rho,\nu^m\rho])\otimes\sigma.\end{aligned}$$

Further
$$\begin{aligned}&\text{s.s.}\left(s_{((m+2)p)}\left(\delta([\nu^m\rho,\nu^{m+1}\rho])\rtimes\delta([\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)\right)\right)\\&=\delta([\nu^{-(m+1)}\rho,\nu^{-m}\rho])\times\delta([\rho,\nu^{m-1}\rho])\otimes\sigma\\&+\nu^{-m}\rho\times\nu^{m+1}\rho\times\delta([\rho,\nu^{m-1}\rho])\otimes\sigma+\delta([\nu^m\rho,\nu^{m+1}\rho])\times\delta([\rho,\nu^{m-1}\rho])\otimes\sigma.\end{aligned}$$

From this we see that the two considered representations have exactly one irreducible subquotient in common. It has in the Jacquet module $\delta([\rho,\nu^{m+1}\rho])\otimes\sigma$. One gets easily that this irreducible subquotient is $\delta([\rho,\nu^{m+1}\rho]_{\tau_i},\sigma)$. This also implies (i). The characterization of $\delta([\rho,\nu^{m+1}\rho]_{\tau_i},\sigma)$ as a unique irreducible subquotient of $\nu^{m+1}\rho\rtimes\delta([\rho,\nu^m\rho]_{\tau_i},\sigma)$ and $\delta([\nu^m\rho,\nu^{m+1}\rho])\rtimes\delta([\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)$ implies (ii). Claim (iii) follows in a standard way using the inductive assumption and characterization of essentially square integrable representations of general linear groups by Jacquet modules. Since the multiplicity of $\delta([\rho,\nu^m\rho])\otimes\sigma$ in the corresponding Jacquet module of $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu\rho\times\tau_i$ is one, we have (iv). One gets (v) from (iii). □

We continue with assumptions from the beginning of this section.

5.3. Lemma. *Let $n, m \in \mathbb{Z}, m \geq n \geq 0$. The representation*

$$(\nu\rho \times \nu^2\rho \times \cdots \times \nu^n\rho) \times (\nu\rho \times \nu^2\rho \times \cdots \times \nu^m\rho) \rtimes \tau_i \tag{5-6}$$

contains a unique irreducible subquotient π such that $s_{((n+m+1)p)}(\pi)$ contains

$$\delta([\nu\rho, \nu^n\rho]) \times \delta([\rho, \nu^m\rho]) \otimes \sigma \tag{5-7}$$

as a subquotient. We denote π by $\delta([\nu^{-n}\rho, \nu^m\rho]_{\tau_i}, \sigma)$. The multiplicity of π in (5-6) is one.

Proof. We have

$$\text{s.s.}\left(s_{((n+m+1)p)}\left((\nu\rho \times \nu^2\rho \times \cdots \times \nu^n\rho) \times (\nu\rho \times \cdots \times \nu^m\rho) \rtimes \tau_i\right)\right) = \sum_{\substack{(\varepsilon_j)\in\{\pm1\}^n \\ (\mu_j)\in\{\pm1\}^m}} (\nu^{\varepsilon_1}\rho \rtimes \nu^{2\varepsilon_2}\rho \times \nu^{3\varepsilon_3}\rho \times \cdots \times \nu^{n\varepsilon_n}\rho) \times (\nu^{\mu_1}\rho \times \nu^{2\mu_2}\rho \times \cdots \times \nu^{m\mu_m}\rho) \times \rho \otimes \tau. \tag{5-8}$$

If some $\varepsilon_j \neq 1$ or $\mu_j \neq 1$, then the corresponding member in the sum have different GL-support from (5-7). If all ε_j are one, then the multiplicity of (5-7) in (5-8) is one ([Z1]). This proves the lemma. $\square$

The representation $\delta([\nu\rho, \nu^n\rho]) \rtimes \tau_i$ contains a unique irreducible subrepresentation $\delta([\rho, \nu^n\rho]_{\tau_i}, \sigma)$ which we have already studied.

In the following theorem we continue with the previous notation. The theorem considers non square integrable tempered representations which are useful in the construction of square integrable representations.

5.4. Theorem. *(i) The representations $\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$ and $\delta([\nu\rho, \nu^n\rho]) \rtimes \delta([\rho, \nu^n\rho]_{\tau_i}, \sigma)$ have exactly one irreducible subquotient in common. This factor is $\delta([\nu^{-n}\rho, \nu^n\rho]_{\tau_i}, \sigma)$.*
(ii) $\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma = \delta([\nu^{-n}\rho, \nu^n\rho]_{\tau_1}, \sigma) \oplus \delta([\nu^{-n}\rho, \nu^n\rho]_{\tau_2}, \sigma)$ and the representations on the right hand side are inequivalent.
(iii) s.s. $\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho, \nu^n\rho]_{\tau_i}, \sigma)\right)\right) = \sum_{k=0}^{n} \delta([\nu^{-k}\rho, \nu^n\rho]) \times \delta([\nu^{1+k}\rho, \nu^n\rho]) \otimes \sigma.$
(iv) $$\delta([\nu^{-1}\rho, \nu^n\rho]_{\tau_i}, \sigma)\tilde{} \cong \delta([\nu^{-1}\rho, \nu^n\rho]_{\tilde{\tau}_i}, \tilde{\sigma}).$$
(v) One can characterize $\delta([\nu^{-n}\rho, \nu^n\rho]_{\tau_i}, \sigma)$ also as a unique common irreducible subquotient of $\delta([\nu\rho, \nu^n\rho]) \times \delta([\nu\rho, \nu^n\rho]) \rtimes \tau_i$ and $\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma$.

Proof. We consider the representation

$$\delta([\nu^{-n}\rho, \nu^{-1}\rho]) \times \rho \times \delta([\nu\rho, \nu^n\rho]) \rtimes \sigma. \tag{5-9}$$

Obviously, in the Grothendieck group we have

$$\delta([\nu^{-n}\rho, \nu^n\rho]) \rtimes \sigma \leq \delta([\nu^{-n}\rho, \nu^{-1}\rho]) \times \rho \times \delta([\nu\rho, \nu^n\rho]) \rtimes \sigma, \tag{5-10}$$

$$\delta([\nu\rho \times \nu^n\rho]) \rtimes \delta([\rho, \nu^n\rho]_{\tau_i}, \sigma) \leq \delta([\nu^{-n}\rho, \nu^{-1}\rho]) \times \rho \times \delta([\nu\rho, \nu^n\rho]) \rtimes \sigma. \tag{5-11}$$

Compute

$$\text{s.s.}\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{-1}\rho])\times\rho\times\delta([\nu\rho,\nu^{n}\rho])\rtimes\sigma\right)\right) = 2\rho\times\left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^{-1}\rho])\times\delta([\nu^{k+1}\rho,\nu^{n}\rho])\right]^{2}\otimes\sigma, \tag{5-12}$$

$$\text{s.s.}\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma\right)\right) = 2\left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^{n}\rho])\times\delta([\nu^{1+k}\rho,\nu^{n}\rho])\right]\otimes\sigma, \tag{5-13}$$

$$\text{s.s.}\left(s_{((2n+1)p)}\left(\delta([\nu\rho,\nu^{n}\rho])\rtimes\delta([\rho,\nu^{n}\rho]_{\tau_i},\sigma)\right)\right) = \left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^{-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^{n}\rho])\right]\times\delta([\rho,\nu^{n}\rho])\otimes\sigma. \tag{5-14}$$

We shall now obtain same consequences from the above formulas. The multiplicity of $\delta([\nu^{-n}\rho,\nu^{n}\rho])\otimes\sigma$ in (5-13) is two (look at the support of GL-part of the representation). The Frobenius reciprocity now implies that the dimension of the intertwining algebra of the (unitarizable) representation $\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma$ is at most two. Therefore, $\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma$ is a multiplicity one representation of length ≤ 2. Also, if π is an irreducible subrepresentation of $\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma$, then $\delta([\nu^{-n}\rho,\nu^{n}\rho])\otimes\sigma$ is a subquotient of $s_{((2n+1)p)}(\pi)$.

Considering $2\delta([\rho,\nu^{n}\rho])\times\delta([\nu\rho,\nu^{n}\rho])\otimes\sigma$ and taking into account supports, one gets that in the Grothendieck group

$$s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma\right)\not\leq s_{((2n+1)p)}\left(\delta([\nu\rho,\nu^{n}\rho])\rtimes\delta([\rho,\nu^{n}\rho]_{\tau_i},\sigma)\right).$$

Thus

$$\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma\not\leq\delta([\nu\rho,\nu^{n}\rho])\rtimes\delta([\rho,\nu^{n}\rho]_{\tau_i},\sigma). \tag{5-15}$$

In a similar way considering $\delta([\nu^{-n}\rho,\nu^{-1}\rho])\times\delta([\rho,\nu^{n}\rho])\otimes\sigma$ one gets

$$\delta([\nu\rho,\nu^{n}\rho])\rtimes\delta([\rho,\nu^{n}\rho]_{\tau_i},\sigma)\not\leq\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma. \tag{5-16}$$

Note that the multiplicity of $\delta([\nu\rho,\nu^{n}\rho])\times\delta([\rho,\nu^{n}\rho])$ in $\rho\times\delta([\nu\rho,\nu^{n}\rho])\times\delta([\nu\rho,\nu^{n}\rho])$ is one (both of these representations are non-degenerate, and the highest derivatives are the same). We can now conclude that the multiplicity of $\delta([\nu\rho,\nu^{n}\rho])\times\delta([\rho,\nu^{n}\rho])\otimes\sigma$ in (5-12) is 2, in (5-13) is 2 and in (5-14) is 1. From the last multiplicities we can conclude that $\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma$ and $\delta([\nu\rho,\nu^{n}\rho])\rtimes\delta([\rho,\nu^{n}\rho]_{\tau_i},\sigma)$ have non-disjoint Jordan-Hölder series. Further, from the above multiplicities follows that some common subquotient must have $\delta([\nu\rho,\nu^{n}\rho])\times\delta([\rho,\nu^{n}\rho])\otimes\sigma$ for a subquotient of corresponding Jacquet modules (the multiplicity must be one). Furthermore, (5-15) implies that $\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma$ is reducible. Since $\delta([\nu^{-n}\rho,\nu^{n}\rho])\rtimes\sigma$ is a multiplicity one representation of length two, (5-15) and (5-16) imply that $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ and $\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)$ have exactly one irreducible subquotient in common. All this implies that the common irreducible subquotient must be $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$. Therefore, (i) holds.

Next we shall see that $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_1},\sigma)\not\cong\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_2},\sigma)$. Suppose that we have an isomorphism. Write $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma=\pi_1\oplus\pi_2$ where π_1 and π_2 are irreducible. We know that $\pi_1\not\cong\pi_2$. It is easy to conclude from (5-13) that $\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma\leq s_{((2n+1)p)}(\pi_i)$ for some i. Lemma 5.3 and its proof imply that the multiplicity is one, so the inequality holds for $i=1$ and 2. Now $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_1},\sigma)\cong\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_2},\sigma)$ implies that there exists $i\in\{1,2\}$ such that $2\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_1},\sigma)+\pi_i\leq(\rho\times\nu\rho\times\nu^2\rho\times\cdots\times\nu^m\rho)\times(\nu\rho\times\nu^2\rho\times\cdots\times\nu^n\rho)\rtimes\sigma$. Lemma 5.3 implies that this can not happen (look at the Jacquet modules corresponding to s_{GL}). This finishes the proof of (ii).

From the Jacquet modules of $\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)$ we know $\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\hookrightarrow\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i$. Thus $\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\hookrightarrow\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i$. Note that

$$\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\times\rho\rtimes\sigma\cong\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes(\tau_1\oplus\tau_2). \tag{5-17}$$

One gets directly that $s_{((2n+1)p)}\left(\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i\right)$ is just a half of the right hand side of (5-12). Looking at (5-13) we can now conclude that

$$\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\not\leq\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i \tag{5-18}$$

Now it is clear that $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ may be characterized as a unique common irreducible subquotient of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ and $\delta([\nu\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\tau_i$. This and(1-3) imply directly the formula for contragredients. Thus (iv) and (v) hold.

From (i), (5-13) and (5-14) we obtain easily that

$$\text{s.s.}\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\right)\right)\leq\sum_{k=0}^{k=n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$$

Since the sum of s.s. $\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\right)\right)$ for $i=1,2$, equals to (5-13) by (ii), in the above inequality we must have the equality. This proves (iii). $\square$

5.5. Theorem. *Suppose that (ρ,σ) satisfies (C0). Let $n,m\in\mathbb{Z},0<n<m$. Then:*
(i) There exists a unique common irreducible subquotient of $\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)$ and $\nu^n\rho\rtimes\delta([\nu^{-(n-1)}\rho,\nu^m\rho]_{\tau_i},\sigma)$. That subquotient is $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$.
(ii) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$ is square integrable.
(iii) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)\tilde{}\cong\delta([\nu^{-n}\rho,\nu^m\rho]_{\tilde\tau_i},\tilde\sigma)$
(iv) s.s. $\left(s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)\right)\right)=\sum_{k=0}^n\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$
(v) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_1},\sigma)\not\cong\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_2},\sigma)$

Proof. We consider the lexicographic order on pairs $\{(n,m)\in\mathbb{Z}\times\mathbb{Z},\quad 0<n<m\}$. We shall prove the theorem by induction with respect to this order. Write first

$$\begin{aligned}\text{s.s.}&\left(s_{((n+m+1)p)}\left(\nu^n\rho\times\nu^m\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)\right)\right)\\ &=(\nu^n\rho\times\nu^m\rho+\nu^{-n}\rho\times\nu^m\rho+\nu^n\rho\times\nu^{-m}\rho+\nu^{-n}\rho\times\nu^{-m}\rho)\\ &\times\left[\sum_{k=0}^{n-1}\delta([\nu^{-k}\rho,\nu^{m-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^{n-1}\rho])\right]\otimes\sigma,\end{aligned} \tag{5-19}$$

$$\text{(5-20)} \quad \text{s.s.}\left(s_{((n+m+1)p)}\left(\nu^n\rho \rtimes \delta([\nu^{-n+1}\rho,\nu^m\rho]_{\tau_i},\sigma)\right)\right)$$
$$= (\nu^n\rho + \nu^{-n}\rho) \times \left[\sum_{k=0}^{n-1} \delta([\nu^{-k}\rho,\nu^m\rho]) \times \delta([\nu^{1+k}\rho,\nu^{n-1}\rho])\right] \otimes \sigma,$$

$$\text{(5-21)} \quad \text{s.s.}\left(s_{((n+m+1)p)}\left(\nu^m\rho \times \delta([\nu^{n}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)\right)\right)$$
$$= (\nu^m\rho \times \nu^{-m}\rho) \times \left[\sum_{k=0}^{n} \delta([\nu^{-k}\rho,\nu^{m-1}\rho]) \times \delta([\nu^{1+k}\rho,\nu^{n}\rho])\right] \otimes \sigma.$$

We shall first find all common irreducible subquotients of (5-20) and (5-21). Since $\nu^{-m}\rho$ does not appear in GL-support of any irreducible representation in (5-20), this term after multiplication in (5-21) will not give anything in common. From the other side, if we fix a member of the sum in (5-21), and consider all $\alpha \in \mathbb{Z}$, such that $\nu^\alpha\rho$ is in the GL-support of that member, then they form a $\mathbb{Z}-$segment. Using this observation we can see that factor $\nu^{-n}\rho$ can give after multiplication in (5-20) something in common with (5-21) only when it is multiplied with $\delta([\nu^{-n+1}\rho,\nu^m\rho])$.

Comparing GL-supports, we see that the following pairs can have something in common:

$$\nu^{-n}\rho \times \delta([\nu^{-n+1}\rho,\nu^m\rho]) \otimes \sigma \quad \text{and} \quad \nu^m\rho \times \delta([\nu^{-n}\rho,\nu^{m-1}\rho]) \otimes \sigma;$$

$$\nu^n\rho \times \delta([\nu^{-k}\rho,\nu^m\rho]) \times \delta([\nu^{1+k}\rho,\nu^{n-1}\rho]) \otimes \sigma \quad \text{and}$$
$$\nu^m\rho \times \delta([\nu^{-k}\rho,\nu^{m-1}\rho]) \times \delta([\nu^{1+k}\rho,\nu^{n}\rho]) \otimes \sigma, \text{ for } k = 0,1,\dots,n-1.$$

From the description of subquotients of generalized principal series representations ([Z1], see also [T1]), we get that irreducible subquotients which are in common are

$$\text{(5-22)} \quad \delta([\nu^{-k}\rho,\nu^m\rho]) \times \delta([\nu^{1+k}\rho,\nu^{n}\rho]) \otimes \sigma, \text{ when } k = 0,1,\dots,n.$$

Multiplicities with which these representations appear in (5-20) and (5-21) are one.

We shall now see the multiplicities of the above representations in (5-19). Considering supports, by a similar analysis as above, we can easily get that they can appear only in the following terms

$$\nu^{-n}\rho \times \nu^m\rho \times \delta([\nu^{-n+1}\rho,\nu^{m-1}\rho]) \otimes \sigma, \quad \text{and}$$
$$\nu^n\rho \times \nu^m\rho \times \delta([\nu^{-k}\rho,\nu^{m-1}\rho]) \times \delta([\nu^{1+k}\rho,\nu^{n-1}\rho]) \otimes \sigma, \text{ when } k = 0,1,\dots,n-1.$$

This implies that the multiplicities of representations of (5-22) in (5-19) are one.

We now claim

$$\text{(5-23)} \quad \nu^n\rho \rtimes \delta([\nu^{-n+1}\rho,\nu^m\rho]_{\tau_i},\sigma) \leq \nu^n\rho \times \nu^m\rho \rtimes \delta([\nu^{-n+1}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma),$$

$$\text{(5-24)} \quad \nu^m\rho \rtimes \delta([\nu^{-n}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma) \leq \nu^n\rho \times \nu^m\rho \rtimes \delta([\nu^{-n+1}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma).$$

If $m > n+1$, then both relations follow from the inductive assumptions. Suppose that $m = n+1$. Then the first relation is again a consequence of the inductive assumption. For (5-24) it is enough to prove that $\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma) \leq \nu^n\rho \rtimes \delta([\nu^{-n+1}\rho,\nu^n\rho]_{\tau_i},\sigma)$. Note that the right hand side of the inequality is $\leq \nu^n\rho\times\nu\rho\times\nu^2\rho\times\cdots\times\nu^{n-1}\rho\times\nu\rho\times\nu^2\rho\times\cdots\times\nu^n\rho\rtimes\tau_i$. Further, using the inductive assumption we see that $s_{((2n+1)p)}\left(\nu^n\rho \rtimes \delta([\nu^{n-1}\rho,\nu^n\rho]_{\tau_i},\sigma)\right)$ contains $\delta([\rho,\nu^n\rho]\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient. This proves the second inequality in the case $m = n+1$.

At this point we can draw same conclusions. Denote $\pi_1 = \nu^n\rho \rtimes \delta([\nu^{-n+1}\rho,\nu^m\rho]_{\tau_i},\sigma)$, $\pi_2 = \nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)$, and $\pi_3 = \nu^n\rho\times\nu^m\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^{m-1}\rho]_{\tau_i},\sigma)$. If π is an irreducible subquotient of π_1 and π_2, then $s_{((2n+m+1)p)}(\pi)$ has for a subquotient at least one representation from (5-22). Conversely, if π is a subquotient of π_3 which has at least one representation from (5-22) as a subquotient of $s_{((n+m+1)p)}(\pi)$, then π has multiplicity one in π_3, and it is a subquotient of both π_1 and π_2. We used that $\pi_1 \leq \pi_3$ (what is just inequality (5-23)), $\pi_2 \leq \pi_3$ ((5-24)), and that all multiplicities of representation from (5-22) in $s_{(2n+m+1)p)}(\pi_i)$ are one. Denote all common irreducible subquotients of π_1 and π_2 by $\vartheta_1,\dots,\vartheta_\ell$, where $\vartheta_i \not\cong \vartheta_j$ for $i \neq j$. We now know that s.s.$(s_{((n+m+1)p)}(\vartheta_1+\cdots+\vartheta_\ell)) = \sum_{k=0}^n \delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma$. From this we see easily that all $\vartheta_1,\dots,\vartheta_\ell$ are square integrable using the square integrability criterion.

It remains to prove $\ell = 1$. This would prove (i) and (iii). Then the formula for the contragredient follows directly from the inductive assumption, Theorems 5.2, 5.4, and the characterization of $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$ in (i).

Take $\vartheta\in\{\vartheta_1,\dots,\vartheta_\ell\}$ which has $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ as a subquotient of $s_{((n+m+1)p)}(\vartheta)$. Then $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ it is actually a direct summand (see the central character). Therefore, $\vartheta\hookrightarrow\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$. This implies $\vartheta\hookrightarrow\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^{-n}\rho\rtimes\sigma$. Take $0\leq k<n$. Then

$$\begin{aligned}
&\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^{-n}\rho\rtimes\sigma \cong \nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^n\rho\rtimes\sigma\\
&\cong \nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^{-k}\rho\times\nu^n\rho\times\nu^{-k-1}\rho\times\nu^{-k-2}\rho\times\cdots\times\nu^{-n+1}\rho\rtimes\sigma\cong\dots\\
&\cong \nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu\rho\times\rho\times\nu^{-1}\rho\times\cdots\times\nu^{-k}\rho\times\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{k+1}\rho\rtimes\sigma.
\end{aligned}$$

Thus $\nu^m\rho\otimes\cdots\otimes\nu\rho\otimes\rho\otimes\nu^{-1}\rho\otimes\cdots\otimes\nu^{-k}\rho\otimes\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu^{k+1}\rho\otimes\sigma$ is a subquotient of $s_{(p)^{n+m+1}}(\vartheta)$. Therefore, $s_{((n+m+1)p)}(\vartheta)$ has an irreducible subquotient which has GL-support $(\nu^m\rho,\cdots,\nu\rho,\rho,\nu^{-1}\rho,\cdots,\nu^{-k}\rho,\nu^n\rho,\nu^{n-1}\rho,\cdots,\nu^{k+1}\rho)$. The only representation in (5-22) with such GL-support, is $\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{k+1}\rho,\nu^n\rho])\otimes\sigma$. Thus, the above representation must be subquotient of $s_{((n+m+1)p)}(\vartheta)$. Since $0\leq k<n$ was arbitrary, we get that $\ell = 1$ (the proof of ℓ =1 we could start from any ϑ_i, any irreducible quotient of $s_{((n+m+1)p)}(\vartheta_i)$; in a similarly way as above we would get that all representations in (5-22) are subquotients of $s_{((n+m+1)p)}(\vartheta_i)$; this would again imply $\ell = 1$).

The claim (v) follows from the following lemma in a similar way as (iv) of Theorem 5.4 followed from the fact that $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ is a multiplicity one representation. □

5.6. Lemma. *If $0\leq n\leq m$, then $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ is a multiplicities one representation.*

Proof. For $n = m$ we know that the lemma holds (Theorem 5.4). It is enough to consider the case $n < m$. We shall prove the lemma by induction on $n+m$. For $n = 0$ and $m = 1$

the lemma follows from the formula for $\mu^*\left(\delta([\rho,\nu\rho])\rtimes\sigma)\right)$ in the proof of Lemma 5.1. Fix $n+m>1$ and suppose that the lemma holds for $n'+m'<n+m$. Observe that $M^*(\delta([\nu^{-n}\rho,\nu^m\rho]))$ can be written as

$$\left[1\otimes\delta([\nu^{-n}\rho,\nu^m\rho])\right]+\left[\nu^m\rho\otimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho])+\nu^n\rho\otimes\delta([\nu^{-n+1}\rho,\nu^m\rho])\right]+X,$$

where X is a sum of members of the form $x_i\otimes y_i$ such that x_i is a representation of some $GL(p\,k,F)$ with $k\geq 2$. This implies

$$\text{s.s.}\left(s_{(p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma\right)\right)=\nu^m\rho\otimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho])\rtimes\sigma+\nu^n\rho\otimes\delta([\nu^{-n+1}\rho,\nu^m\rho])\rtimes\sigma.$$

The inductive assumption and $n\neq m$, imply that the above representation is a multiplicity one representation (observe that $\delta([\nu\rho,\nu^m\rho])\rtimes\sigma$ is irreducible by Theorem 9.1 of [T7]). Now the lemma follows directly since each irreducible subquotient π of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ must have $s_{(p)}(\pi)\neq 0$. $\square$

The above lemma follows also from Proposition 3.10 of [J], using [A2] or [ScSt].

5.7. Remark. One can easily see that $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)\cong\delta([\nu^{-n'}\rho',\nu^{m'}\rho']_{\tau'_{i'}},\sigma')$ implies $\rho\cong\rho'$, $n=n'$, $m=m'$ and $\sigma\cong\sigma'$ (then we have shown that also $\tau_i\cong\tau'_{i'}$).

5.8. Theorem. *Let $n,m\in\mathbb{Z}$, $m>n\geq 0$.. Write $\rho\rtimes\sigma=\tau_1\oplus\tau_2$, with τ_1 and τ_2 irreducible. Then*
(i) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$, $(i=1,2)$, is a subrepresentation of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$. There are no other irreducible subrepresentations of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$.
(ii) $\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$ is a subrepresentation of $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$, and there is no other irreducible subrepresentation in $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$.

Proof. The proof is a variation of the proof of Theorem 4.3. We shall give only the main points of the proof. Set $\pi_i=\delta([\nu^{-n}\rho,\nu^m\rho]_{\tau_i},\sigma)$. Theorems 5.2 ((iii)), 5.5 ((iv)) and [C] (Theorem 7.3.2 imply that $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ is a direct summand in $s_{GL}(\pi)$ and $s_{GL}(\pi_-)$. Therefore, we have embeddings $\pi_i\hookrightarrow\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$, $i=1,2$. Assume that there is an irreducible subrepresentation π' of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ different from (images of) π_i, $i=1,2$. Then $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ is a quotient of $s_{GL}(\pi')$, which implies (using also (v) of Theorem 5.5) that multiplicity of $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma\right)$ is at least 3. One checks directly that this multiplicity is 2. This completes the proof of (i).

Multiplicity of $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)$ is 2, and in $s_{GL}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\right)$ is at least 1 $(i=1,2)$. Thus, these two multiplicities are both 1. The fact $\delta([\nu^{-n}\rho,\nu^m\rho])\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho])$ and the above discussion, imply that either $\pi_i\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ for $i=1,2$, or $\pi_i\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_{3-i}},\sigma)$ for $i=1,2$. We shall see that the last possibility can not occur. Note that $\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)\leq\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\delta([\rho,\nu^n\rho]_{\tau_i},\sigma)\leq\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\times\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu\rho\times\tau_i$ by (i) of Theorem 5.4 and Theorem 5.2. Now Lemma 5.3 implies that π_i is a subquotient of $\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ for $i=1,2$. The above discussion about multiplicities implies now $\pi_i\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_{\tau_i},\sigma)$ for $i=1,2$.

The uniqueness in (ii) one gets in the same way as in Theorem 4.3 from

$$(5\text{-}25)\qquad M^*\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\right)=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\circ m^*\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\right)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\Bigg(\sum_{a=n}^{m}\delta([\nu^{a+1}\rho,\nu^m\rho])\otimes\delta([\nu^{n+1}\rho,\nu^a\rho])\Bigg)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\Bigg(\sum_{a=n}^{m}\delta([\nu^{n+1}\rho,\nu^a\rho])\otimes\delta([\nu^{a+1}\rho,\nu^m\rho])\Bigg)$$

$$=\sum_{a=n}^{m}\sum_{b=a}^{m}\delta([\nu^{-a}\rho,\nu^{-n-1}\rho])\times\delta([\nu^{b+1}\rho,\nu^m\rho])\otimes\delta([\nu^{a+1}\rho,\nu^b\rho]),$$

and

$$(15\text{-}26)\quad \mu^*\left(\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)\le\Bigg(\prod_{i=-n}^{n}(1\otimes\nu^i\rho+\nu^i\rho\otimes 1+\nu^{-i}\rho\otimes 1)\Bigg)\rtimes(1\otimes\sigma).\quad\square$$

5.9. Proposition. *Let $n\in\mathbb{Z}$, $n\ge 0$ and $\alpha\in\mathbb{R}$.*
(i) Assume that (ρ,σ) satisfies (C0). Suppose that $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ contains an irreducible square integrable subquotient, say π. Then π is equivalent either to a representation listed in Theorem 5.2 or Theorem 5.5.
(ii) If $\rho\not\cong\tilde\rho$, then $\nu^\alpha\delta([\rho,\nu^n\rho])\rtimes\sigma$ can not contain a square integrable subquotient.

Proof. One proves the above proposition in a similar way as Proposition 4.4. One needs only to use Proposition 3.11 of [J] instead of Proposition 3.6 from the same paper, which was used in the proof of Proposition 4.4. $\square$

6. Reducibility at 1, I

In this section ρ will be an irreducible unitarizable cuspidal representation of $GL(p,F)$ and σ an irreducible cuspidal representation of S_q such that $\nu\rho\rtimes\sigma$ reduces and $\nu^\alpha\rho\rtimes\sigma$ is irreducible for $\alpha\in\mathbb{R}\backslash\{\pm1\}$. In other words, we assume that (ρ,σ) satisfies (C1).

6.1. Theorem. *For a positive integer n the representation $\rho\times\delta([\nu\rho,\nu^n\rho],\sigma)$ splits into a sum of two non-equivalent irreducible tempered representations. They are not square integrable. Denote them by π_1 and π_2. Then $\delta([\rho,\nu^n\rho])\otimes\sigma$ is a subquotient either of $s_{((n+1)p)}(\pi_1)$ or of $s_{((n+1)p)}(\pi_2)$. Denote the irreducible tempered representation which has $\delta([\rho,\nu^n\rho])\otimes\sigma$ for a subquotient of the Jacquet module by $\delta([\rho,\nu^n\rho],\sigma)$. The other irreducible tempered representation will be denoted by $\delta([\rho,\nu^n\rho]_-,\sigma)$. Then:*
(i) s.s. $\left(s_{((n+1)p)}\left(\delta([\rho,\nu^n\rho],\sigma)\right)\right)=\delta([\rho,\nu^n\rho])\otimes\sigma+\delta([\nu\rho,\nu^n\rho])\times\rho\otimes\sigma.$
(ii) s.s. $\left(s_{((n+1)p)}\left(\delta([\rho,\nu^n\rho]_-,\sigma)\right)\right)=L\left(\rho,\delta([\nu\rho,\nu^n\rho])\right)\otimes\sigma.$
(iii) $\delta([\rho,\nu^n\rho],\sigma)\tilde{}\cong\delta([\rho,\nu^n\rho],\tilde\sigma)$, $\delta([\rho,\nu^n\rho]_-,\sigma)\tilde{}\cong\delta([\rho,\nu^n\rho]_-,\tilde\sigma)$.

(iv) The representation $\delta([\rho,\nu^n\rho],\sigma)$ can be characterized as a unique common irreducible subquotient of $\delta([\rho,\nu^n\rho])\rtimes\sigma$ and $\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$.

Proof. Write

$$\mu^*\left(\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)\right)=1\otimes\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$$
$$+\left[2\rho\otimes\delta([\nu\rho,\nu^n\rho],\sigma)+\nu^n\rho\otimes\rho\rtimes\delta([\nu\rho,\nu^{n-1}\rho],\sigma)\right]+\cdots+\left[2\rho\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma\right],$$
$$\text{s.s.}\left(s_{((n+1)p)}\left(\delta([\rho,\nu^n\rho])\rtimes\sigma\right)\right)=\left[\sum_{k=0}^{n+1}\delta([\nu^{-k+1}\rho,\rho])\times\delta([\nu^k\rho,\nu^n\rho])\right]\otimes\sigma.$$

From the Frobenius reciprocity we can conclude that $\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$ is a multiplicity one representation of length ≤ 2. Then, the common irreducible factors in the Jacquet modules $s_{((n+1)p)}\left(\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)\right)$ and $s_{((n+1)p)}\left(\delta([\rho,\nu^n\rho])\rtimes\sigma\right)$ are $\delta([\rho,\nu^n\rho])\otimes\sigma$ and $L\left(\rho,\delta([\nu\rho,\nu^n\rho])\right)\otimes\sigma$. The multiplicity of $\delta([\rho,\nu^n\rho])\otimes\sigma$ in both Jacquet modules is 2. The multiplicity of $L\left((\rho,[\nu\rho,\nu^n\rho])\right)\otimes\sigma$ in the first Jacquet module is two, while in the second one is one. Note that

$$\delta([\rho,\nu^n\rho])\rtimes\sigma\not\leq\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma),\quad\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)\not\leq\delta([\rho,\nu^n\rho])\rtimes\sigma,$$
$$\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)\leq\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma,\quad\delta([\rho,\nu^n\rho])\rtimes\sigma\leq\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma,$$
$$\text{s.s.}\left(s_{((n+1)p)}\left(\rho\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma\right)\right)=2\rho\times\left[\sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^{-1}\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\right]\otimes\sigma.$$

The multiplicities of $\delta([\rho,\nu^n\rho])$ and $L\left(\rho,\delta([\nu\rho,\nu^n\rho])\right)$ in the above Jacquet module are both equal to two. We can now conclude that $\rho\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$ and $\delta([\rho,\nu^n\rho])\rtimes\sigma$ have exactly one irreducible subquotient in common, say π_1, and that $\text{s.s.}\left(s_{((n+1)p)}(\pi_1)\right)=\delta([\rho,\nu^n\rho])\otimes\sigma+\rho\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$. Denote the other summand of $\rho\times\delta([\nu\rho,\nu^n\rho],\sigma)$ by π_2. Then $s_{((n+1)p)}(\pi)=L\left(\rho,\delta([\nu\rho,\nu^n\rho])\right)$. All the remaining claims of the theorem now follow automatically. $\square$

We need the following lemma for a lower estimate of a Jacquet module in the following theorem.

6.2. Lemma. *Suppose that (ρ,σ) satisfies (C1). Let π be an irreducible subquotient of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$. Then*

$$\text{s.s.}\left(s_{((2n+1)p)}(\pi)\right)\geq\sum_{k=1}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{k+1}\rho,\nu^n\rho])\otimes\sigma.$$

Proof. Note that each term of the sum on the right hand side of the above inequality is irreducible.

Recall that π must be a subrepresentation of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$. Frobenius reciprocity implies $s_{((2n+1)p)}(\pi)\geq\delta([\nu^{-n}\rho,\nu^n\rho])\otimes\sigma$. If $n=1$, then the lemma is proved. Thus suppose that $n>1$. Now

$$\pi\hookrightarrow\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\hookrightarrow\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^{-n}\rho\rtimes\sigma$$
$$\cong\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{-n+2}\rho\times\nu^{-n+1}\rho\times\nu^n\rho\rtimes\sigma.$$

Thus $s_{(p,p,\dots,p)}(\pi)$ has $\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu^{-n+2}\rho\otimes\nu^{-n+1}\rho\otimes\nu^n\rho\otimes\sigma$ as subquotient. Since this subquotient of the Jacquet module must come as a subquotient of the Jacquet module of an irreducible subquotient of $s_{((2n+1)p)}(\pi)$ (because of the transitivity of process of taking Jacquet modules), and $\pi\leq\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$, we see that $\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu^{-n+2}\rho\otimes\nu^{-n+1}\rho\otimes\nu^n\rho\otimes\sigma$ must came from Jacquet module of $s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)$. Recall that (5-13) gives a formula for semi simplification of the last representation. Considering the right hand side of (5-13), looking at the GL-supports, we see that $\nu^n\rho\otimes\nu^{n-1}\rho\otimes\cdots\otimes\nu^{-n+2}\rho\otimes\nu^{-n+1}\rho\otimes\nu^n\rho\otimes\sigma$ can come only from $\delta([\nu^{-n+1}\rho,\nu^n\rho])\times\nu^n\rho\otimes\sigma$. This implies $s_{((2n+1)p)}(\pi)\geq\delta([\nu^{-n+1}\rho,\nu^n\rho])\times\nu^n\rho\otimes\sigma$. In the some way, one proves for the other terms the inequality. This completes the proof of the lemma. □

6.3. Theorem. *Suppose that (ρ,σ) satisfies (C1). Let n be a positive integer. Then representation $\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^{-n}\rho\rtimes\sigma$ contains a unique irreducible subquotient $\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)$ which has $\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ for a subquotient of $s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\right)$. Further:*
(i) The multiplicity of $\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ in $s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\right)$ is two.
(ii) The multiplicity of $\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)$ in $\nu^n\rho\times\nu^{n-1}\rho\times\cdots\times\nu^{-n+1}\rho\times\nu^{-n}\rho\rtimes\sigma$ is one.
(iii) The representation $\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)$ may be characterized as a unique common irreducible subquotient of $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ and $\delta([\rho,\nu^n\rho])\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$.
(iv) $\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\tilde{}\cong\delta([\nu^{-n}\rho,\nu^n\rho],\tilde{\sigma})$.

$$\text{(v)}\quad s.s.\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\right)\right)=\sum_{k=-1}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma$$

$$=2\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma+\sum_{k=1}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$$

(vi) The representation $\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$ is a multiplicity one representation of length two. Denote the other irreducible subquotient by $\delta([\nu^{-n}\rho,\nu^n\rho]_-,\sigma)$ (see (iii)). Then

$$s.s.\left(s_{((2n+1)p)}\left(\delta([\nu^{-n}\rho,\nu^n\rho]_-,\sigma)\right)\right)=\sum_{k=1}^{n}\delta([\nu^{-k}\rho,\nu^n\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$$

Proof. The proof of the theorem is similar to the proof of Theorem 3.2 (and Theorem 5.4). Therefore, we shall only sketch the proof (the complete proof can be found in [T6]). Denote $\pi_1=\delta([\rho,\nu^n\rho])\times\delta([\nu\rho,\nu^n\rho])\rtimes\sigma$, $\pi_2=\delta([\rho,\nu^n\rho])\rtimes\delta([\nu\rho,\nu^n\rho],\sigma)$, $\pi_3=\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma$. Then $\pi_2,\pi_3\leq\pi_1$. From the formula for s.s. $\left(s_{((2n+1)p)}(\pi_3)\right)$, we see that π_3 is a multiplicity one representation of length ≤ 2. Further, s.s. $\left(s_{((2n+1)p)}(\pi_3)\right)\not\leq$ s.s. $\left(s_{((2n+1)p)}(\pi_2)\right)$ implies $\pi_3\not\leq\pi_2$. The multiplicity of $\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma$ in $s_{((2n+1)p)}(\pi_i)$, $i=1,2,3$, is 2. From this we conclude that π_3 reduces., and that there exists a common irreducible subquotient π of π_2 and π_3 which has $\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma$ in the Jacquet module. We can also conclude that the multiplicity in the Jacquet module is 2. Now the last lemma and the formula for s.s. $\left(s_{((2n+1)p)}(\pi_3)\right)$ (see (5-13)) imply (v). All other claims follow now easily. □

6.4. Proposition. *Let $n, m \in \mathbb{Z}$, $m \geq n \geq 0$. Then*
(i) $s_{((n+m+1)p)}(\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma)$ contains $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient. The multiplicity is two.
(ii) If π is a subquotient of $\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$ such that $s_{((n+m+1)p)}(\pi)$ contains $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient, then $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ has in $s_{((n+m+1)p)}(\pi)$ multiplicity two. The multiplicity of π in $\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$ is one. We denote π by $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)$ (note that the above definition in the cases of $n=m$ or $n=0$ agrees with our old definitions in that cases).

Proof. We have proved (i) already. For (ii), it is enough to see that if π is a subquotient of $\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$ such that $s_{((n+m+1)p)}(\pi)$ contains $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient, then the multiplicity of $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ in $s_{((n+m+1)p)}(\pi)$ is two. Theorems 6.1 and 6.3 imply that it is enough to consider only the case $0<n<m$. Suppose that there exists a subquotient π_1 of $\nu^{-n}\rho\times\nu^{-n+1}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$ such that $s_{((n+m+1)p)}(\pi_1)$ contains $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ as a subquotient with multiplicity one. Set $\vartheta=\nu^{-n}\rho\times\cdots\times\nu^{m-1}\rho\times\nu^m\rho\rtimes\sigma$. Then there exists a subrepresentation $\vartheta_1\subseteq\vartheta_2\subseteq\vartheta$ such that $\vartheta_2/\vartheta_1\cong\pi_1$. Because of (i), there exists a subquotient π_2 of ϑ_1 or ϑ/ϑ_2 which has $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma$ for a subquotient of $s_{((n+m+1)p)}(\pi_2)$ (note that we do not claim that $\pi_1\not\cong\pi_2$). We now know $\pi_1+\pi_2\leq\vartheta$ and $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma\leq s_{((n+m+1)p)}(\pi_i)$ for $i=1,2$.

Consider now $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\vartheta$. Set $\varphi=\sum_{k=n}^m\delta([\nu^{-k}\rho,\nu^{-(n+1)}\rho])\times\delta([\nu^{k+1}\rho,\nu^m\rho])$. Now $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes(\pi_1+\pi_2)\leq\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^{-m}\rho\rtimes\sigma$. This implies

$$\text{(6-1)}\quad s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes(\pi_1+\pi_2)\right)\\ \leq s_{((2m+1)p)}(\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\nu^{-m}\rho\rtimes\sigma).$$

From the other side (using (1-4)), we have

$$\begin{aligned} s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\pi_i\right) &\geq \varphi\times\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma\\ &\geq\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^m\rho])\otimes\sigma.\end{aligned}$$

This fact, (i) and (6-1) imply that $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^m\rho])\otimes\sigma$ has multiplicity one in $s_{((2n+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\pi_i\right)$. This contradicts to (ii) in the case $m=n$ which we know that holds (Theorem 6.4). □

6.5. Lemma. *For $0\leq n\leq m$ we have*

$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\leq\sum_{k=-1}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho]).$$

6.6. Remark. Note that in the above sum only the first term (corresponding to $k=-1$) is not always irreducible. It is reducible when $n<m$. In that case, that term is a multiplicity one representation of length two. In the Grothendieck group we have $\delta([\nu\rho,\nu^m\rho])\times\delta([\rho,\nu^n\rho])\otimes\sigma=\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])+L\left(\delta([\nu\rho,\nu^m\rho]),\delta([\rho,\nu^n\rho])\right).$

Proof. For $n = m$ or $n = 0$ we know that the lemma holds (Theorems 6.3 and 6.4). Therefore, it is enough to consider the case of $m > n > 0$. We shall prove this case by induction (the lexicographic order is considered on pairs (n, m)). First we can conclude from Jacquet modules that $\delta([\nu^{-n}\rho, \nu^m\rho], \sigma) \leq \nu^m\rho \rtimes \delta([\nu^{-n}\rho, \nu^{m-1}\rho], \sigma)$ and $\delta([\nu^{-n}\rho, \nu^m\rho], \sigma) \leq \nu^n\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^m\rho], \sigma)$. Now we can write a natural upper bound for $s_{((n+m+1)p)}\left(\nu^m\rho \rtimes \delta([\nu^{-n}\rho, \nu^{m-1}\rho], \sigma)\right)$ using the inductive assumption, and also a natural upper bound for $s_{((n+m+1)p)}\left(\nu^n\rho \rtimes \delta([\nu^{-n+1}\rho, \nu^m\rho], \sigma)\right)$ using the inductive assumption. Then we find all common irreducible subquotients of that upper bounds, and also the multiplicities of that common irreducible subquotients. As a consequence, we get the estimate of the lemma. Since we have already done estimates of this type in the proofs of Theorems 3.3 and 5.5, we omit here details (all details can be found in the preprint [T6]). $\square$

6.7. Lemma. *For $0 < n < m$ we have*

$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)\right) \geq 2\delta([\rho, \nu^m\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma + \sum_{k=1}^{n} \delta([\nu^{-k}\rho, \nu^m\rho]) \times \delta([\nu^{1+k}\rho, \nu^n\rho]) \otimes \sigma.$$

Proof. We know $s_{((n+m+1)p)}(\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)) \geq 2\delta([\rho, \nu^m\rho]) \times \delta([\nu\rho, \nu^n\rho]) \otimes \sigma$ from Proposition 6.4. Suppose that we know $s_{((n+m+1)p)}(\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)) \geq \delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$. We shall first show that this implies the lemma. One needs to consider only the case of $n > 1$. Since $\delta([\nu^{-n}\rho, \nu^m\rho]) \otimes \sigma$ has different central character from the other irreducible subquotients in the Jacquet module $s_{((n+m+1)p)}(\delta([\nu^{-n}\rho, \nu^m\rho], \sigma))$, we conclude that it is a direct summand in the Jacquet module (use Lemma 6.5). Thus $\delta([\nu^{-n}\rho, \nu^m\rho], \sigma) \hookrightarrow \delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma$. Now we shall use an argument similar to the one that we have already used in the proof of Lemma 6.2. We have

$$\delta([\nu^{-n}\rho, \nu^m\rho], \sigma) \hookrightarrow \delta([\nu^{-n}\rho, \nu^m\rho]) \rtimes \sigma \hookrightarrow \nu^m\rho \times \cdots \times \nu^{-n+1}\rho \times \nu^{-n}\rho \rtimes \sigma \cong \nu^m\rho \times \cdots \times \nu^{-n+1}\rho \times \nu^n\rho \rtimes \sigma.$$

This implies that $\nu^m\rho \otimes \cdots \otimes \nu^{-n+1}\rho \otimes \nu^n\rho \otimes \sigma$ is in the Jacquet module. Further, the second term $\delta([\nu^{-n+1}\rho, \nu^m\rho]) \times \nu^n\rho \otimes \sigma$ in the sum must be in the Jacquet module (this is the only possible term by Lemma 6.5 which is in the Jacquet module and which has $\nu^m\rho \otimes \cdots \otimes \nu^{-n+1}\rho \otimes \nu^n\rho \otimes \sigma$ in suitable Jacquet module). One gets further terms in a similar fashion.

Note that $\delta([\nu^{-m}\rho, \nu^m\rho]) \otimes \sigma \leq s_{((2m+1)p)}(\delta([\nu^{-m}\rho, \nu^m\rho], \sigma))$, and $\delta([\nu^{-m}\rho, \nu^m\rho], \sigma) \leq \delta([\nu^{n+1}\rho, \nu^m\rho]) \rtimes \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)$. This implies

$$\delta([\nu^{-m}\rho, \nu^m\rho]) \otimes \sigma \leq s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho, \nu^m\rho]) \rtimes \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)\right). \tag{6-2}$$

Write $s_{((n+m+1)p)}(\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)) = \vartheta \otimes \sigma$. Then

$$\text{s.s.}\left(s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho, \nu^m\rho]) \rtimes \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)\right)\right) = \left[\sum_{k=n}^{m} \delta([\nu^{-k}\rho, \nu^{-n-1}\rho]) \times \delta([\nu^{1+k}\rho, \nu^m\rho])\right] \times \vartheta \otimes \sigma.$$

Lemma 6.5 and (6-2) imply that $\delta([\nu^{-n}\rho,\nu^m\rho]) \leq \vartheta$ (consider the term $\delta([\nu^{-m}\rho,\nu^m\rho])\otimes\sigma$). This ends the proof of the lemma. $\square$

6.8. Theorem. *Suppose that (ρ,σ) satisfies (C1) (then $\rho\cong\tilde{\rho}$). For $n,m\in\mathbb{Z}$, $0<n<m$, the representation $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)$ is square integrable. Further, $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{} \cong \delta([\nu^{-n}\rho,\nu^m\rho],\tilde{\sigma})$ and*

$$\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho]) + \sum_{k=0}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma$$
$$\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right) \leq \sum_{k=-1}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$$

Proof. It remains only to prove the formula for the contragredient. We proceed by induction. Suppose that $m>n>0$ and that the theorem holds for $m'+n'<m+n$. Note that $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}$ is a common irreducible subquotient of $\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho],\tilde{\sigma})$ and $\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho],\tilde{\sigma})$. The last two representations can have at most two common irreducible subquotients (this follows from the proof of Lemma 6.5). One is $\delta([\nu^{-n}\rho,\nu^m\rho],\tilde{\sigma})$. If there is only one irreducible subquotient in common, then the proof is complete. If there are two, denote the second one by π. Then $s_{((n+m+1)p)}(\pi)$ is irreducible. Therefore, for the proof in this case, it is enough to show that $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}\right)$ is not irreducible. Recall that

$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}\right)\cong\left[\underline{s}_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\right]\tilde{}$$

where $\underline{s}$ denotes the Jacquet module with respect to the choice of lower triangular matrices for standard minimal parabolic subgroup ([C], Corollary 4.2.5). But the lower parabolic subgroup ${}^tP_{((n+m+1)p)}$ is conjugated to $P_{((n+m+1)p)}$. Therefore, the length is the same. This implies that $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\tilde{}\right)$ is reducible. Now the proof is complete. $\square$

6.9. Remark. Proposition 3.10 of [J], together with [A2] or [ScSt], imply

$$\text{s.s.}\left(s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right)\right) = \sum_{k=-1}^{n}\delta([\nu^{-k}\rho,\nu^m\rho])\times\delta([\nu^{1+k}\rho,\nu^n\rho])\otimes\sigma.$$

7. Reducibility at 1, II

We continue in this section to denote by ρ an irreducible unitarizable cuspidal representation of $GL(p,F)$ and by σ an irreducible cuspidal representation of S_q, such that (ρ,σ) satisfies (C1).

7.1. Proposition. *There exists a unique irreducible subquotient $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$ of $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ which satisfies conditions*

$$\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma\leq s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right), \tag{7-1}$$
$$\nu\rho\times\delta([\rho,\nu^2\rho])\otimes\sigma\not\leq s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right). \tag{7-2}$$

Multiplicity of $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$ *in* $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ *is 1. Further* $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\not\cong\delta([\nu^{-1}\rho,\nu^2\rho],\sigma)$, $s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)=\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma$ *and* $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$ *is square integrable.*

Proof. Write

$$\begin{aligned}(7\text{-}3)\quad \text{s.s.}\left(s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma\right)\right)&=\big[\delta([\nu^{-2}\rho,\nu\rho])+\delta([\nu^{-1}\rho,\nu\rho])\times\nu^2\rho\\ &+\delta([\rho,\nu\rho])\times\delta([\nu\rho,\nu^2\rho])+\nu\rho\times\delta([\rho,\nu^2\rho])+\delta([\nu^{-1}\rho,\nu^2\rho])\big]\otimes\sigma.\end{aligned}$$

We see from this formula that the multiplicity of $\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma$ in (7-3) is 2. Now we can conclude from Theorem 6.8 that there exists a unique irreducible subquotient $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$ of $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ satisfying (7-1) and (7-2).

Since

$$\text{s.s.}\left(s_{(4p)}\left(\nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma\right)\right)=(\nu^{-2}\rho+\nu^2\rho)\times 2\left(\delta([\nu^{-1}\rho,\nu\rho])+\nu\rho\times\delta([\rho,\nu\rho])\right)\otimes\sigma,$$

multiplicity of $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ in the above representation is 2. Since

$$\begin{gathered}\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma\le\nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma,\\ \nu^2\rho\rtimes\delta([\nu^{-1}\rho,\nu\rho]_-,\sigma)\le\nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho])\rtimes\sigma,\\ \delta([\nu^{-1}\rho,\nu^2\rho],\sigma)\le\nu^2\rho\rtimes\delta([\nu^{-1}\rho,\nu\rho],\sigma),\\ \delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma\le s_{(4p)}\big(\delta([\nu^{-1}\rho,\nu^2\rho],\sigma)\big),\\ \delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma\le s_{(4p)}\big(\nu^2\rho\rtimes\delta([\nu^{-1}\rho,\nu\rho]_-,\sigma)\big),\\ \delta([\nu^{-1}\rho,\nu\rho],\sigma)\not\cong\delta([\nu^{-1}\rho,\nu\rho]_-,\sigma),\end{gathered}$$

we see that it must be $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\le\nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho]_-,\sigma)$. This implies

$$s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)\le(\nu^{-2}\rho+\nu^2\rho)\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma.$$

Since $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\le\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$, we can conclude further

$$(7\text{-}4)\qquad s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)\le\delta([\nu^{-2}\rho,\nu\rho])\otimes\sigma+\nu^2\rho\times\delta([\nu^{-1}\rho,\nu\rho])\otimes\sigma.$$

Consider now the unique irreducible quotient $L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)$ of $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma$ (such irreducible quotient is unique by the properties of the Langlands classification). Clearly, $L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)\not\cong\delta([\nu^{-1}\rho,\nu^2\rho],\sigma)$, since the later representation is square integrable. From Frobenius reciprocity we can conclude that $\delta([\nu^{-2}\rho,\nu\rho])\otimes\sigma$ is a subquotient of $s_{(4p)}\left(L\left(\delta([\nu^{-1}\rho,\nu^2\rho],\sigma)\right)\right)$. Multiplicity of $\delta([\nu^{-2}\rho,\nu\rho])\otimes\sigma$ in (7-3) is one. Therefore, one can characterize $L\left(\delta([\nu^{-1}\rho,\nu^2\rho],\sigma\right)$ using this subquotient of the Jacquet module.

Consider $\delta([\rho,\nu^2\rho])\times\nu\rho\rtimes\sigma$. Clearly $\delta([\nu^{-1}\rho,\nu^2\rho])\rtimes\sigma\le\delta([\rho,\nu^2\rho])\times\nu\rho\rtimes\sigma$. Further

$$\begin{aligned}&\text{s.s.}(s_{(4p)}(\delta([\rho,\nu^2\rho])\rtimes\delta(\nu\rho,\sigma)))=\\ &\qquad\left(\delta([\nu^{-2}\rho,\rho])+\delta([\nu^{-1}\rho,\rho])\times\nu^2\rho+\rho\times\delta([\nu\rho,\nu^2\rho])+\delta([\rho,\nu^2\rho])\right)\times\nu\rho\otimes\sigma.\end{aligned}$$

Note that the multiplicities of $\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma$ and $\nu\rho\times\delta([\rho,\nu^2\rho])\otimes\sigma$ in the above representation are 1 and 2 respectively. This implies $\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\leq\delta([\rho,\nu^2\rho])\rtimes L(\nu\rho,\sigma)$ (use Theorem 6.8). Applying Jacquet functor to this inequality, we get

$$\begin{aligned}&s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)\\&\qquad\leq\left(\delta([\nu^{-2}\rho,\rho])+\delta([\nu^{-1}\rho,\rho])\times\nu^2\rho+\rho\times\delta([\nu^1\rho,\nu^2\rho])+\delta([\rho,\nu^2\rho])\right)\times\nu^{-1}\rho\otimes\sigma.\end{aligned}$$

Multiplicity of $\delta([\nu^{-2}\rho,\nu\rho])\otimes\sigma$ in the above representation is 0. Therefore, we can conclude $L\left(\delta([\nu^{-1}\rho,\nu^2\rho],\sigma\right)\not\cong\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)$.

One has the following embeddings

$$L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)\hookrightarrow\delta([\nu^{-2}\rho,\nu\rho])\rtimes\sigma\hookrightarrow\nu\rho\times\rho\times\nu^{-1}\rho\times\nu^{-2}\rho\rtimes\sigma\cong\nu\rho\times\rho\times\nu^{-1}\rho\times\nu^2\rho\rtimes\sigma$$

(passing to contragredients one gets the first embedding). Therefore, $\nu\rho\otimes\rho\otimes\nu^{-1}\rho\otimes\nu^2\rho\otimes\sigma$ is a subquotient of the Jacquet module of $L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)$. From (7-3) we can now conclude that $L\left(\delta([\nu^{-1}\rho,\nu\rho]),\nu^2\rho\right)\otimes\sigma$ is a subquotient of $s_{(4p)}\left(L\left(\delta([\nu^{-1}\rho,\nu^2\rho]),\sigma\right)\right)$. This, (7-4) and (7-3) imply $s_{(4p)}\left(\delta([\nu^{-1}\rho,\nu^2\rho]_-,\sigma)\right)=\delta([\nu^{-1}\rho,\nu^2\rho])\otimes\sigma$. This implies square integrability. □

7.2. Proposition. *Suppose $m\geq 3$. Then there exists a unique irreducible subquotient $\delta([\nu^{-1}\rho,\nu^m\rho]_-,\sigma)$ of $\delta([\nu^{-1}\rho,\nu^m\rho])\rtimes\sigma$ which satisfies following conditions*

$$\delta([\nu^{-1}\rho,\nu^m\rho])\otimes\sigma\leq s_{((m+2)p)}\left(\delta([\nu^{-1}\rho,\nu^m\rho]_-,\sigma)\right)\tag{7-5}$$

$$\nu\rho\times\delta([\rho,\nu^m\rho])\otimes\sigma\not\leq s_{((m+2)p)}\left(\delta([\nu^{-1}\rho,\nu^m\rho]_-,\sigma)\right).\tag{7-6}$$

Multiplicity of $\delta([\nu^{-1}\rho,\nu^m\rho]_-,\sigma)$ in $\delta([\nu^{-1}\rho,\nu^m\rho])\rtimes\sigma$ is 1. Also $\delta([\nu^{-1}\rho,\nu^m\rho]_-,\sigma)\not\cong\delta([\nu^{-1}\rho,\nu^m\rho],\sigma)$ and $s_{((m+2)p)}\left(\delta([\nu^{-1}\rho,\nu^m\rho]_-,\sigma)\right)=\delta([\nu^{-1}\rho,\nu^m\rho])\otimes\sigma$. The representation $\delta([\nu^{-1}\rho,\nu^m\rho]_-,\sigma)$ is square integrable.

Proof. We prove the lemma by induction. Write

$$\text{s.s.}\left(s_{((m+2)p)}(\delta([\nu^{-1}\rho,\nu^m\rho])\rtimes\sigma)\right)=\sum_{i=-2}^{m}\delta([\nu^{-i}\rho,\nu\rho])\times\delta([\nu^{i+1}\rho,\nu^m\rho])\otimes\sigma.\tag{7-7}$$

Multiplicity of $\delta([\nu^{-1}\rho,\nu^m\rho])\otimes\sigma$ in (7-7) is 2. Now Theorem 6.8 implies that there exists a unique irreducible subquotient $\delta([\nu^{-1}\rho,\nu^m\rho]_-,\sigma)$ of $\delta([\nu^{-1}\rho,\nu^m\rho])\rtimes\sigma$ which satisfies (7-5) and (7-6).

Further, $\delta([\nu^{-1}\rho,\nu^m\rho])\rtimes\sigma\leq\nu^m\rho\times\delta([\nu^{-1}\rho,\nu^{m-1}\rho])\rtimes\sigma$. We see again easily that the multiplicity of $\delta([\nu^{-1}\rho,\nu^m\rho])\otimes\sigma$ in $s_{((m+2)p)}\left(\nu^m\rho\times\delta([\nu^{-1}\rho,\nu^{m-1}\rho])\rtimes\sigma\right)$ is 2 since

$$\begin{aligned}&\text{s.s.}\left(s_{((m+2)p)}\left(\nu^m\rho\times\delta([\nu^{-1}\rho,\nu^{m-1}\rho])\rtimes\sigma\right)\right)\\&\qquad=(\nu^{-m}\rho+\nu^m\rho)\times\left[\sum_{i=-2}^{m-1}\delta([\nu^{-i}\rho,\nu\rho])\times\delta([\nu^{i+1}\rho,\nu^{m-1}\rho])\right]\otimes\sigma.\end{aligned}$$

Now

$$\begin{aligned}
\delta([\nu^{-1}\rho,\nu^{m}\rho],\sigma) &\leq \nu^{m} \rtimes \delta([\nu^{-1}\rho,\nu^{m-1}\rho],\sigma),\\
\delta([\nu^{-1}\rho,\nu^{m}\rho])\otimes\sigma &\leq s_{((m+2)p)}\left(\delta([\nu^{-1}\rho,\nu^{m}\rho],\sigma)\right),\\
\delta([\nu^{-1}\rho,\nu^{m}\rho])\otimes\sigma &\leq s_{((m+2)p)}\left(\nu^{m}\rho \rtimes \delta([\nu^{-1}\rho,\nu^{m-1}\rho]_{-},\sigma)\right),\\
\delta([\nu^{-1}\rho,\nu^{m-1}\rho]_{-},\sigma) &\not\cong \delta([\nu^{-1}\rho,\nu^{m-1}\rho],\sigma)
\end{aligned}$$

imply

$$\delta([\nu^{-1}\rho,\nu^{m}\rho]_{-},\sigma) \leq \nu^{m}\rho \times \delta([\nu^{-1}\rho,\nu^{m-1}\rho]_{-},\sigma).$$

This, together with the inductive assumption, or the preceding proposition if $m = 3$, implies

$$s_{((m+2)p)}\left(\delta([\nu^{-1}\rho,\nu^{m}\rho]_{-},\sigma)\right) \leq (\nu^{-m}\rho+\nu^{m}\rho)\times\delta([\nu^{-1}\rho,\nu^{m-1}\rho])\otimes\sigma.$$

Since $\delta([\nu^{-1}\rho,\nu^{m}\rho]_{-},\sigma) \leq \delta([\nu^{-1}\rho,\nu^{m}\rho]) \rtimes \sigma$, we get easily using (7-7)

$$s_{((m+2)p)}\left(\delta([\nu^{-1}\rho,\nu^{m}\rho]_{-},\sigma)\right) \leq \delta([\nu^{-1}\rho,\nu^{m}\rho])\otimes\sigma$$

(here we needed the assumption $m \geq 3$). This implies square integrability. Also, obviously the equality must hold in the above relation. This finishes the proof. □

7.3. Theorem. *Let $n, m \in \mathbb{Z}, 1 < n < m$. Then there exists a unique irreducible subquotient $\delta([\nu^{-n}\rho,\nu^{m}\rho]_{-},\sigma)$ of $\delta([\nu^{-n}\rho,\nu^{m}\rho]) \rtimes \sigma$ which satisfies conditions*

$$\delta([\nu^{-1}\rho,\nu^{m}\rho])\times\delta([\nu^{2}\rho,\nu^{n}\rho])\otimes\sigma \leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{m}\rho]_{-},\sigma)\right), \tag{7-8}$$

$$\delta([\rho,\nu^{m}\rho])\times\delta([\nu\rho,\nu^{n}\rho])\otimes\sigma \not\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{m}\rho]_{-},\sigma)\right). \tag{7-9}$$

Multiplicity of $\delta([\nu^{-n}\rho,\nu^{m}\rho]_{-},\sigma)$ in $\delta([\nu^{-n}\rho,\nu^{m}\rho]) \rtimes \sigma$ is 1 and $\delta([\nu^{-n}\rho,\nu^{m}\rho]_{-},\sigma) \not\cong \delta([\nu^{-n}\rho,\nu^{m}\rho],\sigma)$. Further

$$\text{s.s.}\left(s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{m}\rho]_{-},\sigma)\right)\right) = \sum_{i=1}^{n}\delta([\nu^{-i}\rho,\nu^{m}\rho])\times\delta([\nu^{i+1}\rho,\nu^{n}\rho])\otimes\sigma. \tag{7-10}$$

The representation $\delta([\nu^{-n}\rho,\nu^{m}\rho]_{-},\sigma)$ is square integrable.

Proof. We shall prove the lemma by induction on n and m. Note that the claim of the theorem holds if $n = 1$ or $n = m$, except that in the later case the representation $\delta([\nu^{-n}\rho,\nu^{n}\rho]_{-},\sigma)$ is not square integrable. First we shall prove the following inequality

$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{m}\rho]_{-},\sigma)\right) \leq \sum_{i=-1}^{n}\delta([\nu^{-i}\rho,\nu^{m}\rho])\times\delta([\nu^{i+1}\rho,\nu^{n}\rho])\otimes\sigma. \tag{7-11}$$

Write

$$\text{s.s.}\left(s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^{m}\rho])\rtimes\sigma\right)\right) = \sum_{i=-m}^{n+1}\delta([\nu^{i}\rho,\nu^{n}\rho])\times\delta([\nu^{-i+1}\rho,\nu^{m}\rho])\otimes\sigma. \tag{7-12}$$

Observe that the multiplicity of $\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^2\rho,\nu^n\rho])\otimes\sigma$ in (7-12) is 2. Note that the multiplicity is 2 also when $n=m$ (see the formula (6-3) also). Now Theorem 6.8 implies that there exists a unique irreducible subquotient $\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)$ of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ satisfying (7-8) and (7-9).

Obviously

$$\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma\leq\nu^m\rho\times\nu^n\rho\times\delta([\nu^{-n+1}\rho,\nu^{m-1}\rho])\rtimes\sigma. \tag{7-13}$$

Since $n>m\geq 2$, it is easy to see that the multiplicity of $\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^2\rho,\nu^n\rho])\otimes\sigma$ in $s_{((n+m+1)p)}\left(\nu^m\rho\times\nu^n\rho\times\delta([\nu^{-n+1}\rho,\nu^{m-1}\rho])\rtimes\sigma\right)$ is 2. Now

$$\begin{aligned}
\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^2\rho,\nu^n\rho])\otimes\sigma &\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right),\\
\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^2\rho,\nu^n\rho])\otimes\sigma &\leq s_{((n+m+1)p)}\left(\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho]_-,\sigma)\right),\\
\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^2\rho,\nu^n\rho])\otimes\sigma &\leq s_{((n+m+1)p)}\left(\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho]_-,\sigma)\right),\\
\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho]_-,\sigma) &\leq \nu^m\rho\times\nu^n\rho\times\delta([\nu^{-n+1}\rho,\nu^{m-1}\rho])\rtimes\sigma,\\
\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho]_-,\sigma) &\leq \nu^m\rho\times\nu^n\rho\times\delta([\nu^{-n+1}\rho,\nu^{m-1}\rho])\rtimes\sigma,\\
\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma &\not\leq s_{((n+m+1)p)}\left(\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho]_-,\sigma)\right),\\
\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma &\not\leq s_{((n+m+1)p)}\left(\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho]_-,\sigma)\right),
\end{aligned}$$

imply that it must be

$$\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\leq\nu^m\rho\rtimes\delta([\nu^{-n}\rho,\nu^{m-1}\rho]_-,\sigma), \tag{7-14}$$

$$\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\leq\nu^n\rho\rtimes\delta([\nu^{-n+1}\rho,\nu^m\rho]_-,\sigma). \tag{7-15}$$

Now in the same way as in the proof of Lemma 6.5, using induction follows the inequality (7-11). Since on the right hand sides of (7-14) and (7-15) there are no representations which in the support have only representations of type $\nu^\alpha\rho$ with $\alpha\geq 0$, we get the stronger inequality

$$s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right)\leq\sum_{i=1}^{n}\delta([\nu^{-i}\rho,\nu^m\rho])\times\delta([\nu^{i+1}\rho,\nu^n\rho])\otimes\sigma. \tag{7-16}$$

Consider $\delta([\nu^{-n}\rho,\nu^m\rho])\times\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\sigma$. One gets easily that the multiplicity of $\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^2\rho,\nu^m\rho])\otimes\sigma$ in $s_{((2m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho])\times\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\sigma\right)$ is 2, since the last representation is equal to

$$\left[\sum_{i=n}^{m}\delta([\nu^{-i}\rho,\nu^{-n-1}\rho])\times\delta([\nu^{i+1}\rho,\nu^m\rho])\right]\times\left[\sum_{i=-m}^{n+1}\delta([\nu^{i}\rho,\nu^{n}\rho])\times\delta([\nu^{-i+1}\rho,\nu^m\rho])\right]\otimes\sigma.$$

Further

$$\begin{aligned}
\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^2\rho,\nu^m\rho])\otimes\sigma &\leq s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right),\\
\delta([\nu^{-1}\rho,\nu^m\rho])\times\delta([\nu^2\rho,\nu^m\rho])\otimes\sigma &\leq s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)\right),\\
\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^n\rho])\otimes\sigma &\not\leq s_{((2m+1)p)}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right).
\end{aligned}$$

One can easily conclude

$$\delta([\nu^{-m}\rho,\nu^m\rho]_-,\sigma)\leq\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma).$$

Using the last relation and $\delta([\nu^{-m}\rho,\nu^m\rho])\otimes\sigma\leq s_{((2m+1)p)}\left(\delta([\nu^{-m}\rho,\nu^m\rho]_-,\sigma)\right)$, one can in the same way as in the second half of the proof of Lemma 6.7 prove that

$$\delta([\nu^{-n}\rho,\nu^m\rho]\otimes\sigma\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right).$$

Using the inequality (7-16), we see that $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ must be a direct summand in $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right)$. Now in the same way as in the first half of the proof of Lemma 6.7 it follows that the formula in the lemma for $s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right)$ holds. This finishes the proof.

7.4. Corollary. *The representation $\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)$ can be characterized as a unique irreducible subquotient of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ which satisfies conditions*

$$\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right),$$
$$\delta([\nu\rho,\nu^n\rho])\times\delta([\rho,\nu^m\rho])\otimes\sigma\not\leq s_{((n+m+1)p)}\left(\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)\right). \quad \square$$

7.5. Theorem. *Let $n,m\in\mathbb{Z}$, $m>n>0$. Then*
(i) $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)$ and $\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)$ are subrepresentations of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$, and $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ does not contain any other irreducible subrepresentation.
(ii) $\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)$ (resp. $\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)$) is a unique irreducible subrepresentation of $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)$ (resp. $\delta([\nu^{n+1}\rho,\nu^m\rho])\rtimes\delta([\nu^{-n}\rho,\nu^n\rho]_-,\sigma)$).

Proof. The proof is a simple modification of the proof of Theorem 4.3. We shall only outline the proof. Put $\pi=\delta([\nu^{-n}\rho,\nu^m\rho],\sigma)$ (resp. $\pi_-=\delta([\nu^{-n}\rho,\nu^m\rho]_-,\sigma)$). Again we conclude from Theorems 6.8 and 7.3, using Theorem 7.3.2 of [C], that $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ is a direct summand in $s_{GL}(\pi)$ and $s_{GL}(\pi_-)$. Therefore, there exist embeddings $\pi\hookrightarrow\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ and $\pi_-\hookrightarrow\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$. If there is an irreducible subrepresentation π' of $\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma$ different from (images of) π and π_-, then $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ would be a quotient of $s_{GL}(\pi')$. Therefore, the multiplicity of $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma\right)$ would be at least 3. This contradicts to the fact that multiplicity of $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{-n}\rho,\nu^m\rho])\rtimes\sigma\right)$ is 2, what one easily prove using (1-4).

Multiplicity of $\delta([\nu^{-n}\rho,\nu^m\rho])\otimes\sigma$ in $s_{GL}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho])\rtimes\sigma\right)$ is 2, in $s_{GL}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)\right)$ and $s_{GL}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_-,\sigma)\right)$ is at least 1. Because of this, the last two multiplicities are both 1. Note that $\delta([\rho,\nu^m\rho])\times\delta([\nu\rho,\nu^m\rho])\otimes\sigma$ is a subquotient of $s_{GL}\left(\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]\sigma)\right)$

Now from $\delta([\nu^{-n}\rho,\nu^m\rho])\hookrightarrow\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho])$ and the above discussion we can conclude that π embeds in $\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho],\sigma)$, and π_- in $\delta([\nu^{n+1}\rho,\nu^m\rho])\times\delta([\nu^{-n}\rho,\nu^n\rho]_-,\sigma)$. At the end, one gets uniqueness in (ii) in the same way as in Theorem 4.3, using formulas (5-25) and (5-26). $\square$

7.6. Proposition. *Let $n \in \mathbb{Z}$, $n \geq 0$ and $\alpha \in \mathbb{R}$.*
(i) Assume that (ρ, σ) satisfies (C1). Suppose that $\nu^\alpha \delta([\rho, \nu^n \rho]) \rtimes \sigma$ contains an irreducible square integrable subquotient, say π. Then π is equivalent either to a representation listed in Theorem 2.1, or Theorem 6.8 or Theorem 7.3.
(ii) If $\rho \not\cong \tilde{\rho}$, then $\nu^\alpha \delta([\rho, \nu^n \rho]) \rtimes \sigma$ can not contain a square integrable subquotient.

Proof. Now one uses Proposition 3.10 of [J]. $\square$

8. A completeness of square integrable representations of segment type

We have proved:

8.1. Theorem. *Let τ be an irreducible essentially square integrable representation of $GL(p, F)$, $p \geq 1$, and let σ be an irreducible cuspidal representation of S_g, $q \geq 0$. Suppose that $\tau \rtimes \sigma$ contains an irreducible square integrable subquotient π. Assume that (C) holds in general. Then π is equivalent to one of the square integrable representations listed in Theorems 2.1, 3.3, 4.2, 5.2, 5.5, 6.8 and 7.3.* $\square$

To hold the claim of the theorem, we did not need to assume that (C) holds in general. Write $\tau = \delta([\rho, \nu^k \rho])$, where ρ is an irreducible cuspidal representation of suitable $GL(i, F)$. It was enough to assume only that (ρ^u, σ) satisfies (C).

9. Square integrable representations corresponding to several segments

For $\Delta \in M(\mathcal{S}(\mathcal{C}))$ denote $\tilde{\Delta} = \{\tilde{\rho} \in \Delta; \rho \in \Delta\}$. We shall say that Δ is selfdual if $\Delta = \tilde{\Delta}$. We say that Δ is balanced if $e(\delta(\Delta)) = 0$. Clearly, a selfdual segment is balanced.

Let X be a set. For a finite multiset $x = (x_1, \dots, x_k)$ in X, we shall denote by $\text{Set}(x) = \{x_1, \dots, x_k\}$ the subset of X corresponding to x (this is the set which one gets from the multiset x forgetting multiplicities of elements which enter x). If one considers a finite multiset x in X as a function $x : X \to \{z \in \mathbb{Z}; z \geq 0\}$ with a finite support, then $\text{Set}(x)$ is just the support of the function x.

In the following proposition we collect some facts about tempered representations that we need in construction of square integrable representations corresponding to several segments in irreducible cuspidal representations of general linear groups. Claim (i) in the following proposition was proved by D. Goldberg in [G]. We present here a different proof of (i), to have the claim (i) proved also in positive characteristic.

9.1. Proposition. *Let σ be an irreducible cuspidal representation of S_q. Let $\Delta_1, \dots, \Delta_k \in \mathcal{S}(\mathcal{C})$ be a sequence of different selfdual segments. Write $\Delta_i = [\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i]$, $i = 1, \dots, k$, where $\rho_i \in \mathcal{C}$, $n_i \in (1/2)\mathbb{Z}$. Suppose that (ρ_i, σ) satisfy (C) and $\delta(\Delta_i) \rtimes \sigma$ reduces, for $i = 1, \dots, k$. Then*
(i) $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \rtimes \sigma$ is a multiplicity one representation of length 2^k.
(ii) The multiplicity of $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \otimes \sigma$ in $s_{GL}(\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \rtimes \sigma)$ is 2^k.
(iii) Let τ be an irreducible subrepresentation of $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \rtimes \sigma$. Then the multiplicity of $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \otimes \sigma$ in $s_{GL}(\tau)$ is one.
(iv) Let τ be as in (iii). If π is any irreducible subquotient of $s_{GL}(\tau)$ different from $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \otimes \sigma$, then

$$\text{(9-1)} \qquad \textit{Set}\,(\textit{supp}_{GL}(\pi)) \subseteq \Delta_1 \cup \dots \cup \Delta_k \textit{ and } \textit{supp}_{GL}(\pi) \neq \Delta_1 + \dots + \Delta_k.$$

9.2. Remark. The condition that $\delta(\Delta_i) \rtimes \sigma$ reduces is equivalent to the following conditions:

$$\text{(9-2)} \qquad \begin{cases} \text{if } (\rho_i, \sigma) \text{ satisfies (C0), then } n_i \in \mathbb{Z}, n_i \geq 0; \\ \text{if } (\rho_i, \sigma) \text{ satisfies (C1/2), then } n_i \in 1/2 + \mathbb{Z}, n_i \geq 1/2; \\ \text{if } (\rho_i, \sigma) \text{ satisfies (C1), then } n_i \in \mathbb{Z}, n_i \geq 1. \end{cases}$$

This is proved in [T7]. Actually, we have proved in the present paper that conditions (9-2) imply reducibility. For the purpose of this paper, it would be enough to work directly with conditions (9-2) in the above proposition. We have chosen rather the conditions that representations $\delta(\Delta_i) \rtimes \sigma$ reduce, because this is the real meaning of conditions (9-2). The condition for reducibility can be expressed in a very natural (and simple) way (see Theorem 9.3).

Recall that if Δ_i is balanced, but not selfdual, then $\delta(\Delta_i) \rtimes \sigma$ is irreducible.

Proof. First we prove (ii). Denote $\beta = \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \otimes \sigma$ (clearly, β is irreducible). Denote

$$\text{(9-3)} \qquad M^*_{GL}\left(\delta([\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i])\right) = \sum_{j=-n_i}^{n_i+1} \delta([\nu^j\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{-j+1}\rho_i, \nu^{n_i}\rho_i]).$$

The above sum runs over $j \in n_i + \mathbb{Z}$ such that $-n_i \leq j \leq n_i + 1$ (such convention we shall also use in the sequel). Then (1-4) implies

$$\text{(9-4)} \qquad \text{s.s.}\left(s_{GL}\left(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma\right)\right) = M^*_{GL}\left(\delta(\Delta_1)\right) \times \cdots \times M^*_{GL}\left(\delta(\Delta_k)\right) \otimes \sigma.$$

Note that for $k = -n_i$ or $n_i + 1$, the term in the sum (9-3) is $\delta(\Delta_i)$. Therefore multiplying these terms in (9-4) we get that the multiplicity of β in $s_{GL}\left(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma\right)$ is at least 2^k.

Now we shall see that β can appear as a subquotient of $s_{GL}\left(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma\right)$ only in the above way. We shall discuss when β can be obtained as a subquotient of the product in (9-4). Choose i_1 such that $\Delta_{i_1} \not\subseteq \Delta_i$ for $i \in \{1, \ldots, k\}, i \neq i_1$ (this choice is possible since Δ_i's are mutually different). If we want to get β in the product of the right hand side of (9-4), then on the i_1-place in the product we must take a term in the sum (9-3) corresponding to $-n_{i_1}$ or $n_{i_1}+1$ (since $\nu^{-n_{i_1}}\rho_{i_1}$ is in $\text{supp}_{GL}(\beta)$, and because neither other terms in the sum except these two, can give $\nu^{-n_{i_1}}\rho_{i_1}$ in the GL-support, nor other terms in the product can give $\nu^{-n_{i_1}}\rho_{i_1}$ in the GL-support, thanks to the condition $\Delta_{i_1} \not\subseteq \Delta_i$ for $i \neq i_1$). This proves that on the i_1-th place β can come only from terms corresponding to $k = -n_{i_1}$ or $n_{i_1} + 1$. Now chose $i_2 \in \{1, \ldots, k\}, i_2 \neq i_1$ such that $\Delta_{i_2} \not\subseteq \Delta_i$ for $i \in \{1, \ldots, n\} \backslash \{i_1, i_2\}$. Then repeating the above type of argument with the GL-support (and $\nu^{-n_2}\rho_{n_2}$), we obtain that we can get β in the product only if on i_2-th place we take a term corresponding to $-i_2$ or $i_2 + 1$ (one needs to work with $\text{supp}_{GL}(\beta) - \Delta_{i_1}$, where $-$ denotes subtraction between multisets). Choosing $i_3, i_4, \ldots, i_k$ in analogous way and continuing with above type of argument, we obtain that β can appear only in the way that we have described before. Therefore, the multiplicity of β in $s_{GL}\left(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma\right)$

is 2^k. This proves (ii). Note that in the proof of (ii) we did not need that (ρ_i, σ)'s satisfy (C).

Now we shall prove (i) by induction (Theorems 4.9, 6.4, 6.5 and 1.9 of [G] imply (i) when $\operatorname{char}(F) = 0$). Suppose that these claims hold for k. After renumeration we can assume that $\Delta_{k+1} \not\subseteq \Delta_i$ for $1 \le i \le k$. Now (ii) implies for the intertwining algebra

$$\dim_{\mathbb{C}}\left(\operatorname{End}\left(\left(\prod_{i=1}^{k+1} \delta(\Delta_i)\right) \rtimes \sigma\right)\right) \le 2^{k+1}. \tag{9-5}$$

Let τ be any irreducible subrepresentation of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma$. Using (1-4) and (2-1) we compute

(9-6)

$$\mu^*\left(\delta(\Delta_{k+1}) \rtimes \tau\right) = M^*\left(\delta(\Delta_{k+1})\right) \rtimes \mu^*(\tau) = M^*\left(\delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])\right) \rtimes \mu^*(\tau)$$

$$= (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*\left(\delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])\right) \rtimes \mu^*(\tau) = (m \otimes 1) \circ (\sim \otimes m^*) \circ s$$

$$\left(\sum_{a_{k+1}=-n_{k+1}-1}^{n_{k+1}} \delta([\nu^{a_{k+1}+1}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}]) \otimes \delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{a_{k+1}}\rho_{k+1}])\right) \rtimes \mu^*(\tau)$$

$$= (m \otimes 1) \circ (\sim \otimes m^*)$$

$$\left(\sum_{a_{k+1}=-n_{k+1}-1}^{n_{k+1}} \delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{a_{k+1}}\rho_{k+1}]) \otimes \delta([\nu^{a_{k+1}+1}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])\right) \rtimes \mu^*(\tau)$$

$$= \left(\sum_{a_{k+1}=-n_{k+1}-1}^{n_{k+1}} \sum_{b_{k+1}=a_{k+1}}^{n_{k+1}} \delta([\nu^{-a_{k+1}}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}]) \times \delta([\nu^{b_{k+1}+1}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])\right.$$

$$\left. \otimes\, \delta([\nu^{a_{k+1}+1}\rho_{k+1}, \nu^{b_{k+1}}\rho_{k+1}])\right) \rtimes \mu^*(\tau)$$

From the above formula we see that multiplicity of $\delta(\Delta_{k+1}) \rtimes \tau$ in $\mu^*\left(\delta(\Delta_{k+1}) \rtimes \tau\right)$ is 2, since $\delta(\Delta_{k+1}) \rtimes \tau$ can come only from terms corresponding to indexes $a_{k+1} = -n_{k+1} - 1$, $b_{k+1} = a_{k+1} = -n_{k+1} - 1$, and $a_{k+1} = n_{k+1}$, $b_{k+1} = a_{k+1} = n_{k+1}$ (consider the term $\nu^{-n_{k+1}}\rho_{k+1}$).

Note that the inductive assumption and (9-5) imply that for the proof of (i) it is enough to prove that $\delta(\Delta_{k+1}) \rtimes \tau$ reduces. Suppose that it does not reduce. We know $\delta(\Delta_{k+1}) \rtimes \sigma = \Psi_1 \oplus \Psi_2$, for irreducible Ψ_1 and Ψ_2. Therefore, $\delta(\Delta_{k+1}) \rtimes \tau \le \left(\prod_{i=1}^{k}(\delta(\Delta_i)\right) \rtimes \Psi_i$ for some $\Psi \in \{\Psi_1, \Psi_2\}$. First we get in the same way as (9-6)

$$\mu^*\left(\left[\prod_{i=1}^{k} \delta(\Delta_i)\right] \rtimes \Psi\right) = \left[\prod_{i=1}^{k} M^*\left(\delta(\Delta_i)\right)\right] \rtimes \mu^*(\Psi) \tag{9-7}$$

$$= \Big[\prod_{i=1}^{k} \Big[\sum_{a_i=-n_i-1}^{n_i} \sum_{b_i=a_i}^{n_i} \delta([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i+1}\rho_i, \nu^{n_i}\rho_i]) \otimes \delta([\nu^{a_i+1}\rho_i, \nu^{b_i}\rho_i]) \Big] \Big] \rtimes \mu^*(\Psi).$$

Further we have

$$\text{(9-8)} \quad \mu^*(\delta(\Delta_{k+1}) \rtimes \sigma) = \Big[\sum_{a_{k+1}=-n_{k+1}-1}^{n_{k+1}} \sum_{b_{k+1}=a_{k+1}}^{n_{k+1}} \delta([\nu^{-a_{k+1}}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}])$$
$$\times \delta([\nu^{b_{k+1}+1}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}]) \otimes \delta([\nu^{a_{k+1}+1}\rho_{k+1}, \nu^{b_{k+1}}\rho_{k+1}]) \Big] \rtimes \big(1 \otimes \sigma\big).$$

Multiplicity of $\delta(\Delta_{k+1}) \otimes \sigma$ in (9-8) is 2. Frobenius reciprocity implies that the multiplicity of $\delta(\Delta_{k+1}) \otimes \sigma$ in $s_{GL}(\Pi)$ is one. Now (9-7), $\Psi \leq \delta(\Delta_{k+1}) \rtimes \sigma$, (9-8) and the inductive assumption imply that the multiplicity of $\delta(\Delta_{k+1}) \otimes \tau$ in (9-7) is one. This contradicts to the multiplicity of $\delta(\Delta_{k+1}) \otimes \sigma$ in (9-6) (one gets this easily analyzing when can be obtained a term of the form $\delta([\nu^{-n_{k+1}}\rho_{k+1}, \nu^{n_{k+1}}\rho_{k+1}]) \otimes \dots$ in (9-8)). Thus, (i) holds.

Claim (iii) follows from the fact that $\Big(\prod_{i=1}^{k} \delta(\Delta_i)\Big) \otimes \sigma$ must be a quotient of $s_{GL}(\tau)$ (what follows from Frobenius reciprocity and unitarizability of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$), using (i) and (ii).

It remains to prove (iv). The first claim in (iv) follows from (9-4) and (9-3). From the above considerations it is easy to see that π as in (iv) can come from a term (say β') in product of the right hand side of (9-4), only if at some place i' enters a term corresponding to $-n_{i'} < j < n_{i'} + 1$. Denote (for above β') the set of all such indexes i' by X (i.e. the set of all indexes i' where enters a term corresponding to $-n_{i'} < j < n_{i'} + 1$). Choose i_0 such that $\Delta_{i_0} \not\subseteq \Delta_i$ for any $i \in X \backslash \{i_0\}$. Now it is easy to see that $\mathrm{supp}_{GL}(\pi) \neq \Delta_1 + \cdots + \Delta_k$ (consider multiplicity of $\nu^{-n_{i_0}}\rho_{i_0}$ in $\mathrm{supp}_{GL}(\pi)$ and in $\Delta_1 + \cdots + \Delta_k$; they are different). □

The following theorem we do not need in this paper. We mention the theorem because it gives an additional explanation of background of conditions in the following proposition.

9.3. Theorem ([T7]). *Let $\Delta = [\nu^{\alpha}\rho, \nu^{\beta}\rho] \in \mathcal{S}(\mathcal{C})$, where $\alpha, \beta \in \mathbb{R}$, and ρ is unitarizable. Assume char$(F) = 0$. Let σ be an irreducible cuspidal representation of S_q. Suppose that (ρ, σ) satisfies (C) if ρ is selfdual (in particular, this holds if σ is non-degenerate). Then $\delta(\Delta) \rtimes \sigma$ reduces if and only if $\rho' \rtimes \sigma$ reduces for some $\rho' \in \Delta$.*

9.4. Proposition. *Let $\Delta_i = [\nu^{-n_i}\rho_i, \nu^{m_i}\rho_i] \in \mathcal{S}(\mathcal{C})$, $i = 1, \dots, k$, where ρ_i are selfdual, $m_i, n_i \in (1/2)\,\mathbb{Z}$, and let σ be an irreducible cuspidal representation of S_q. Assume that (ρ_i, σ) satisfy (C) for $i = 1, \dots, k,$. Suppose that the following three conditions hold*

(a) $m_i > n_i$ for $i = 1, \dots, k$.

$$\text{(b)} \qquad \begin{cases} \text{If } (\rho_i, \sigma) \text{ satisfies (C1/2), then } m_i, n_i \in 1/2 + \mathbb{Z},\ m_i \geq 1/2, n_i \geq -1/2. \\ \text{If } (\rho_i, \sigma) \text{ satisfies (C0), then } m_i, n_i \in \mathbb{Z},\ m_i \geq 1, n_i \geq 0. \\ \text{If } (\rho_i, \sigma) \text{ satisfies (C1), then } m_i, n_i \in \mathbb{Z},\ m_i \geq 1, n_i \geq -1 \text{ and } n_i \neq 0. \end{cases}$$

(c) If $\rho_i \cong \rho_j$ for some $i \neq j$, the either $m_i < n_j$ or $m_j < n_i$.

Denote $l = card(\{\Delta_i \cap \tilde{\Delta}_i; i = 1, \dots, n\} \backslash \{\emptyset\}) = card(\{i; 1 \le i \le n$ and $n_i \ge 0\})$. Then:
(i) Multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}\left(\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma\right)$ is 2^l.
(ii) Let τ be an irreducible subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$. Multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}\left(\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau\right)$ is one.
(iii) Let τ be as in (ii). The representation $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau$ has a unique irreducible subquotient π_τ such that $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma$ is a subquotient of $s_{GL}(\pi_\tau)$. Further, multiplicity of π_τ in $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau$ is one. We shall denote π_τ by

$$\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau.$$

Multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau)$ is one.
(iv) $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ is a subquotient of $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \rtimes \sigma$.
(v) If π is a subquotient of $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \rtimes \sigma$ such that $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \otimes \sigma$ is a subquotient $s_{GL}(\pi)$, then π is isomorphic to some $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$.
(vi) If $\Delta'_1, \dots, \Delta'_{k'}$ and σ' is some system which satisfies (a) - (c), and τ' is an irreducible subrepresentation of $\left(\prod_{i=1}^{k'} \delta(\Delta'_i \cap \tilde{\Delta}'_i)\right) \rtimes \sigma'$, then $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau \cong \delta(\Delta'_1, \dots, \Delta'_{k'}, \sigma')_{\tau'}$ implies $\{\Delta_1, \dots, \Delta_k\} = \{\Delta'_1, \dots, \Delta'_{k'}\}$ and $\sigma \cong \sigma'$.

Later we shall prove that in (vi) also τ must be equivalent τ'.

9.5. Remarks. Assume that char $(F) = 0$. Then conditions on $\Delta_1, \dots, \Delta_k \in \mathcal{S}(\mathcal{C})$ and σ in the last proposition are equivalent to the following conditions
(α) $\delta(\Delta_i) \rtimes \sigma$ reduces for $i = 1, \dots, k$.
(β) If $1 \le i \le k$ and $\Delta_i \cap \tilde{\Delta}_i \neq \emptyset$, then $\delta(\Delta_i \cap \tilde{\Delta}_i) \rtimes \sigma$ reduces.
(γ) $e\,(\delta(\Delta_i)) > 0$ for $i = 1, \dots, k$.
(δ) If $\Delta_i \cap \Delta_j \neq \emptyset$ for some $1 \le i \neq j \le k$, then

$$\Delta_i \cup \tilde{\Delta}_i \subsetneqq \Delta_j \cap \tilde{\Delta}_j \text{ or } \Delta_j \cup \tilde{\Delta}_j \subsetneqq \Delta_i \cap \tilde{\Delta}_i.$$

(ϵ) (ρ_i, σ) satisfies (C) for $i = 1, \dots, k$.

If σ is non-degenerate, then conditions in the last proposition are equivalent to (α) - (δ) only.

Note that condition (β) is almost automatically fulfilled when (α) holds. The only exception is when (ρ_i, σ) satisfies (C1) (i.e. $\nu\rho \rtimes \sigma$ reduces) and n_i=0.

Proof. Assume that $\Delta_1, \dots, \Delta_k$ and σ satisfy conditions (a) - (c) in the proposition.

The proof of (i) and (ii) is similar to the proof of (ii) and (iii) of Proposition 9.1. We shall modify that proof to the present situation. Denote $\beta = \delta(\Delta_1) \times \dots \times \delta(\Delta_k) \otimes \sigma$ (condition (c) provides that β is irreducible) and

$$M^*_{GL}\left(\delta([\nu^{-n_i}\rho_i, \nu^{m_i}\rho_i])\right) = \sum_{j=-n_i}^{m_i+1} \delta([\nu^{-j+1}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{j}\rho_i, \nu^{m_i}\rho_i]) \tag{9-9}$$

(the sum is over $j \in -n_i + \mathbb{Z}$ such that $-n_i \le j \le m_i + 1$). Now as before

$$s_{GL}\left(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma\right) = M^*_{GL}\left(\delta(\Delta_1)\right) \times \cdots \times M^*_{GL}\left(\delta(\Delta_k)\right) \otimes \sigma. \tag{9-10}$$

For $j = -n_i$, the term in the sum (9-9) is $\delta(\Delta_i)$. If $n_i < 0$ then this is the only term of the sum (9-9) where $\delta(\Delta_i)$ can be a subquotient (all other terms have different support from the support of $\delta(\Delta_i)$). If $n_i \ge 0$ then $-n_i < n_i+1$, and the term for $j = n_i+1$ in the sum is $\delta([\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{n_i+1}\rho_i, \nu^{m_i}\rho_i])$, which has $\delta(\Delta_i)$ for a subquotient (the multiplicity is one). These are the only two terms in the sum where $\delta(\Delta_i)$ can appear as a subquotient (again, all other terms have different support from the support of $\delta(\Delta_i)$). Multiplying above $\delta(\Delta_i)$'s in (9-10), we get that multiplicity of β in $s_{GL}\left(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma\right)$ is at least 2^l.

We shall show that β can appear as a subquotient of $s_{GL}\left(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma\right)$ only in the above way. Now we shall examine when β can appear in (9-10). Take i_1 such that $\Delta_{i_1} \not\subseteq \Delta_i$ for $i \in \{1, \dots, k\}, i \ne i_1$ (this choice is possible because of (c) and (a)). Since $\nu^{-n_{i_1}}\rho_{i_1}$ is in $\mathrm{supp}_{GL}(\beta)$, and $\nu^{\alpha}\rho_i$ is not in the GL-support for any $\alpha < -n_i$, we see that if we want that β appear as a subquotient of the product on the right hand side of (9-10), then on the i_1-place must be a term corresponding to $-n_{i_1}$ or $n_{i_1}+1$ if $n_i \ge 0$ (the choice of i_1 and (c) guarantees that no other terms in the product can give $\nu^{-n_{i_1}}\rho_{i_1}$ in the GL-support). The term corresponding to $n_{i_1}+1$ we can have only if $n_{i_1} \ge -1/2$. If $n_{i_1} = -1/2$, then $-n_{i_1} = n_{i_1}+1$. This proves that if we want to get β, then on the i_1-th place we need to take a term corresponding to $j = -n_{i_1}$ or $n_{i_1}+1$ (the possibility $n_{i_1}+1$ can happen only if $n_{i_1} \ge -1/2$, and if $n_{i_1} = -1/2$, then $-n_{i_1} = n_{i_1}+1$).

Further we chose $i_2 \in \{1, \dots, k\}, i_2 \ne i_1$, such that $\Delta_{i_2} \not\subseteq \Delta_i$ for $i \in \{1, \dots, k\} \backslash \{i_1, i_2\}$. Then repeating the above type of argument with the support, we get that we can get β in the product only if on the i_2-th place we take a term corresponding to $-i_2$ or i_2+1 (the possibility i_2+1 we need to take into account only if $n_{i_2} \ge 0$, see such discussion for i_1). Actually, one needs to consider now $\mathrm{supp}_{GL}(\beta) - \Delta_{i_1}$. Continuing choosing $i_3, i_4, \dots$ in a similar way, and repeating the above type of argument, we obtain that we can get β only in the way that we have described already above. Therefore, the multiplicity of β in $s_{GL}\left(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma\right)$ is 2^l, what is the claim of (i).

If $n_i \ge -1/2$, denote

$$M^*_{GL}\left(\delta([\nu^{n_i+1}\rho_i, \nu^{m_i}\rho_i])\right) = \sum_{j_i=n_i+1}^{m_i+1} \delta([\nu^{-j_i+1}\rho_i, \nu^{-n_i-1}\rho_i]) \times \delta([\nu^{j_i}\rho_i, \nu^{m_i}\rho_i]). \tag{9-11}$$

For $n_i = -1$ put

$$M^*_{GL}\left(\delta([\nu\rho_i, \nu^{m_i}\rho_i])\right) = \sum_{j_i=1}^{m_i+1} \delta([\nu^{-j_i+1}\rho_i, \nu^{-1}\rho_i]) \times \delta([\nu^{j_i}\rho_i, \nu^{m_i}\rho_i]). \tag{9-12}$$

Then

$$\mathrm{s.s.}\left(s_{GL}\left(\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau\right)\right) = \left(\prod_{i=1}^{k} M^*_{GL}(\Delta_i \backslash \tilde{\Delta}_i)\right) \times s_{GL}(\tau). \tag{9-13}$$

Here $\times$ on the right hand side multiplies $\prod_{i=1}^k M^*_{GL}(\Delta_i\backslash\tilde{\Delta}_i)$ with the terms on the left hand side of $\otimes$ which show up in $s_{GL}(\tau)$ (more precisely, of s.s. $(s_{GL}(\tau))$). From (iii) of Proposition 9.1 and (9-12) we get that β is a subquotient of (9-13). Namely, in the product of the right hand side of (9-13), one takes in (9-11) term corresponding to $j_i = n_i + 1$ if $n_i \geq -1/2$, and $j_i = 1$ if $n_i = -1$, and one takes $\left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \otimes \sigma$ from $s_{GL}(\tau)$ (see (iii) of Proposition 9.1). In this way one gets (in the Grothendieck group) $\left(\prod_{i=1}^k \left(\delta(\Delta_i\backslash\tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i)\right)\right) \rtimes \sigma$, which contains $\beta = \left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma$ as a subquotient, of multiplicity one.

Now we shall show that β can appear only in this way. Suppose that β is a subquotient of some $\gamma = \left(\prod_{i=1}^k \delta([\nu^{-j_i+1}\rho_i, \nu^{-n_i-1}\rho_i]) \times \delta([\nu^{j_i}\rho_i, \nu^{m_i}\rho_i])\right) \times s_{GL}(\tau)$, where $n_i + 1 \leq j_i \leq m_i + 1$ if $n_i \geq -1/2$ and $1 \leq j_i \leq m_i + 1$ if $n_i = -1$. Suppose that for some i_1, $n_{i_1} + 1 < j_{i_1}$ if $n_{i_1} \geq -1/2$, or $1 < j_{i_1}$ if $n_{i_1} = -1$.

Note that beginnings of segments $\delta(\Delta_i)$ are $\nu^{-n_i}\rho_i$. If $\nu^{-j_{i_1}+1}\rho_{i_1} = \nu^{-n_i}\rho_i$, then $\rho_{i_1} \cong \rho_i$ and $-j_{i_1} + 1 = -n_i$ (i.e. $j_{i_1} = n_i + 1$).

We shall now show that j_{i_1} can not be $n_i + 1$. Suppose that it is.This implies $n_{i_1} < n_i$ if $n_{i_1} \geq -1/2$, and $n_i + 1 > 1$ if $n_{i_1} = -1$. Therefore $n_{i_1} < n_i < m_i$. Now (c) implies $n_{i_1} < m_{i_1} < n_i < m_i$. The choice of j_{i_1}implies $j_{i_1} = n_i \leq m_{i_1} + 1$. This is a contradiction.

Therefore, $[\nu^{-j_{i_1}}\rho_{i_1}, \nu^{-n_{i_1}-1}\rho_{i_1}]$ must be possible to link from left with a disjoint segment in γ. Clearly, such segment can not be $[\nu^{j_i}\rho_i, \nu^{m_i}\rho_i]$ since $m_i > 0$ (see (a) and (c)). Suppose that such a segment is $[\nu^{-j_i+1}\rho_i, \nu^{-n_i-1}\rho_i]$. Then $-n_i = -j_{i_1}+1$, i.e. $j_{i_1} = n_1+1$ (and $\rho_i \cong \rho_{i_1}$). We have already seen that this is not possible.

The last possibility is to link with some segment from $s_{GL}(\tau)$. Their ends are $\nu^{n_i}\rho_i$ (see (9-2) and (9-3)). Therefore $n_i = -j_{i_1}$ and n_i is non-negative (and $\rho_i \cong \rho_{i_1}$). Thus $j_{i_1} \leq 0$. From the other side, we know $j_{i_1} > n_{i_1} \geq 1/2$ if $n_{i_1} \geq -1/2$, and $j_{i_1} > 1$ if $n_{i_1} = -1$. This is a contradiction. Therefore, β must be a subquotient of $\gamma = \left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \times s_{GL}(\tau)$.

Suppose that π is an irreducible subquotient of $s_{GL}(\tau)$ such that β is a subquotient of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \times \pi$. Then

$$\mathrm{supp}_{GL}(\pi) + \sum_{i=1}^k (\Delta_i\backslash\tilde{\Delta}_i) = \mathrm{supp}_{GL}(\beta) = \sum_{i=1}^k \Delta_i,$$

where we consider sets $\Delta_i\backslash\tilde{\Delta}_i$ and Δ_i as multisets in an obvious way. The above relation uniquely determines $\mathrm{supp}_{GL}(\pi)$. Since $\sum_{i=1}^k \Delta_i \cap \tilde{\Delta}_i$ satisfies the above relation, $\mathrm{supp}_{GL}(\pi) = \sum_{i=1}^k \Delta_i \cap \tilde{\Delta}_i$. We get $\pi = \left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \otimes \sigma$ from (iv) of Proposition 9.1. This is what we wanted to show. Therefore, this proves that multiplicity of β in $s_{GL}\left(\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \rtimes \tau\right)$ is 1, what is the claim of (ii). A direct consequence of (ii) is (iii).

Write $(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)) \rtimes \sigma = \oplus_{i=1}^{2^l} \tau_i$ where τ_i are irreducible. Then we have the following relations in the Gorthendieck group:

$$\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \rtimes \left(\oplus_{j=1}^{2^l} \tau_j\right) = \left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$$

$$= \left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma \geq \left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma$$

Since multiplicity of $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$ in $s_{GL}\left(\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \left(\oplus_{j=1}^{2^l} \tau_j\right)\right)$ and in $s_{GL}\left(\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma\right)$ is 2^l by (i) and (ii), and $\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \rtimes \sigma \leq \left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \left(\oplus_{j=1}^{2^l} \tau_j\right)$, (i), (ii), (iii) and Proposition 9.1 imply (iv). Similar argumentation gives (v).

We get (vi) using the fact that if two parabolically induced representations by irreducible cuspidal representations ρ' and ρ'' have an irreducible subquotient in common, then ρ' and ρ'' must be associate (see [C]). $\square$

The main aim of the rest of this section is to prove that representations $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ introduced in the last proposition are square integrable. By the way, we shall get a number of useful and interesting facts about these representations. We shall first prove three lemmas.

9.6. Lemma. *Fix an irreducible cuspidal representation σ of S_q. Let $\rho \in \mathcal{C}$ be selfdual. Assume that (ρ, σ) satisfies (C). Let $n_i, m_i \in (1/2)\,\mathbb{Z}$, $i = 1, \dots, k$, such that $m_i - n_j \in \mathbb{Z}$ for any $i, j \in \{1, \dots, k\}$, and*

$$n_1 < m_1 < n_2 < m_2 < n_3 < m_3 < \cdots < m_{k-1} < n_k < m_k.$$

Denote $\Delta_i = [\nu^{-n_i}\rho, \nu^{m_i}\rho]$. Suppose:

$$\begin{cases} \text{If } (\rho,\sigma) \text{ satisfies (C1/2), then } n_1 \in 1/2 + \mathbb{Z} \text{ and } n_1 \geq -1/2. \\ \text{If } (\rho,\sigma) \text{ satisfies (C0), then } n_1 \in \mathbb{Z} \text{ and } n_1 \geq 0. \\ \text{If } (\rho,\sigma) \text{ satisfies (C1), then } n_1 \in \mathbb{Z},\ n_1 \geq -1 \text{ and } n_1 \neq 0. \end{cases}$$

Let τ be an irreducible subrepresentation of $\left(\prod_{i=1}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$. Then:

(i) If $k \geq 2$ and $i' \in \{1, \dots, k\}$, then there exists an irreducible subrepresentation τ' of $\left(\prod_{1 \leq i \leq k, i \neq i'} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ such that

$$\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau \leq \delta(\Delta_{i'}) \rtimes \delta(\Delta_1, \dots, \Delta_{i'-1}, \Delta_{i'+1}, \dots, \Delta_k, \sigma)_{\tau'}.$$

(ii) There exists a positive integer c, depending on $\Delta_1, \dots, \Delta_k$ and σ, such that

$$\text{(9-14)} \quad s_{GL}\left(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau\right) \leq c\left[\prod_{i=1}^{k}\left[\sum_{a_i=-n_i}^{|n_i|} \sum_{b_i=-n_i}^{m_i+1} \delta([\nu^{a_i+1}\rho, \nu^{n_i}]) \times \delta([\nu^{b_i}\rho, \nu^{m_i}\rho])\right]\right] \otimes \sigma.$$

(iii) If π is and irreducible subquotient of $s_{GL}\left(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau\right)$ which satisfies $\pi \not\cong \left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma$, then $supp_{GL}(\pi) \neq supp_{GL}\left(\left(\prod_{i=1}^{k} \delta(\Delta_i)\right) \otimes \sigma\right)$

9.7. Remark. The product on the right hand side of (9-14) is not contained in a single $R_q(S)$. The inequality (9-14) holds also if one write the right hand side as a sum of products, and drop all products for which there does not exist an irreducible subquotient of the left hand side $s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)$, with the same GL-support (one can use also $s_{GL}\left(\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\sigma\right)$ instead of $s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)$). The formula (9-14) is convenient for inductive arguments.

Proof. For $k=1$ we know from the previous sections (3, 4, 5, 6 and 7) that the lemma holds. Therefore, we shall suppose that $k\geq 2$.

If $n_1<0$ define $\epsilon(\Delta_1)=1$. Take otherwise $\epsilon(\Delta_1)=0$.

Suppose first $n_1\geq 0$ or $i'>1$. Proposition 9.1 implies that we can write

$$\left[\prod_{1\leq i\leq k, i\neq i'}\delta(\Delta_i\cap\tilde{\Delta}_i)\right]\rtimes\sigma=\bigoplus_{j=1}^{2^{k-1-\epsilon(\Delta_1)}}\tau_j',\quad \left[\prod_{1\leq i\leq k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right]\rtimes\sigma=\bigoplus_{j=1}^{2^{k-\epsilon(\Delta_1)}}\tau_j$$

where τ_j and τ_j' irreducible. Similarly as in the proof of Proposition 9.4 we get in the Grothendieck group

$$\left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\left(\bigoplus_{r=1}^{2^{k-\epsilon(\Delta_1)}}\tau_r\right)=\left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\times\left(\prod_{i=1}^{k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$$

$$=\delta(\Delta_{i'}\backslash\tilde{\Delta}_{i'})\times\delta(\Delta_{i'}\cap\tilde{\Delta}_{i'})\times\left(\prod_{1\leq i\leq k, i\neq i'}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\times\left(\prod_{1\leq i\leq k, i\neq i'}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$$

$$\geq\delta(\Delta_{i'})\times\left(\prod_{1\leq i\leq k, i\neq i'}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\times\left(\prod_{1\leq i\leq k, i\neq i'}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$$

$$=\delta(\Delta_{i'})\times\left(\prod_{1\leq i\leq k, i\neq i'}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\left(\bigoplus_{j=1}^{2^{k-1-\epsilon(\Delta_1)}}\tau_j'\right)$$

$$\geq\delta(\Delta_{i'})\times\left(\sum_{j=1}^{2^{k-1-\epsilon(\Delta_1)}}\delta(\Delta_1,\dots,\Delta_{i'-1},\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau_j'}\right)$$

$$=\sum_{j=1}^{2^{k-1-\epsilon(\Delta_1)}}\delta(\Delta_{i'})\times\delta(\Delta_1,\dots,\Delta_{i'-1},\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau_j'}.$$

From (1-4), (iii) of Proposition 9.4 and (9-9) follows that the multiplicity of $\left(\prod_{i=1}^{k}\delta(\Delta_i)\right)\otimes\sigma$ in $s_{GL}\left(\delta(\Delta_{i'})\times\delta(\Delta_1,\dots,\Delta_{i'-1},\Delta_{i'+1},\Delta_k,\sigma)_{\tau_j'}\right)$ is ≥ 2. The above inequalities and (iii) of Proposition 9.4 imply that the multiplicity is 2. From multiplicities one concludes that each $\delta(\Delta_1,\dots,\Delta_k,\sigma)_{\tau_r}\leq\delta(\Delta_{i'})\times\delta(\Delta_1,\dots,\Delta_{i'-1},\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau_j'}$ for some τ_j'. This proves (i) in the case that $n_1\geq 0$ or $i'>1$.

Suppose now $n_1<0$ and $i'=1$. Write

$$\left(\prod_{2\leq i\leq k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma=\left(\prod_{1\leq i\leq k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma=\bigoplus_{j=1}^{2^{k-1}}\tau_j$$

where τ_j are irreducible. Then

$$\left(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\left(\overset{2^{k-1}}{\underset{i=1}{\oplus}}\tau_i\right)=\delta(\Delta_1)\times\left(\prod_{i=2}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\left(\overset{2^{k-1}}{\underset{j=1}{\oplus}}\tau_j\right)$$

$$\geq\delta(\Delta_1)\times\left(\sum_{j=1}^{2^{k-1}}\delta(\Delta_2,\Delta_3,\dots,\Delta_k,\sigma)_{\tau_j}\right)=\sum_{j=1}^{2^{k-1}}\delta(\Delta_1)\times\delta(\Delta_2,\Delta_3,\dots,\Delta_k,\sigma)_{\tau_j}.$$

Using (1-4), (iii) of Proposition 9.4 and (9-9) we get that multiplicity of $\left(\prod_{i=1}^{k}\delta(\Delta_i)\right)\otimes\sigma$ in $s_{GL}\left(\delta(\Delta_1)\times\delta(\Delta_2,\Delta_3,\dots,\Delta_k,\sigma)_{\tau_j}\right)$ is ≥ 1. The above inequalities and (iii) of Proposition 9.4 imply that the multiplicity is 1. This implies (i) in this case ($n_1<0$ and $i'=1$). Thus the proof of (i) is complete.

Using (i), we shall prove (ii) by induction with respect to k. Let $k\geq 2$ and suppose that (ii) holds for $k'<k$. From (i) we know that

$$\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\leq\delta([\nu^{-n_k}\rho,\nu^{m_k}\rho])\rtimes\delta(\Delta_1,\dots,\Delta_{k-1},\sigma)_{\tau'}$$

for some irreducible subquotient τ' of $\left(\prod_{i=i}^{k-1}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$. The inductive assumption and (1-4) imply

$$\text{(9-15)}\quad s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\leq\left(\sum_{j_k=-n_k}^{m_k+1}\delta([\nu^{-j_k+1}\rho,\nu^{n_k}\rho])\times\delta([\nu^{j_k}\rho,\nu^{m_k}\rho])\right)$$
$$\times c_1\left(\prod_{i=1}^{k-1}\left(\sum_{a_i=-n_i}^{|n_i|}\sum_{b_i=-n_i}^{m_i+1}\delta([\nu^{a_i+1}\rho,\nu^{n_i}])\times\delta([\nu^{b_i}\rho,\nu^{m_i}\rho])\right)\right)\otimes\sigma.$$

Further $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\leq\delta(\Delta_1)\rtimes\delta(\Delta_2,\dots,\Delta_k,\sigma)_{\tau''}$ for some irreducible subquotient τ'' of $\left(\prod_{i=2}^{k}\delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$, implies

$$\text{(9-16)}\quad s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\leq\left(\sum_{j_1=-n_1}^{m_1+1}\delta([\nu^{-j_1+1}\rho,\nu^{n_1}\rho])\times\delta([\nu^{j_1}\rho,\nu^{m_1}\rho])\right)$$
$$\times c_2\left(\prod_{i=2}^{k}\left(\sum_{a_i=-n_i}^{n_i}\sum_{b_i=-n_i}^{m_i+1}\delta([\nu^{a_i+1}\rho,\nu^{n_i}])\times\delta([\nu^{b_i}\rho,\nu^{m_i}\rho])\right)\right)\otimes\sigma.$$

The above formula shows that $\nu^{-m_k}\rho,\nu^{-m_k+1}\rho,\dots,\nu^{-n_k-1}\rho$ are not in $\mathrm{supp}_{GL}(\pi)$ for any irreducible subquotient π of $s_{GL}(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau)$. Therefore, we can sharpen the estimate (9-15) to the following estimate

$$s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)\leq\left(\sum_{j_k=-n_k}^{n_k+1}\delta([\nu^{-j_k+1}\rho,\nu^{n_k}\rho])\times\delta([\nu^{j_k}\rho,\nu^{m_k}\rho])\right)$$
$$\times c_1\left(\prod_{i=1}^{k-1}\left(\sum_{a_i=-n_i}^{|n_i|}\sum_{b_i=-n_i}^{m_i+1}\delta([\nu^{a_i+1}\rho,\nu^{n_i}])\times\delta([\nu^{b_i}\rho,\nu^{m_i}\rho])\right)\right)\otimes\sigma.$$

It is clear that

$$\Big[\sum_{j_k=-n_k}^{n_k}\delta([\nu^{-j_k+1}\rho,\nu^{n_k}\rho])\times\delta([\nu^{j_k}\rho,\nu^{m_k}\rho])\Big]$$
$$\times c_1\Big[\prod_{i=1}^{k-1}\Big[\sum_{a_i=-n_i}^{|n_i|}\sum_{b_i=-n_i}^{m_i+1}\delta([\nu^{a_i+1}\rho,\nu^{n_i}])\times\delta([\nu^{b_i}\rho,\nu^{m_i}\rho])\Big]\Big]\otimes\sigma$$
$$\le c\Big[\prod_{i=1}^{k}\Big[\sum_{a_i=-n_i}^{|n_i|}\sum_{b_i=-n_i}^{m_i+1}\delta([\nu^{a_i+1}\rho,\nu^{n_i}])\times\delta([\nu^{b_i}\rho,\nu^{m_i}\rho])\Big]\Big]\otimes\sigma,$$

for some c. Therefore, to prove (ii), it is enough to prove that if π is an irreducible subquotient of

$$\text{(9-17)}\quad \delta([\nu^{-n_k}\rho,\nu^{n_k}\rho])\times\delta([\nu^{n_k+1}\rho,\nu^{m_k}\rho])$$
$$\times\Big(\prod_{i=1}^{k-1}\Big(\sum_{a_i=-n_i}^{|n_i|}\sum_{b_i=-n_i}^{m_i+1}\delta([\nu^{a_i+1}\rho,\nu^{n_i}])\times\delta([\nu^{b_i}\rho,\nu^{m_i}\rho])\Big)\Big)\otimes\sigma.$$

and right hand side of (9-16), then π is an irreducible subquotient of the right hand side of (9-14). Now we write (9-17) in a slightly different way

$$\delta([\nu^{-n_k}\rho,\nu^{n_k}\rho])\times\delta([\nu^{n_k+1}\rho,\nu^{m_k}\rho])$$
$$\text{(9-18)}\qquad \times\Big(\sum_{a_1=-n_1}^{|n_1|}\sum_{b_1=-n_1}^{m_1+1}\sum_{a_2=-n_2}^{n_2}\sum_{b_2=-n_2}^{m_2+1}\cdots\sum_{a_{k-1}=-n_{k-1}}^{n_{k-1}}\sum_{b_{k-1}=-n_{k-1}}^{m_{k-1}+1}$$
$$\text{(9-19)}\qquad \prod_{i=1}^{k-1}\Big(\delta([\nu^{a_i+1}\rho,\nu^{n_i}])\times\delta([\nu^{b_i}\rho,\nu^{m_i}\rho])\Big)\Big)\otimes\sigma.$$

Now we shall point out some properties of the factors in the line (9-19). Consider segments $\Delta_i'=[\nu^{a_i+1}\rho,\nu^{n_i}\rho]$, $\Delta_i''=[\nu^{b_i}\rho,\nu^{m_i}\rho]$ for $i=1,\dots,k-1$, where $-n_i\le a_i\le n_i$ and $-n_i\le b_i\le m_i+1$. We consider the following multisets

$$\text{(9-20)}\qquad a=(\Delta_1',\Delta_1'',\Delta_2',\Delta_2'',\dots,\Delta_{k-1}',\Delta_{k-1}'')$$

(if some $\Delta_i'=\emptyset$ or $\Delta_i''=\emptyset$, then we omit $\emptyset$ from the above definition of a). Let X be a set of all such multisets. Denote $\Delta_k^{\dagger}=[\nu^{-n_k}\rho,\nu^{n_k}]$, $\Delta_k^{\dagger\dagger}=[\nu^{n_k+1}\rho,\nu^{m_k}\rho]$. Using conditions on n_i and m_i in the lemma one checks directly that the following properties hold:

1. If $a\in X$ and $b\le a$, then $b\in X$ (it is enough to check this when $b\prec a$, and checking for $b\prec a$ is direct using $n_1<m_1<n_2<m_2<n_3<\dots$).
2. $\Delta_i',\Delta_i''\subseteq\Delta_k^{\dagger}\subseteq\Delta_k$, for any $1\le i\le k-1$.
3. For any $1\le i\le k-1$, neither Δ_i' nor Δ_i'' is linked with $\Delta_k^{\dagger\dagger}$.
4. Linking $\Delta_k^{\dagger}$ and $\Delta_k^{\dagger\dagger}$ one gets Δ_k.

Let π be a common irreducible subquotient of the right hand side of (9-16) and of (9-17). Write $\pi = L(\Gamma_1, \dots, \Gamma_t)$ with $\Gamma_i \in \mathcal{S}(\mathcal{C})$. Because π is a subquotient of the right hand side of (9-17), (1) - (4) directly imply that $(\Gamma_1, \dots, \Gamma_t) = a + (\Delta_k^\dagger, \Delta_k^{\dagger\dagger})$ or $a + (\Delta_k)$ for some $a \in X$. Note that in both cases in the GL-support of π is $\nu^{-n_k}\rho$. We shall use now that $\pi = L(\Gamma_1, \dots, \Gamma_t)$ is a subquotient of the right hand side of (9-16). To get $\nu^{-n_k}\rho$ in the GL-support, we see from (9-16) that π must be a subquotient of some

$$
\begin{aligned}
\text{(9-21)} \quad & \gamma = \delta([\nu^{-j_1+1}\rho, \nu^{n_1}\rho]) \times \delta([\nu^{j_1}\rho, \nu^{m_1}\rho]) \\
& \times \left(\prod_{i=2}^{k-1} \delta([\nu^{a_i+1}\rho, \nu^{n_i}]) \times \delta([\nu^{b_i}\rho, \nu^{m_i}\rho]) \right) \times \delta([\nu^{a_k+1}\rho, \nu^{n_k}]) \times \delta([\nu^{-n_k}\rho, \nu^{m_k}\rho]) \otimes \sigma.
\end{aligned}
$$

where j_1, a_i and b_i satisfy conditions of (9-16). Note that $[\nu^{-n_k}\rho, \nu^{m_k}\rho]$ contains each segment which enters (9-21). This implies that $(\Gamma_1, \dots, \Gamma_t) = a + (\Delta_k)$. Without lost of generality we can assume $\Gamma_t = \Delta_k$. Thus, π is a subquotient of $\delta(\Gamma_1) \times \dots \delta(\Gamma_{t-1}) \times \delta(\Delta_k) \otimes \sigma$, where $(\Gamma_1, \dots, \Gamma_{k-1} \in X$. It is obvious that $\delta(\Gamma_1) \times \dots \delta(\Gamma_{t-1}) \times \delta(\Delta_k) \otimes \sigma$ is $\leq$ of the right hand side of (9-14). This finishes the proof of (ii).

We shall prove (iii) by induction. Suppose that (iii) holds for $k-1$. Let π be an irreducible subquotient of $s_{GL}\left(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau\right)$ such that $\mathrm{supp}_{GL}(\pi) = \sum_{i=1}^k \Delta_i$ ($= \mathrm{supp}_{GL}\left(\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma\right)$). Since $\nu^{-n_k}\rho$ is not in $\mathrm{supp}(\pi)$, π can appear as a subquotient of the right hand side of (9-14) if $\pi \leq \gamma$, where γ is as in (9-21). Since Δ_k contains each segment which enters in the definition (9-21) of γ, $\pi = \delta(\Delta_k) \times \pi'$, where π' is an irreducible representation of some $GL(p', F) \times S_q$. Since

$$
\begin{aligned}
\text{(9-22)} \quad & \text{s.s.}\left(s_{GL}\left(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau\right)\right) \\
& \leq \left(\sum_{j_k=-n_k}^{n_k+1} \delta([\nu^{-j_k+1}\rho, \nu^{n_k}\rho]) \times \delta([\nu^{j_k}\rho, \nu^{m_k}\rho]) \right) \times s_{GL}\left(\delta(\Delta_1, \dots, \Delta_{k-1}, \sigma)_{\tau'}\right),
\end{aligned}
$$

for some τ',

$$
\text{(9-23)} \quad \delta(\Delta_k) \times \pi' \leq \delta([\nu^{-j_k+1}\rho, \nu^{n_k}\rho]) \times \delta([\nu^{j_k}\rho, \nu^{m_k}\rho]) \times s_{GL}\left(\delta(\Delta_1, \dots, \Delta_{k-1}, \sigma)_{\tau'}\right).
$$

Since $\nu^{-n_k}\rho$ is in the support of the left hand side of (9-23), $j_k = -n_k$, i.e. $\delta(\Delta_k) \times \pi' \leq \delta(\Delta_k) \times s_{GL}\left(\delta(\Delta_1, \dots, \Delta_{k-1}, \sigma)_{\tau'}\right)$. This implies $\pi' \leq s_{GL}\left(\delta(\Delta_1, \dots, \Delta_{k-1}, \sigma)_{\tau'}\right)$. Since $\mathrm{supp}_{GL}(\pi) = \Delta_k + \mathrm{supp}_{GL}(\pi') = \sum_{i=1}^k \Delta_i$, then $\mathrm{supp}_{GL}(\pi') = \sum_{i=1}^{k-1} \Delta_i$. Now the inductive assumption implies $\pi' = \left(\prod_{i=1}^{k-1} \delta(\Delta_i)\right) \otimes \sigma$. This finishes the proof of (iii). Therefore the proof of lemma is complete. $\square$

Let $\rho', \rho'' \in \mathcal{C}$. We shall say that they are are strongly $\mathbb{Z}$-disconnected if there dose not exist $\Delta \in \mathcal{S}(\mathcal{C})$ such that $\rho', \rho'' \in \Delta$ or $\rho', (\rho'')\tilde{} \in \Delta$. For $\Gamma_1, \Gamma_2 \in \mathcal{S}(\mathcal{C})$ we say that they are strongly $\mathbb{Z}$-disconnected any $\rho_1 \in \Gamma_1$ is strongly $\mathbb{Z}$-disconnected with any $\rho_2 \in \Gamma_2$.

9.8. Lemma. *Let $\rho'_1, \dots, \rho'_{k'}$, $\rho''_1, \dots, \rho''_{k''} \in \mathcal{C}$ and let σ be an irreducible cuspidal representation of S_q. Suppose:*

(a) Any ρ_i' is strongly $\mathbb{Z}$-disconnected with any ρ_j''.
(b) π' is an irreducible subquotient of $\rho_1' \times \cdots \times \rho_{k'}' \rtimes \sigma$ and π'' is an irreducible subquotient of $\rho_1'' \times \cdots \times \rho_{k''}'' \rtimes \sigma$. Write s.s.$(s_{GL}(\pi')) = \gamma' \otimes \sigma$ and s.s.$(s_{GL}(\pi'')) = \gamma'' \otimes \sigma$.
(c) π is a representation which satisfies $\pi \leq \rho_1' \times \cdots \times \rho_{k'}' \rtimes \pi''$ and $\pi \leq \rho_1'' \times \cdots \times \rho_{k''}'' \rtimes \pi'$.
Then there exists a positive integer d such that $s_{GL}(\pi) \leq d(\gamma' \times \gamma'' \otimes \sigma)$.

Proof. From (1-4) and (b) follows

$$s_{GL}(\pi) \leq \left(\prod_{i=1}^{k'}(\rho_i' + (\rho_i')\tilde{\ })\right) \times \gamma'' \otimes \sigma, \quad s_{GL}(\pi) \leq \left(\prod_{j=1}^{k''}(\rho_j'' + (\rho_j'')\tilde{\ })\right) \times \gamma' \otimes \sigma. \tag{9-24}$$

Let β be an irreducible subquotient of $s_{GL}(\pi)$. Then (9-24) and (a) imply $\beta = \alpha' \times \phi'' \otimes \sigma = \alpha'' \times \phi' \otimes \sigma$ where α' is an irreducible subquotient of γ', α'' is an irreducible subquotient of γ'', ϕ' is an irreducible subquotient of $\prod_{i=1}^{k'}(\rho_i' + (\rho_i')\tilde{\ })$ and ϕ'' of $\prod_{j=1}^{k''}(\rho_j'' + (\rho_j'')\tilde{\ })$. Obviously supp$(\phi')$ consists only of elements from $\{\rho_i', \tilde{\rho}_i'; 1 \leq i \leq k'\}$, while supp$(\phi'')$ consists only of elements from $\{\rho_j'', \tilde{\rho}_j''; 1 \leq j \leq k''\}$. Also supp$(\alpha')$ consists only of elements from $\{\rho_i', \tilde{\rho}_i'; 1 \leq i \leq k'\}$ and supp(α'') consists only of elements from $\{\rho_j'', \tilde{\rho}_j''; 1 \leq j \leq k''\}$. Further supp$_{GL}(\beta) = $ supp$(\alpha') + $ supp$(\phi'') = $ supp$(\alpha'') + $ supp(ϕ'). Now (a) implies supp$(\alpha') = $ supp(ϕ') and supp$(\phi'') = $ supp(α'').

Now we use the following fact from the representation theory of general linear groups. Let $X_1, X_2 \subseteq \mathcal{C}$. Suppose that any element of X_1 is strongly $\mathbb{Z}$-disconnected with any any element of X_2 (a weaker condition would be enough for what follows). Let $\lambda_1, \lambda_1', \lambda_2, \lambda_2'$ be irreducible representations of general linear groups such that supp(λ_1) and supp(λ_1') consists only of elements from X_1 and supp(λ_2) and supp(λ_2') consists only of elements from X_2. Then $\lambda_1 \times \lambda_2 \cong \lambda_1' \times \lambda_2'$ implies $\lambda_1 \cong \lambda_1'$ and $\lambda_2 \cong \lambda_2'$ (this follows easily from [Z1], see also [Z2]). The above fact implies $\alpha' \cong \phi'$ and $\phi'' \cong \alpha''$. Therefore $\beta \cong \alpha' \times \alpha'' \otimes \sigma$. This implies $\beta \leq (\gamma_1 \times \gamma_2) \otimes \sigma$. From this follows the claim of the lemma. □

9.9. Lemma. *Suppose that $\Delta_1, \ldots, \Delta_k, \sigma$ and τ satisfy assumptions of Proposition 9.4. Then*
(i) Let $1 \leq i' \leq k$. There exists an irreducible subrepresentation τ' of $\left(\prod_{i=1}^{i'} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$ such that

$$\delta(\Delta_1, \Delta_2, \ldots, \Delta_k, \sigma)_\tau \leq \left(\prod_{i=1}^{i'} \delta(\Delta_i)\right) \rtimes \delta(\Delta_{i'+1}, \Delta_{i'+2}, \ldots, \Delta_k, \sigma)_{\tau'}$$

(note that the order of Δ_i's is now again arbitrary).
(ii) For some positive integer c holds

$$s_{GL}\left(\delta(\Delta_1, \ldots, \Delta_k, \sigma)_\tau\right) \leq c\left(\prod_{i=1}^{n}\left(\sum_{a_i=-n_i}^{|n_i|}\sum_{b_i=-n_i}^{m_i+1} \delta([\nu^{a_i+1}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i}\rho_i, \nu^{m_i}\rho_i])\right)\right) \otimes \sigma.$$

(iii) If π is and irreducible subquotient of $s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)$ which satisfies $\pi \not\cong \left(\prod_{i=1}^k \delta(\Delta_i)\right)\otimes\sigma$, then $supp_{GL}(\pi) \neq \sum_{i=1}^k \Delta_i$.
(iv) $\left(\prod_{i=1}^k \delta(\Delta_i)\right)\otimes\sigma$ is a direct summand in $s_{GL}\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)$.

Proof. Denote $l_1 = \text{card}(\{i; i'+1 \le i \le n \text{ and } \Delta_i \cap \tilde{\Delta}_i \neq \emptyset\}$, $l_2 = \text{card}(\{i; 1 \le i \le i' \text{ and } \Delta_i \cap \tilde{\Delta}_i \neq \emptyset\}$ and $l = \text{card}(\{i; 1 \le i \le n \text{ and } \Delta_i \cap \tilde{\Delta}_i \neq \emptyset\}$. Then $l_1 + l_2 = l$. By Proposition 9.1 we can write

$$\left(\prod_{i=i'+1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma = \oplus_{i=1}^{2^{l_1}} \tau_i', \quad \left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma = \oplus_{i=1}^{2^l} \tau_i.$$

Now in the Grothendieck group we have

$$\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\oplus_{j=1}^{2^l} \tau_j\right) \rtimes \sigma = \left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$$

$$= \left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^{i'} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \times \left(\prod_{i=i'+1}^k \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma$$

$$= \left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\prod_{i=1}^{i'} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \left(\oplus_{j=1}^{2^{l_1}} \tau_j'\right)$$

$$\ge \left(\prod_{i=1}^{i'} \delta(\Delta_i)\right) \times \left(\prod_{i=i'+1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \left(\oplus_{j=1}^{2^{l_1}} \tau_j'\right)$$

$$= \left(\prod_{i=1}^{i'} \delta(\Delta_i)\right) \rtimes \left(\oplus_{j=1}^{2^{l_1}} \left(\prod_{i=i'+1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau_j'\right)$$

$$\ge \left(\prod_{i=1}^{i'} \delta(\Delta_i)\right) \rtimes \left(\sum_{j=1}^{2^{l_1}} \delta(\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau_j'}\right)$$

$$\ge \sum_{j=1}^{2^{l_1}} \left(\prod_{i=1}^{i'} \delta(\Delta_i)\right) \rtimes \delta(\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau_j'}.$$

The multiplicity of $\left(\prod_{j=1}^k \delta(\Delta_i)\right)\otimes\sigma$ in $s_{GL}\left(\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \times \left(\oplus_{i=1}^{2^l} \tau_i\right) \rtimes \sigma\right)$ is 2^l by (ii) of Proposition 9.4. One gets easily from (ii) of Proposition 9.4 and (1-4) that the multiplicity of $\left(\prod_{j=1}^k \delta(\Delta_i)\right)\otimes\sigma$ in $s_{GL}\left(\sum_{j=1}^{2^{l_1}} \left(\prod_{i=1}^{i'} \delta(\Delta_i)\right) \rtimes \delta(\Delta_{i'+1},\dots,\Delta_k,\sigma)_{\tau_j'}\right)$ is at least $2^{l_1}2^{l_2} = 2^l$. The above inequalities imply that the multiplicity is exactly 2^l. Now we can conclude that (i) holds.

We prove (ii) by induction. For $k=1$, (ii) holds. Let $k>1$. If $\Delta_i \cap \Delta_j \neq \emptyset$ for all $1 \leq i < j \leq k$, then Lemma 9.6 implies (ii). Therefore we can suppose that $\Delta_i \cap \Delta_j = \emptyset$ for some $1 \leq i < j \leq n$. This implies that we can make a partition $\{\Delta_1, \dots, \Delta_k\}$ into a union $X \cup Y$ of two non-empty sets of segments in a such a way that any segment in X is strongly $\mathbb{Z}$-disconnected with any segment in Y. Now using (i) and applying applying Lemma 9.8, the inductive assumption implies (ii).

From (iii) of Lemma 9.6, using Lemma 9.8, one easily obtains (iii). We can also prove (iii) directly in a similar way as we proved (iii) in Lemma 9.6 (after renumeration one can assume that $\Delta_k \not\subseteq \Delta_i$ for $i=1,\dots,k-1$; after this one proceeds in analogously as in Lemma 9.6).

At the end, (ii) of Lemma 9.6 and (iii) imply (iv) (use Theorem 7.3.2 of [C]). $\square$

9.10. Theorem. *Let $\Delta_1, \dots, \Delta_k, \sigma$ and τ be as in Proposition 9.4. Then*
(i) $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ are square integrable representations.
(ii) If π is a subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$, then $\pi \cong \delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ for some τ. Also, each $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ is isomorphic to a subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$.
(iii) $(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau)\tilde{} \cong \delta(\Delta_1, \dots, \Delta_k, \tilde{\sigma})_{\tilde{\tau}}$.

Proof. One gets (i) from (i) of the last lemma using the square integrability criterion (one needs from [Z1] description of Jacquet modules of the right hand side of the inequality in (i) of the last lemma).

If π is an irreducible subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$, then Frobenius reciprocity implies $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \otimes \sigma \leq s_{GL}(\pi)$. Now (v) of Proposition 9.4 implies that π is isomorphic to some $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$. Further, (iii) of Lemma 9.9 and Frobenius reciprocity imply that each representation $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ is a subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i)\right) \rtimes \sigma$. This proves (ii).

We shall use now notation analogous to the notation which we have introduced for general linear groups and groups S_q, with the difference that the lower triangular matrices are fixed to play the role of the standard minimal parabolic subgroup. Then this new notation will be the same as our standard notation, except that we shall underline this new notation. So, we are going to work with $\underline{\times}, \underline{\rtimes}, \underline{s}_{GL}, \dots$. More details regarding this notation can be found in section 4 of [T2] and section 6 of [T4]. From $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau \hookrightarrow \delta(\Delta_1) \times \dots \times \delta(\Delta_k) \rtimes \sigma$, Propositions 4.1 of [T2] and 6.1 of [T4], we get $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau \hookrightarrow \delta(\Delta_1)\tilde{}\,\underline{\times} \cdots \underline{\times}\delta(\Delta_k)\tilde{}\,\underline{\rtimes}\sigma$. Therefore, there exists an epimorphism $\underline{s}_{GL}\left(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau\right) \twoheadrightarrow \delta(\Delta_1)\tilde{}\,\underline{\times} \cdots \underline{\times}\delta(\Delta_k)\tilde{} \otimes \sigma$. Thus $\delta(\Delta_1)\underline{\times} \dots, \underline{\times}\delta(\Delta_k) \otimes \tilde{\sigma} \hookrightarrow \left(\underline{s}_{GL}\left(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau\right)\right)\tilde{}$. Since $\left(\underline{s}_{GL}\left(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau\right)\right)\tilde{} \cong s_{GL}\left((\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau)\tilde{}\right)$ by Corollary 4.2.5 of [C], and $\delta(\Delta_1)\underline{\times} \dots, \underline{\times}\delta(\Delta_k) = \delta(\Delta_1) \times \dots \times \delta(\Delta_k)$, we get that $\delta(\Delta_1) \times \dots \times \delta(\Delta_k) \otimes \tilde{\sigma}$ is a subrepresentation of $s_{GL}\left((\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau)\tilde{}\right)$.

Recall that $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$ is a subquotient of $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tau$. Therefore, $(\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau)\tilde{}$ is a subquotient of $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\tilde{}\right) \rtimes \tilde{\tau}$. The last representation has the same Jordan-Hölder series as $\left(\prod_{i=1}^k \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \rtimes \tilde{\tau}$ (use (1-3)). From the definition of

$\delta(\Delta_1,\dots,\Delta_k,\tilde\sigma)_{\tilde\tau}$ in Proposition 9.4, we get $\delta(\Delta_1,\dots,\Delta_k,\tilde\sigma)_{\tilde\tau}\cong(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau)\tilde{}$. □

We end this section with two propositions which give some interesting additional information about representations $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau$

9.11. Proposition. *Suppose that $\Delta_1,\dots,\Delta_k,\sigma$ and τ satisfy the assumptions of Proposition 9.4. Then*
(i) Multiplicity of $\left(\prod_{i=1}^k\delta(\Delta_i\backslash\tilde\Delta_i)\right)\otimes\tau$ in $\mu^*\left(\left(\prod_{i=1}^k\delta(\Delta_i)\right)\rtimes\sigma\right)$ is one.
(ii) The representation $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau$ is a subrepresentation of $\left(\prod_{i=1}^k\delta(\Delta_i\backslash\tilde\Delta_i)\right)\rtimes\tau$.
(iii) If $\tau'\not\cong\tau''$, then $\delta(\Delta_1,\dots,\Delta_k,\sigma)_{\tau'}\not\cong\delta(\Delta_1,\dots,\Delta_k,\sigma)_{\tau''}$.

Proof. First we compute

$$M^*\left(\delta([\nu^{-n_i}\rho,\nu^{m_i}\rho])\right)=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\circ m^*\left(\delta([\nu^{-n_i}\rho,\nu^{m_i}\rho])\right)\tag{9-25}$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\left(\sum_{a_i=-n_i-1}^{m_i}\delta([\nu^{a_i+1}\rho_i,\nu^{m_i}\rho_i])\otimes\delta([\nu^{-n_i}\rho_i,\nu^{a_i}\rho_i])\right)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\left(\sum_{a_i=-n_i-1}^{m_i}\delta([\nu^{-n_i}\rho_i,\nu^{a_i}\rho_i])\otimes\delta([\nu^{a_i+1}\rho_i,\nu^{m_i}\rho_i])\right)$$

$$=\sum_{a_i=-n_i-1}^{m_i}\sum_{b_i=a_i}^{m_i}\delta([\nu^{-a_i}\rho_i,\nu^{n_i}\rho_i])\times\delta([\nu^{b_i+1}\rho_i,\nu^{m_i}\rho_i])\otimes\delta([\nu^{a_i+1}\rho_i,\nu^{b_i}\rho_i]).$$

By (1-4) we have

$$\mu^*\left(\left(\prod_{i=1}^k\delta(\Delta_i\right)\rtimes\sigma\right)=\left(\prod_{i=1}^k\left(\sum_{a_i=-n_i-1}^{m_i}\sum_{b_i=a_i}^{m_i}\right.\right.\tag{9-26}$$
$$\left.\left.\delta([\nu^{-a_i}\rho_i,\nu^{n_i}\rho_i])\times\delta([\nu^{b_i+1}\rho_i,\nu^{m_i}\rho_i])\otimes\delta([\nu^{a_i+1}\rho_i,\nu^{b_i}\rho_i])\right)\right)\rtimes(1\otimes\sigma).$$

Conditions (a) - (c) in Proposition 9.4 imply that $\beta=\left(\prod_{i=1}^k\delta(\Delta_i\backslash\tilde\Delta_i)\right)\otimes\tau$ is irreducible. Suppose that β is a subquotient of the right hand side of (9-26) Then β is a subquotient of some

$$\left(\prod_{i=1}^k\delta([\nu^{-a_i}\rho_i,\nu^{n_i}\rho_i])\times\delta([\nu^{b_i+1}\rho_i,\nu^{m_i}\rho_i])\otimes\delta([\nu^{a_i+1}\rho_i,\nu^{b_i}\rho_i])\right)\rtimes(1\otimes\sigma),\tag{9-27}$$

where

$$-n_i-1\le a_i\le m_i \text{ and } a_i\le b_i\le m_i.\tag{9-28}$$

Denote (9-27) by $\gamma \otimes \gamma'$. Since $\beta = \left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \otimes \tau$ is irreducible, and it is a subquotient of $\gamma \otimes \gamma'$, $\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)$ is a subquotient of γ. In particular

$$\operatorname{supp}(\gamma) = \operatorname{supp}\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right), \tag{9-29}$$

i.e. $$\sum_{i=1}^{k} \left([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i] + [\nu^{b_i+1}\rho_i, \nu^{m_i}\rho_i]\right) = \sum_{i=1}^{k} (\Delta_i \backslash \tilde{\Delta}_i) \quad \left(= \sum_{i=1}^{k} [\nu^{n_i+1}\rho_i, \nu^{m_i}\rho_i]\right).$$

Now chose i_1 such that $\Delta_{i_1} \not\subseteq \Delta_i$ for any $1 \le i \le$, $i_1 \neq i$. Since we have $\nu^{n_{i_1}+1}\rho_i, \nu^{n_{i_1}+2}\rho_i, \dots, \nu^{m_{i_1}}\rho_i$ in the support of γ, $b_{i_1} + 1 \le n_{i_1} + 1$ (i.e. $b_{i_1} \le n_{i_1}$). Since $\nu^{-n_{i_1}-1}\rho_i$ is not in the support of γ, $b_{i_1} + 1 \ge n_{i_1} + 1$ (i.e. $b_{i_1} \ge n_{i_1}$) and $-a_{i_1} > n_{i_1}$ (i.e. $-n_{i_1} > a_{i_1}$) . Thus $a_{i_1} = -n_{i_1} - 1$ (what follows now from (9-28)), and $b_{i_1} = n_{i_1}$.

From (9-29) follows

$$\sum_{1 \le i \le k, i \neq i_1} \left([\nu^{-a_i}\rho_i, \nu^{n_i}\rho_i] + [\nu^{b_i+1}\rho_i, \nu^{m_i}\rho_i]\right) = \sum_{1 \le i \le k, i \neq i_1} (\Delta_i \backslash \tilde{\Delta}_i).$$

Chose $i_2 \in \{1, \dots, k\} \backslash \{i_1\}$ such that $\Delta_{i_2} \not\subseteq \Delta_i$ for any $i \in \{1, \dots, k\} \backslash \{i_1, i_2\}$. Now one gets in the same way as above that must be $a_{i_2} = -n_{i_2} - 1$ and $b_{i_2} = n_{i_2}$ in (9-27). Continuing this process we get that $\left(\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right) \otimes \tau$ must be a subquotient of

$$\left(\prod_{i=1}^{k} \left(\delta([\nu^{n_{i_1}+1}\rho, \nu^{m_i}\rho]) \otimes \delta([\nu^{-n_{i_1}}\rho, \nu^{n_{i_1}}\rho])\right)\right) \rtimes (1 \otimes \sigma)$$

(we have obtained that this is the only term on the right hand side of (9-26) where β can be a subquotient). From this and Proposition 9.1 we get (i).

Now we shall list some obvious properties of the segments that we have considered.

(1) Among segments $\Delta_i \cap \tilde{\Delta}_i, \Delta_i \backslash \tilde{\Delta}_i, 1 \le i \le k$, the only pairs of linked segments are $\Delta_i \cap \tilde{\Delta}_i, \Delta_i \backslash \tilde{\Delta}_i$ when $\Delta_i \backslash \tilde{\Delta}_i \neq \emptyset$ (this follows easily from conditions on segments Δ_i).

(2) From [Z1] we know $\delta(\Delta_i) \hookrightarrow \delta(\Delta_i \backslash \tilde{\Delta}_i) \times \delta(\Delta_i \cap \tilde{\Delta}_i)$.

From (1) and (2) we obtain

$$\left[\prod_{i=1}^{k} \delta(\Delta_i)\right] \rtimes \sigma \hookrightarrow \left[\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right] \times \left[\prod_{i=1}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right] \rtimes \sigma \cong \left[\prod_{i=1}^{k} \delta(\Delta_i \backslash \tilde{\Delta}_i)\right] \rtimes \left(\oplus_{j=1}^{2^l} \tau_j\right),$$

where $\left(\prod_{i=1}^{k} \delta(\Delta_i \cap \tilde{\Delta}_i)\right) \rtimes \sigma = \oplus_{j=1}^{2^l} \tau_j$ is the decomposition in the sum of irreducible representations. Therefore, using (ii) of Theorem 9.10, we get that each $\delta(\Delta_1, \dots, \Delta_k, \sigma)_\tau$

is isomorphic to a subrepresentation of some $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \rtimes \tau_j$. Frobenius reciprocity implies

$$\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right) \otimes \tau_j \leq \mu^*\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right).$$

Now $\tau=\tau_j$ (if not, then the multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\otimes\tau_j$ in $\left(\prod_{i=1}^k \delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$ would be at least two, which contradicts to (i)). Therefore $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau$ is (an irreducible) subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\tau$, what is the claim of (ii).

At the end, one gets (iii) from $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\otimes\tau \leq \mu^*\left(\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau\right)$. □

9.12. Proposition. *Suppose that $\Delta_1,\dots,\Delta_k,\sigma$ and τ are as in Proposition 9.4. Then*
(i) Multiplicity of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\otimes\tau$ in $\mu^*\left(\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\tau\right)$ is one.
(ii) $\delta(\Delta_1,\dots,\Delta_k,\sigma)_\tau$ is a unique irreducible subrepresentation of $\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\rtimes\tau$.

Proof. Denote $\beta=\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)$ and $\gamma=\left(\prod_{i=1}^k \delta(\Delta_i\backslash\tilde{\Delta}_i)\right)\times\left(\prod_{i=1}^k \delta(\Delta_i\cap\tilde{\Delta}_i)\right)\rtimes\sigma$. To prove (i), it is enough to prove that multiplicity of $\beta\otimes\tau$ in $\mu^*(\gamma)$ is one (note that $\beta\otimes\tau$ is irreducible). Compute

$$M^*\left(\delta([\nu^{n_j+1}\rho_j,\nu^{m_j}\rho_j])\right)=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\circ m^*\left(\delta([\nu^{n_j+1}\rho_j,\nu^{m_j}\rho_j])\right)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\Bigg(\sum_{a_j=n_j}^{m_j}\delta([\nu^{a_j+1}\rho_j,\nu^{m_j}\rho_j])\otimes\delta([\nu^{n_j+1}\rho_j,\nu^{a_j}\rho_j])\Bigg)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\Bigg(\sum_{a_j=n_j}^{m_j}\delta([\nu^{n_j+1}\rho_j,\nu^{a_j}\rho_j])\otimes\delta([\nu^{a_j+1}\rho_j,\nu^{m_j}\rho_j])\Bigg)$$

$$=\sum_{a_j=n_j}^{m_j}\sum_{b_j=a_j}^{m_j}\delta([\nu^{-a_j}\rho_j,\nu^{-n_j-1}\rho_j])\times\delta([\nu^{b_j+1}\rho_j,\nu^{m_j}\rho_j])\otimes\delta([\nu^{a_j+1}\rho_j,\nu^{b_j}\rho_j]),$$

$$M^*\left(\delta([\nu^{-n_i}\rho_i,\nu^{n_i}\rho_i])\right)=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\circ m^*\left(\delta([\nu^{-n_i}\rho_i,\nu^{n_i}\rho_i])\right)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\circ s\Bigg(\sum_{a_i'=-n_i-1}^{n_i}\delta([\nu^{a_i'+1}\rho_i,\nu^{n_i}\rho_i])\otimes\delta([\nu^{-n_i}\rho_i,\nu^{a_i'}\rho_i])\Bigg)$$

$$=(m\otimes 1)\circ(\sim\otimes m^*)\Bigg(\sum_{a_i'=-n_i-1}^{n_i}\delta([\nu^{-n_i}\rho_i,\nu^{a_i'}\rho_i])\otimes\delta([\nu^{a_i'+1}\rho_i,\nu^{n_i}\rho_i])\Bigg)$$

$$=\sum_{a_i'=-n_i-1}^{n_i}\sum_{b_i'=a_i'}^{n_i}\delta([\nu^{-a_i'}\rho_i,\nu^{n_i}\rho_i])\times\delta([\nu^{b_i'+1}\rho_i,\nu^{n_i}\rho_i])\otimes\delta([\nu^{a_i'+1}\rho,\nu^{b_i'}\rho]),$$

$$\mu^*(\gamma) = \mu^*\bigg(\bigg(\prod_{j=1}^{k}(\delta([\nu^{n_j+1}\rho_j, \nu^{m_j}\rho_j])\bigg) \times \bigg(\prod_{i=1}^{k}\delta([\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i])\bigg) \rtimes \sigma\bigg) \tag{9-30}$$

$$= \prod_{j=1}^{k}\prod_{i=1}^{k}\bigg(\sum_{a_j=n_j}^{m_j}\sum_{b_j=a_j}^{m_j}\delta([\nu^{-a_j}\rho_j, \nu^{-n_j-1}\rho_j]) \times \delta([\nu^{b_j+1}\rho_j, \nu^{m_j}\rho_j])$$

$$\times \sum_{a_i'=-n_i-1}^{n_i}\sum_{b_i'=a_i'}^{n_i}\delta([\nu^{-a_i'}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i'+1}\rho_i, \nu^{n_i}\rho_i])$$

$$\otimes\delta([\nu^{a_j+1}\rho_j, \nu^{b_j}\rho_j]) \times \delta([\nu^{a_i'+1}\rho_i, \nu^{b_i'}\rho_i])\bigg) \otimes \sigma$$

Suppose that $\beta \otimes \tau$ is a subquotient of some

$$\gamma' \otimes \gamma'' = \prod_{j=1}^{k}\prod_{i=1}^{k}\bigg(\delta([\nu^{-a_j}\rho_j, \nu^{-n_j-1}\rho_j]) \times \delta([\nu^{b_j+1}\rho_j, \nu^{m_j}\rho_j])$$

$$\times\delta([\nu^{-a_i'}\rho_i, \nu^{n_i}\rho_i]) \times \delta([\nu^{b_i'+1}\rho_i, \nu^{n_i}\rho_i]) \otimes \delta([\nu^{a_j+1}\rho_j, \nu^{b_j}\rho_j]) \times \delta([\nu^{a_i'+1}\rho_i, \nu^{b_i'}\rho_i])\bigg) \otimes \sigma,$$

where a_j, a_j, a_i' and b_i' are as in (9-30). Since $\beta \otimes \tau$ is a subquotient of $\gamma' \otimes \gamma''$, β is a subquotient of γ'. Then $\operatorname{supp}(\gamma') = \operatorname{supp}(\beta)$.

Chose $i_1 \in \{1, \dots, k\}$ such that $\Delta_{i_1} \not\subseteq \Delta_i$ for $i \in \{1, \dots, k\}\backslash\{i_1\}$. Then $\operatorname{supp}(\gamma') = \operatorname{supp}(\beta)$ implies

$$b_{i_1} + 1 \leq n_{i_1} + 1, \quad -n_{i_1} \leq -a_{i_1}, \quad n_{i_1} + 1 \leq -a_{i_1}', \quad n_{i_1} + 1 \leq b_{i_1}' + 1,$$

since $\nu^{n_{i_1}+1}\rho_{i_1}, \dots, \nu^{m_{i_1}}\rho_{i_1}$ are in $\operatorname{supp}(\beta)$ and $\nu^{-n_{i_1}-1}\rho_{i_1}, \nu^{\pm n_{i_1}}\rho_{i_1}$ are not in $\operatorname{supp}(\beta)$. This implies $a_{i_1} = n_{i_1}, b_{i_1} = n_{i_1}, a_{i_1}' = -n_{i_1} - 1, b_{i_1}' = n_{i_1}$. One continues in a similar way as in the proof of Proposition 9.1, and get that for all i, $a_i = b_i = n_i$, $a_i' = -n_i - i$, $b_i' = n_i$. Then $\gamma' \otimes \gamma'' = \Big(\prod_{i=1}^{k}\delta(\Delta_i\backslash\tilde{\Delta}_i)\Big) \otimes \Big(\prod_{i=1}^{k}\delta(\Delta_i \cap \tilde{\Delta}_i)\Big) \rtimes \sigma$. Proposition 9.1 implies that multiplicity of $\beta \otimes \tau$ in $\gamma' \otimes \gamma''$ is one. This finishes the proof of (i).

Using Frobenius reciprocity one directly gets (ii) from (i), and from (ii) of Proposition 9.11. $\square$

Note that (i) of the above proposition implies (i) of Proposition 9.11.

9.13. Remark. Theorem 4.3 implies that irreducible square integrable representations $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)$ and $\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma)$, $n < m$, considered in sections 3 and 4, in the notation of this section are

$$\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma) = \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)_{\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)},$$

$$\delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho]_-, \sigma) = \delta([\nu^{-n-1/2}\rho, \nu^{m+1/2}\rho], \sigma)_{\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)}$$

(tempered representations $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho], \sigma)$ and $\delta([\nu^{-n-1/2}\rho, \nu^{n+1/2}\rho]_-, \sigma)$ are defined in Theorem 3.2). Similarly, by Theorem 5.8 representations $\delta([\nu^{-n}\rho, \nu^m\rho]_{\tau_i}, \sigma)$ considered in the fifth section are

$$\delta([\nu^{-n}\rho, \nu^m\rho]_{\tau_i}, \sigma) = \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)_{\delta([\nu^{-n}\rho, \nu^n\rho]_{\tau_i}, \sigma)}$$

(tempered representations $\delta([\nu^{-n}\rho, \nu^n\rho]_{\tau_i}, \sigma)$ are defined in Theorem 5.4). Representations $\delta([\nu^{-n}\rho, \nu^m\rho], \sigma)$ and $\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma)$ considered in sections 6 and 7 are by Theorem 7.5

$$\delta([\nu^{-n}\rho, \nu^m\rho], \sigma) = \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)_{\delta([\nu^{-n}\rho, \nu^n\rho], \sigma)},$$
$$\delta([\nu^{-n}\rho, \nu^m\rho]_-, \sigma) = \delta([\nu^{-n}\rho, \nu^m\rho], \sigma)_{\delta([\nu^{-n}\rho, \nu^n\rho]_-, \sigma)}$$

(tempered representations $\delta([\nu^{-n}\rho, \nu^n\rho], \sigma)$ and $\delta([\nu^{-n}\rho, \nu^n\rho]_-, \sigma)$ are defined in Theorem 6.3).

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Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia